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ON THE DECOMPOSITIONS OF COMPLETE GRAPHS INTO CYCLES AND STARS ON THE SAME NUMBER OF EDGES

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Abstract

Let C_m and S_m denote a cycle and a star on m edges, respectively. We investigate the decomposition of the complete graphs, K_n, into cycles and stars on the same number of edges. We give an algorithm that determines values of n, for a given value of m, where K_n is $\{C_m, S_m\}$-decomposable. We show that the obvious necessary condition is sufficient for such decompositions to exist for different values of m.

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1. Introduction

In [6], Alspach conjectured that a multiple length cycle each of length at most n, decomposition of λK_n exists, if $2|\lambda(n-1)$ and the number of edges λK_n is equal the total number of edges of all the cycles. Multiple articles discussed special cases of the conjecture. Results of Alspach, Gavlas, and Šanja solve the conjecture when all the cycles have the same length and $\lambda = 1$ in [7, 9]. In [8], Bryant, Horsley, Maenhaut and Smith extended the results for a general $\lambda \geq 1$.

A graph-pair of order t consists of two non-isomorphic graphs G and H on t non-isolated vertices for which $G \cup H \cong K_t$. In [3], Abueida and Daven showed that there exists a $\{K_m, K_{1,m}\}$-decomposition of λK_n for all $m \geq 3$, $\lambda \geq 1$, and $n \equiv 0, 1 \pmod m$. For graph-pairs of order 4 and 5, G and H, Abueida, Daven, and Roblee (in [2, 4]) determined the values of n for which there exists $\{G, H\}$-decomposition of λK_n for $\lambda \geq 1$. In [5], Abueida and O'Neil showed that there exists a $\{C_m, K_{1,m-1}\}$-decomposition of λK_n for $m = 3$, 4, and 5 and $n \geq m+1$.

Recently, Shyu [10] gave decompositions of the complete graph K_n into p copies of P_{k+1} and q copies of S_{k+1} when $n \geq 4k$, $k(p+q) = \binom{n}{2}$, and either k is even and $p \geq \frac{k}{2}$, or k is odd and $p \geq k$. In [11], Shyu investigated the decomposition of K_n into paths and cycles. He obtained necessary and sufficient condition for decomposing K_n into p copies of P_5 and q copies of C_4 for all possible values of $p \geq 0$ and $q \geq 0$.

A graph G is said to be $\{C_m, S_m\}$-decomposable if there exists a decomposition of G into edge-disjoint subgraphs where each subgraph is isomorphic to either C_m or S_m and where there is at least one copy of C_m and at least one copy of S_m. In this paper we give an algorithm that determines values of n, for a given value of m, where K_n is $\{C_m, S_m\}$-decomposable. We also show that the obvious necessary edge condition is sufficient for some small values of m. Namely, the main results are:

Theorem 1. *For integers m, n with $4 \leq m < n$ and m even, if $n \equiv 0, 1 \pmod{2m}$ and $n \geq 4m$, then K_n is $\{C_m, S_m\}$-decomposable.*

Theorem 2. *For all $n \geq 4m$ such that $m | n(n-1)/2$, if m is even or n is odd, then K_n is $\{C_m, S_m\}$-decomposable.*

2. Preliminaries

As per convention, K_n denotes the complete graph on n vertices. In addition, C_m denotes a cycle of length m, and S_m denotes a star with m edges, that is, $S_m \cong K_{1,m}$. Cycles are denoted by $(v_0, v_1, \ldots, v_n)$, where $\{v_i v_{i+1} : 0 \leq i < n\} \cup \{v_n v_0\}$ are the edges of the cycle. Stars are denoted by $(v_0; v_1, \ldots, v_n)$, where $\{v_0 v_i : 1 \leq i \leq n\}$ are the edges of the star. The set of positive integers is denoted by $\mathbb{N}$.

If a graph G is the union of edge-disjoint subgraphs H_1 and H_2, then we write $G = H_1 \oplus H_2$. If $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, where $H_1 \cong H_2 \cong \cdots \cong H_k \cong H$, we write $G = kH$; the expression $0H$ denotes the null graph.

To facilitate discussion, let Δ_m denote the set of all $\{C_m, S_m\}$-decomposable graphs and let Δ_m^* denote the set of all C_m-decomposable graphs, S_m-decomposable graphs, and $\{C_m, S_m\}$-decomposable graphs. We note that there may be multiple decompositions for a given $G \in \Delta_m$.

To avoid verbosity, let it be understood that all variables introduced are positive integers, unless otherwise specified. In addition, for the sake of consistency, we shall reserve the variables m and n to discuss the $\{C_m, S_m\}$-decomposition of K_n. As such, it should be understood that $3 \le m < n$ whenever $K_n \in \Delta_m$. Clearly, for K_n to be $\{C_m, S_m\}$-decomposable, the number of edges in K_n must be a multiple of m. As such, given any $m \ge 3$, if we want to find all n such that $K_n \in \Delta_m$, we only need to consider all n such that $m|n(n-1)/2$. In this section, we develop an algorithm to determine $\{n : m|n(n-1)/2\}$ for any given m.

We begin with two elementary facts.

Proposition 3. *If* $\gcd(a,b) = 1$ *and* $b \ge 2$*, then there exists a unique integer* $0 \le x < b$ *such that* $ax \equiv 1 \pmod{b}$*.*

Proposition 4. *If* $\gcd(a,b) = 1$ *and* $c|ab$*, then* $c = \gcd(c,a) \cdot \gcd(c,b)$*.*

Proof. Suppose $a = \prod_{i=1}^{A} p_i^{\alpha_i}$ and $b = \prod_{i=1}^{B} q_i^{\beta_i}$ are prime factorizations with positive exponents. Then $\gcd(a,b) = 1$ implies that $\{p_i : 1 \le i \le A\} \cap \{q_i : 1 \le i \le B\} = \varnothing$. Since $c|ab$, we have $c = \prod_{i=1}^{A} p_i^{\gamma_i} \cdot \prod_{i=1}^{B} q_i^{\delta_i}$, where $0 \le \gamma_i \le \alpha_i$ and $0 \le \delta_i \le \beta_i$. Evidently, $\gcd(c,a) = \prod_{i=1}^{A} p_i^{\gamma_i}$ and $\gcd(c,b) = \prod_{i=1}^{B} q_i^{\delta_i}$. ∎

It is trivial that $m|n(n-1)/2$ if and only if $2m|n(n-1)$. We can strengthen this condition by imposing an additional restriction: If m is odd, then $m|n(n-1)/2$ if and only if $m|n(n-1)$. We now introduce a new variable M for convenience. The rest of the results in this section should be applied with $M = 2m$ if m is even, and $M = m$ if m is odd.

Proposition 5. *For* $n \ge 0$ *and* $M \ge 1$*, let* $0 \le r < M$ *be the unique integer such that* $n \equiv r \pmod{M}$*. Then* $M|n(n-1)$ *if and only if* $M|r(r-1)$*.*

Proof. Simply note that $n(n-1) \equiv r(r-1) \pmod{M}$. ∎

For the remainder of this section, the variables n, M, and r are defined as in the above proposition.

Theorem 6. *Let* $A_M = \{a < M : a|M, \gcd(a, M/a) = 1\}$*. For every* $a \in A_M$*, let* x_a *denote the unique integer in* $\{0, 1, 2, \ldots, M/a - 1\}$ *such that* $ax_a \equiv 1 \pmod{M/a}$*. Then* $M|n(n-1)$ *if and only if* $r \in R_M = \{ax_a : a \in A_M\} \cup \{0\}$*.*

Proof. By Proposition 5, it suffices to show that $M|r(r-1)$ if and only if $r \in R_M$. We first demonstrate sufficiency: if $r = 0$, then the conclusion is trivial; otherwise, since $a|ax_a$ and $M/a|ax_a - 1$, it follows that $M|ax_a(ax_a - 1)$. Conversely, if $r = 0$, then we are done; otherwise, let $d = \gcd(M, r)$ and $d' = \gcd(M, r-1)$. Since $\gcd(r, r-1) = 1$, we have $M = dd'$ by Proposition 4. Thus, $\gcd(d, M/d) = \gcd(d, d') = 1$, so $d \in A_M$. Now, $r = dx$ for some x, which implies that $r - 1 = dx - 1$, hence $d'|dx - 1$. But $dx = r < M = dd'$ gives $x < d'$, in which case Proposition 3 guarantees the uniqueness of x, that is, $x = x_d$. ∎

Proposition 7. *For integers* r *and* M*, we have* $1 < r < M$ *and* $M|r(r-1)$ *if and only if* $1 < M + 1 - r < M$ *and* $M|(M+1-r)(M-r)$*.*

Proof. Some elementary algebraic manipulation shows that the inequalities $1 < r < M$ and $1 < M + 1 - r < M$ are equivalent. Furthermore, $r(r-1) \equiv (-r)(1-r) \equiv (M-r)(M+1-r) \pmod{M}$. ■

Now, we use an example to illustrate how Theorem 6 is applied. Then, we show how Proposition 7 simplifies half the work.

Example 8. We shall find all n such that $30|n(n-1)/2$. Since $m = 30$ is even, we use $M = 2m = 60$. By Proposition 5, it suffices to examine all integers less than 60. For a number of this magnitude, a brute-force approach is tedious but not difficult; however, Theorem 6 provides us with a more sophisticated method.

We begin by writing $60 = 2^2 \cdot 3 \cdot 5$. Then $A_{60} = \{1, 2^2, 3, 5, 2^2 \cdot 3, 2^2 \cdot 5, 3 \cdot 5\} = \{1, 3, 4, 5, 12, 15, 20\}$. Next, we find x_a for each $a \in A_{60}$. For $M = 60$, inspection is probably the quickest way to solve for these inverses. For larger numbers, the Euclidean algorithm with back substitution is needed.

$1 \cdot 1 \equiv 1 \pmod{60}$, $3 \cdot 7 \equiv 1 \pmod{20}$, $4 \cdot 4 \equiv 1 \pmod{12}$, $12 \cdot 3 \equiv 1 \pmod{5}$, $15 \cdot 3 \equiv 1 \pmod{4}$, $20 \cdot 2 \equiv 1 \pmod{3}$.

Thus, $R_{60} = \{0, 1, 16, 21, 25, 36, 40, 45\}$, so $30|n(n-1)/2$ if and only if $n \equiv 0, 1, 16, 21, 25, 36, 40, 45 \pmod{60}$.

Now, we shall see how this process of obtaining R_{60} can be simplified. It is trivial that $0, 1 \in R_M$ for any M. For the remaining $r \in R_{60}$, observe that $16 + 45 = 21 + 40 = 25 + 36 = 61 = 60 + 1$. But this should come as no surprise because it simply follows from Proposition 7. This means that, aside from 0 and 1, we only need the "first half" of R_{60} in order to obtain the "second half" by means of subtraction, instead of the less efficient Euclidean algorithm.

The next two corollaries follow immediately from Theorem 6.

Corollary 9. *If $m = 2^k$ for some k, then $m|n(n-1)/2$ if and only if $n \equiv 0, 1 \pmod{2m}$.*

Proof. Since m is even, we use $M = 2m$. It is not difficult to see that $A_{2m} = \{1\}$, so $R_{2m} = \{0, 1\}$. ■

Corollary 10. *If m is odd and has exactly one prime factor, then $m|n(n-1)/2$ if and only if $n \equiv 0, 1 \pmod{m}$.*

Proof. Since m is odd, we use $M = m$. Again, $A_m = \{1\}$, so $R_m = \{0, 1\}$. ■

3. Decompositions

The first two lemmas follow from observing that $K_{a+1} = K_a \oplus S_a$, and more generally, $K_{a+b} = K_a \oplus K_b \oplus K_{a,b}$.

Lemma 11. *If $K_{km} \in \Delta_m^*$ with at least one copy of C_m, then $K_{km+1} \in \Delta_m$.*

Proof. Observe that $K_{km+1} = K_{km} \oplus S_{km} = K_{km} \oplus kS_m$. ∎

Lemma 12. *If $K_{am}, K_b \in \Delta_m^*$ and there exist $\{C_m, S_m\}$-decompositions of K_{am}, and of K_b, with at least one copy of C_m, then $K_{am+b} \in \Delta_m$.*

Proof. Observe that $K_{am+b} = K_{am} \oplus K_b \oplus K_{am,b} = K_{am} \oplus K_b \oplus bS_{am} = K_{am} \oplus K_b \oplus abS_m$. ∎

We make use of the following well-known theory in obtaining our results:

Theorem 13 [7, 9]**.** *For any positive integers m and n, there exists a C_m-decomposition of K_n if and only if n is odd, $3 \leq m \leq n$, and $n(n-1) \equiv 0 \pmod{2m}$.*

Theorem 14 [13]**.** *K_a is S_m-decomposable if and only if $a \geq 2m$ and $m | a(a-1)/2$.*

Corollary 15 [13]**.** *K_{2m} is S_m-decomposable.*

Theorem 16 [12]**.** *For any positive integers a, b and m, there exists a C_{2m}-decomposition of $K_{a,b}$ if and only if a and b are even, $m \geq 2$, $a \geq m$, $b \geq m$, and $ab \equiv 0 \pmod{2m}$.*

Corollary 17 [12]**.** *For any positive even integer m, $K_{2m,2m}$ is C_m-decomposable.*

The next two results provide conditions for $\{C_m, S_m\}$-decompositions when m is odd.

Theorem 18. *For integers m, n with $3 \leq m < n$ and m odd, if $n \equiv 0, 1 \pmod{m}$, then $K_n \in \Delta_m$.*

Proof. By Theorem 13, K_m is C_m-decomposable. Consequently, $K_{m+1} \in \Delta_m$ by Lemma 11. Next, Lemma 12 implies that $K_{2m} = K_{m+m} \in \Delta_m$. We complete the proof by applying Lemma 12 inductively. ∎

Combining Theorem 18 and Corollary 10, we obtain

Theorem 19. *Suppose $3 \leq m < n$ are integers with $m \in \{p^k : p$ is an odd prime, $k \in \mathbb{N}\}$. Then $K_n \in \Delta_m$ if and only if $n \equiv 0, 1 \pmod{m}$.*

Lemma 20. *If $K_a \in \Delta_m$, then $K_{a+2km} \in \Delta_m$ for all $k \geq 0$.*

Proof. This is trivial when $k = 0$. Since K_{2m} is S_m-decomposable by Corollary 15, and $K_{a+2km} \in \Delta_m$ by the induction hypothesis, it follows from Lemma 12 that $K_{a+2(k+1)m} = K_{2m+(a+2km)} \in \Delta_m$. ∎

The next theorem is analogous to Theorem 18 in the case that m is even.

Theorem 21. *For integers m, n with $4 \leq m < n$ and m even, if $n \equiv 0, 1 \pmod{2m}$ and $n \geq 4m$, then $K_n \in \Delta_m$.*

Proof. By Lemma 20, it suffices to show that $K_{4m}, K_{4m+1} \in \Delta_m$. Corollary 15 and Corollary 17 imply that $K_{4m} = K_{2m+2m} = K_{2m} \oplus K_{2m} \oplus K_{2m,2m} \in \Delta_m$. By Lemma 11, $K_{4m+1} \in \Delta_m$. ∎

Now, we shall temporarily remove the restriction of parity and examine the conditions for $\{C_m, S_m\}$-decompositions when m is arbitrary. The following discussion refers to Proposition 7 with $M = 2m$. Recall that $M = 2m$ is in fact a weaker condition than $M = m$ in the case that m is odd. Thus, the following results apply to all m, regardless of parity.

Since r and $2m + 1 - r$ have different parities, $r \neq 2m + 1 - r$. Moreover, if $r \leq m$, then $2m+1-r \geq m+1$, and if $r \geq m+1$, then $2m+1-r \leq m$. Thus, every $r \in \{2, \ldots, m\}$ has exactly one *complement* $2m + 1 - r \in \{m + 1, \ldots, 2m - 1\}$, and vice versa.

The significance of this idea of complements is best illustrated through an example. Recall from Example 8 that 16 and 45 are complements of each other for $M = 60$. It turns out that if $K_{60k+16} \in \Delta_{30}$, then $K_{60k+45} \in \Delta_{30}$, and if $K_{60k+45} \in \Delta_{30}$, then $K_{60(k+1)+16} \in \Delta_{30}$. In general, for any $1 < r \leq m$, if $K_{2km+r} \in \Delta_m$, then $K_{2km+(2m+1-r)} \in \Delta_m$, and if $K_{2km+(2m+1-r)} \in \Delta_m$, then $K_{2(k+1)m+r} \in \Delta_m$. Evidently, if we can find an appropriate "starting point", then this chain of implications gives us an infinite list of values of n such that $K_n \in \Delta_m$. We now prove this.

Lemma 22. *If a is odd, then K_a is $S_{(a-1)/2}$-decomposable.*

Proof. Let $V(K_a) = \{v_i : 0 \leq i < a\}$. Then
$$\{(v_i; v_{i+1 \,(\mathrm{mod}\ a)}, \ldots, v_{i+(a-1)/2 \,(\mathrm{mod}\ a)}) : 0 \leq i < a\}$$
is one possible set of stars into which K_a can be decomposed. ∎

An example should make the previous lemma abundantly clear. For instance, K_5 can be decomposed into 5 copies of S_2: $(v_0; v_1, v_2)$, $(v_1; v_2, v_3)$, $(v_2; v_3, v_4)$, $(v_3; v_4, v_0)$, and $(v_4; v_0, v_1)$.

Lemma 23. *If a is odd and $K_{bm-(a-1)/2} \in \Delta_m^*$ with at least one cycle C_m, then $K_{bm+(a+1)/2} \in \Delta_m$.*

Proof. Notice that $K_{bm+(a+1)/2} = K_{bm-(a-1)/2} \oplus K_a \oplus K_{bm-(a-1)/2,a}$. For each $v \in V(K_a)$, we have b copies of S_m: The first $b - 1$ copies are the edges from v to $(b - 1)m$ vertices in $K_{bm-(a-1)/2}$; the last copy is obtained by combining an $S_{(a-1)/2}$ (from Lemma 22) with the remaining $m - (a - 1)/2$ edges from v to $K_{bm-(a-1)/2}$. This gives an S_m-decomposition of $K_a \oplus K_{bm-(a-1)/2,a}$, which completes the proof. ∎

The next corollary proves our above claim that if $K_{60k+16} \in \Delta_{30}$, then $K_{60k+45} \in \Delta_{30}$.

Corollary 24. *If* $1 < r \leq m$ *and* $K_{2km+r} \in \Delta_m$*, then* $K_{2km+(2m+1-r)} \in \Delta_m$*.*

Proof. Apply Lemma 23 with $a = 2m + 1 - 2r$ and $b = 2k + 1$. ■

Naturally, the next corollary proves our second claim that if $K_{60k+45} \in \Delta_{30}$, then $K_{60(k+1)+16} \in \Delta_{30}$.

Corollary 25. *If* $1 < r \leq m$ *and* $K_{2km+(2m+1-r)} \in \Delta_m$*, then* $K_{2(k+1)m+r} \in \Delta_m$*.*

Proof. Apply Lemma 23 with $a = 2r - 1$ and $b = 2k + 2$. ■

Theorem 26. *For all* $n \geq 4m$ *such that* $m|n(n-1)/2$*, if* m *is even or* n *is odd, then* $K_n \in \Delta_m$*.*

Proof. First, $K_n = K_{2m} \oplus K_{n-2m} \oplus K_{2m,n-2m}$. By Corollary 15, K_{2m} is S_m-decomposable.

If n is even (and also is m from the hypothesis of the theorem), then so is $n - 2m$. Since $n \geq 4m$ gives $n - 2m \geq 2m$, and m divides

$$\frac{(n-2m)(n-2m-1)}{2} = \frac{n(n-1)}{2} - 2mn + 2m^2 + m,$$

it follows that K_{n-2m} is S_m-decomposable by Theorem 14. Now, it is clear that m is even, $m \leq 2m$, $m \leq n - 2m$ and $m|2m(n-2m)$, so $K_{2m,n-2m}$ is C_m-decomposable by Theorem 16. Thus, $K_n \in \Delta_m$.

It remains to show that $K_n \in \Delta_m$ when n is odd (for any parity of m), in which case $n - 2m$ is odd. We have shown that $m|(n-2m)(n-2m-1)/2$, so K_{n-2m} is C_m-decomposable by Theorem 13. Furthermore, $K_{2m,n-2m}$ is clearly S_m-decomposable, so $K_n \in \Delta_m$. ■

If we relax the hypothesis so that $n > 5m$, then the restrictions on the parities of m and of n can be removed.

Corollary 27. *If* $n > 5m$ *and* $m|n(n-1)/2$*, then* $K_n \in \Delta_m$*.*

Proof. We only have to examine the case in which m is odd and n is even. Note that $K_n = K_m \oplus K_m \oplus K_{n-2m} \oplus K_{m,n-2m} \oplus K_{m,n-2m} \oplus K_{m,m}$. By Theorem 13, each copy of K_m is C_m-decomposable. Similar to arguments in the proof of Theorem 26, as $n - 2m > 3m > 2m$, and $m|n-2m)(n-2m-1)/2$, we use Theorem 14 to show that K_{n-2m} is S_m-decomposable. Finally, it is clear that $K_{m,n-2m}$ and $K_{m,m}$ are S_m-decomposable. Hence, $K_n \in \Delta_m$. ■

Note that this corollary also implies that for every m, the list of all n such that $K_n \in \Delta_m$ is "almost" complete in the sense that there are only finitely many n for which the $\{C_m, S_m\}$-decomposability of K_n is unknown.

We now present results on $\{C_m, S_m\}$-decompositions for specific values of m. Henceforth, we adopt the labeling convention $V(K_n) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$.

Theorem 28. *$K_n \in \Delta_4$ if and only if $n \equiv 0, 1 \pmod 8$.*

Proof. One possible $\{C_4, S_4\}$-decomposition of K_8 is: (v_0, v_1, v_2, v_3), (v_4, v_5, v_6, v_7), (v_1, v_5, v_2, v_6), $(v_0; v_2, v_4, v_5, v_6)$, $(v_3; v_1, v_5, v_6, v_7)$, $(v_4; v_1, v_2, v_3, v_6)$, $(v_7; v_0, v_1, v_2, v_5)$. By Lemma 11, $K_9 \in \Delta_4$. Theorem 21 completes the proof of sufficiency. Necessity follows from Corollary 9. ■

From the above proof, it is not difficult to see that in general, for any $m = 2^k$, if we can find a $\{C_m, S_m\}$-decomposition of K_{2m}, then it immediately follows that $K_n \in \Delta_m$ if and only if $n \equiv 0, 1 \pmod{2m}$.

We digress momentarily to present a result whose proof is similar in nature to a later lemma.

Proposition 29. *If m is even and G is an S_m-decomposable graph in which every vertex is of odd degree, then the center of every star (into which G decomposes) is a leaf of another star (into which G decomposes).*

Proposition 30. *If m and n are even and $K_n \in \Delta_m$, then the number of stars S_m is at least three.*

Proof. Write $K_n = G_C \oplus G_S$, where G_C is C_m-decomposable and G_S is S_m-decomposable. Since the degree of any vertex in K_n is odd, and that of any vertex in G_C is even, every vertex in G_S must be of odd degree. Consequently, the center of each star must be a leaf of at least one other star. It is then easy to see that three is the minimum number of stars required to satisfy this condition. ■

Proposition 31. *If $K_{2m} \in \Delta_m$ and the number of S_m is three, then there is only one way to obtain the three stars (every other way is isomorphic to this).*

Proof. Note that each vertex can be the center of at most one star. Without loss of generality, let $(v_0; v_1, v_2, \ldots, v_m)$ be the first star. Now, v_0 must be the leaf of another star. Again, without any loss of generality, let v_{m+1} be the center of the second star. We shall momentarily skip the other leaves of the second star. There must be a third star centered at a leaf of the first star, say v_1. The star centered at v_1 must have a leaf at v_{m+1}. At this point, we have added the following edges: $v_0v_1, \ldots, v_0v_m, v_{m+1}v_0, v_1v_{m+1}$. Now, there are $m-1$ vertices, namely $v_2, \ldots, v_m$, with degree 1, and $m-2$ vertices, namely $v_{m+2}, \ldots, v_{2m-1}$, with degree 0. After completing the second star, we must have exactly $m-1$ vertices with even degree, so that each of these vertices can be a leaf for the third star. But there is only one way to do this: use $m/2$ vertices in $\{v_2, \ldots, v_m\}$ and $m/2-1$ vertices in $\{v_{m+2}, \ldots, v_{2m-1}\}$ as leaves for the second star. ■

Before we present a necessary and sufficient condition for n so that $K_n \in \Delta_6$, we first need the following lemma.

Lemma 32. *$K_9 \notin \Delta_6$.*

Proof. Suppose $K_9 \in \Delta_6$. Since K_9 has 36 edges, it follows that the total number of copies of C_6's and S_6's is 6.

Case 1. One copy of C_6 and five copies of S_6. Let $(v_0, v_1, v_2, v_3, v_4, v_5)$ be the 6-cycle. Since there are five copies of S_6, some v_k with $0 \leq k \leq 5$ is *not* the center of a star. Since v_k must have degree 8 and the cycle contributes 2 to the degree, v_k must be a leaf for 6 other stars, contradicting that there are five copies of S_6.

Case 2. Two copies of C_6 and four copies of S_6. Let $(v_0, v_1, v_2, v_3, v_4, v_5)$ be the first 6-cycle. Since there are four copies of S_6, some v_k with $0 \leq k \leq 5$ *is* the center of a star. Without loss of generality, assume $k = 0$. Then the star centered at v_0 is $(v_0; v_2, v_3, v_4, v_6, v_7, v_8)$. Now, if v_1 is not the center of a star, then either (a) it is a leaf for 6 other stars, contradicting that there are four copies of S_6, or (b) it is a leaf for 4 other stars, as well as a vertex in the second cycle, again contradicting that there are four copies of S_6, so there must be a star centered at v_1. There is another star centered at v_5 for identical reasons. A slightly modified version of the above argument shows that there are stars centered at v_6, v_7, and v_8 as well. This gives us 6 stars, contradicting that there are four copies of S_6.

Case 3. Five copies of C_6 and one copy of S_6. Let $(v_0; v_1, v_2, v_3, v_4, v_5, v_6)$ be the S_6. Then $v_1, \ldots, v_6$ all have odd degree 1, so it is impossible to add cycles to obtain a complete graph.

Case 4. Four copies of C_6 and two copies of S_6. Let $(v_0; v_1, v_2, v_3, v_4, v_5, v_6)$ be the first S_6. Note that each star switches the parity of the degree of exactly 6 vertices (the parity of the degree of the center does not change as it increases by 6, the parity of the degree of each leaf changes as it increases by 1). In order to decompose the rest of the graph into cycles, we need the degree of every vertex to be even after adding the second star. Without loss of generality, we can only have one such star: $(v_7; v_1, v_2, v_3, v_4, v_5, v_6)$. Now, the cycle through v_0 must contain the sequence $(\ldots, v_7, v_0, v_8, \ldots)$, and the cycle through v_7 must contain the sequence $(\ldots, v_0, v_7, v_8, \ldots)$, rendering it impossible to have a 6-cycle through v_0 or v_7, which is a contradiction.

Case 5. Three copies of each C_6 and S_6. Let $(v_0; v_1, v_2, v_3, v_4, v_5, v_6)$ be the first S_6. We shall skip the second star momentarily. After adding the third star, every vertex must have even degree, which means that exactly 6 vertices must have odd degree after adding the second star. But there are exactly 6 vertices with odd degree after adding the *first* star. The only way to preserve the number of vertices with odd degree after adding the second star is to switch the parities of 3 odd vertices and 3 even vertices. The even vertices are v_0, v_7, and v_8. Without loss of generality, let the odd vertices be v_1, v_2, and v_3. To switch the parities of these 6 vertices, the second star must be centered at v_4, v_5, or v_6, but it is impossible to add another edge from any of these vertices to v_0. ■

Theorem 33. *$K_n \in \Delta_6$ if and only if $n \equiv 0, 1, 4, 9 \pmod{12}$ and $n \geq 12$.*

Proof. If $K_n \in \Delta_6$, then $6|n(n-1)/2$. Equivalently, $12|n(n-1)$. Now, $A_{12} = \{1, 2^2, 3\} = \{1, 3, 4\}$ and $1 \cdot 1 \equiv 1 \pmod{12}$, $3 \cdot 3 \equiv 1 \pmod{4}$, $4 \cdot 1 \equiv 1 \pmod{3}$.

Thus, $R_{12} = \{0, 1, 4, 9\}$, so $n \equiv 0, 1, 4, 9 \pmod{12}$. By Lemma 32, $n \neq 9$, so $n \geq 12$.

Conversely, by Lemma 20, we only have to show that $K_{12}, K_{13}, K_{16}, K_{21} \in \Delta_6$.

One possible $\{C_6, S_6\}$-decomposition of K_{12} is: $(v_0; v_1, v_2, v_3, v_4, v_5, v_6)$, $(v_{11}; v_0, v_1, v_2, v_6, v_9, v_{10})$, $(v_5; v_1, v_2, v_6, v_7, v_8, v_{11})$, $(v_1, v_2, v_3, v_4, v_5, v_9)$, $(v_1, v_3, v_5, v_{10}, v_6, v_8)$, $(v_0, v_7, v_1, v_{10}, v_2, v_8)$, $(v_1, v_4, v_2, v_7, v_3, v_6)$, $(v_0, v_9, v_2, v_6, v_4, v_{10})$, $(v_3, v_8, v_4, v_7, v_6, v_9)$, $(v_3, v_{10}, v_7, v_9, v_8, v_{11})$, $(v_4, v_9, v_{10}, v_8, v_7, v_{11})$.

By Lemma 11, $K_{13} \in \Delta_6$.

To see that $K_{16} \in \Delta_6$, start with the above decomposition of K_{12}, add the cycle $(v_0, v_{13}, v_1, v_{14}, v_2, v_{12})$, and add the following stars: $(v_i; v_6, v_7, v_8, v_9, v_{10}, v_{11})$ for $12 \leq i \leq 15$, and $(v_{12}; v_1, v_3, v_4, v_5, v_{13}, v_{15})$, $(v_{13}; v_2, v_3, v_4, v_5, v_{14}, v_{15})$, $(v_{14}; v_0, v_3, v_4, v_5, v_{12}, v_{15})$, $(v_{15}; v_0, v_1, v_2, v_3, v_4, v_5)$.

To see that $K_{21} \in \Delta_6$, simply apply Corollary 24: start with the above decomposition of K_{16} and add the following stars: $(v_i; v_4, v_5, v_6, v_7, v_8, v_9)$ and $(v_i; v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15})$ for $16 \leq i \leq 20$, and $(v_{16}; v_0, v_1, v_2, v_3, v_{17}, v_{18})$, $(v_{17}; v_0, v_1, v_2, v_3, v_{18}, v_{19})$, $(v_{18}; v_0, v_1, v_2, v_3, v_{19}, v_{20})$, $(v_{19}; v_0, v_1, v_2, v_3, v_{20}, v_{16})$, $(v_{20}; v_0, v_1, v_2, v_3, v_{16}, v_{17})$. ■

Theorem 34. *If $n \equiv 0, 1, 5, 16 \pmod{20}$ and $n \geq 25$, then $K_n \in \Delta_{10}$.*

Proof. By Lemma 20 and Theorem 21, we only have to show that $K_{25}, K_{36} \in \Delta_{10}$.

To see that $K_{25} \in \Delta_{10}$, start with an S_{10}-decomposition of K_{20} (this is possible by Corollary 15), add the cycle $(v_0, v_{21}, v_1, v_{22}, v_2, v_{23}, v_3, v_{24}, v_4, v_{20})$, and add the following stars: $(v_i; v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19})$ for $20 \leq i \leq 24$, and $(v_{20}; v_1, v_2, v_3, v_5, v_6, v_7, v_8, v_9, v_{21}, v_{22})$, $(v_{21}; v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{22}, v_{23})$,

$(v_{22}; v_3, v_4, v_0, v_5, v_6, v_7, v_8, v_9, v_{23}, v_{24}), (v_{23}; v_4, v_0, v_1, v_5, v_6, v_7, v_8, v_9, v_{24}, v_{20}),$
$(v_{24}; v_0, v_1, v_2, v_5, v_6, v_7, v_8, v_9, v_{20}, v_{21})$.

To see that $K_{36} \in \Delta_{10}$, simply apply Corollary 24: start with the above decomposition of K_{25} and add the following stars: $(v_i; v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14})$ and $(v_i; v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24})$ for $25 \leq i \leq 35$, and

$$\begin{aligned}
&(v_{25}; v_0, v_1, v_2, v_3, v_4, v_{26}, v_{27}, v_{28}, v_{29}, v_{30}),\\
&(v_{26}; v_0, v_1, v_2, v_3, v_4, v_{27}, v_{28}, v_{29}, v_{30}, v_{31}),\\
&(v_{27}; v_0, v_1, v_2, v_3, v_4, v_{28}, v_{29}, v_{30}, v_{31}, v_{32}),\\
&(v_{28}; v_0, v_1, v_2, v_3, v_4, v_{29}, v_{30}, v_{31}, v_{32}, v_{33}),\\
&(v_{29}; v_0, v_1, v_2, v_3, v_4, v_{30}, v_{31}, v_{32}, v_{33}, v_{34}),\\
&(v_{30}; v_0, v_1, v_2, v_3, v_4, v_{31}, v_{32}, v_{33}, v_{34}, v_{35}),\\
&(v_{31}; v_0, v_1, v_2, v_3, v_4, v_{32}, v_{33}, v_{34}, v_{35}, v_{25}),\\
&(v_{32}; v_0, v_1, v_2, v_3, v_4, v_{33}, v_{34}, v_{35}, v_{25}, v_{26}),\\
&(v_{33}; v_0, v_1, v_2, v_3, v_4, v_{34}, v_{35}, v_{25}, v_{26}, v_{27}),\\
&(v_{34}; v_0, v_1, v_2, v_3, v_4, v_{35}, v_{25}, v_{26}, v_{27}, v_{28}),\\
&(v_{35}; v_0, v_1, v_2, v_3, v_4, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}).
\end{aligned}$$

■

Theorem 35. *If* $n \equiv 0, 1, 9, 16 \pmod{24}$ *and* $n \geq 33$, *then* $K_n \in \Delta_{12}$.

Proof. By Lemma 20 and Theorem 21, we only have to show that $K_{33}, K_{40} \in \Delta_{12}$.

To see that $K_{33} \in \Delta_{12}$, start with an S_{12}-decomposition of K_{24} (this is possible by Corollary 15), add the cycles $(v_0, v_{25}, v_1, v_{26}, v_2, v_{27}, v_3, v_{28}, v_4, v_{29}, v_5, v_{24})$, $(v_6, v_{28}, v_7, v_{29}, v_8, v_{30}, v_9, v_{31}, v_{10}, v_{32}, v_{11}, v_{27})$, and $(v_{12}, v_{31}, v_{13}, v_{32}, v_{14}, v_{24}, v_{15}, v_{25}, v_{16}, v_{26}, v_{17}, v_{30})$, and add the following stars:

$$\begin{aligned}
&(v_{24}; v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_9, v_{25}, v_{26}, v_{27}, v_{28}),\\
&(v_{24}; v_{10}, v_{11}, v_{12}, v_{13}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}),\\
&(v_{25}; v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{26}, v_{27}, v_{28}, v_{29}),\\
&(v_{25}; v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}),\\
&(v_{26}; v_0, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{27}, v_{28}, v_{29}, v_{30}),\\
&(v_{26}; v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}),\\
&(v_{27}; v_0, v_1, v_4, v_5, v_7, v_8, v_9, v_{10}, v_{28}, v_{29}, v_{30}, v_{31}),\\
&(v_{27}; v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}),\\
&(v_{28}; v_0, v_1, v_2, v_5, v_8, v_9, v_{10}, v_{11}, v_{29}, v_{30}, v_{31}, v_{32}),\\
&(v_{28}; v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}),\\
&(v_{29}; v_0, v_1, v_2, v_3, v_6, v_9, v_{10}, v_{11}, v_{24}, v_{30}, v_{31}, v_{32}),\\
&(v_{29}; v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}),\\
&(v_{30}; v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_{24}, v_{25}, v_{31}, v_{32}),\\
&(v_{30}; v_{10}, v_{11}, v_{13}, v_{14}, v_{15}, v_{16}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}),\\
&(v_{31}; v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_{24}, v_{25}, v_{26}, v_{32}),\\
&(v_{31}; v_8, v_{11}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}),\\
&(v_{32}; v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_{24}, v_{25}, v_{26}, v_{27}),
\end{aligned}$$

$(v_{32}; v_8, v_9, v_{12}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23})$.
We know that $K_{40} \in \Delta_{12}$ by Corollary 24. ■

Theorem 36. *If* $n \equiv 0, 1, 8, 21 \pmod{28}$ *and* $n \geq 36$, *then* $K_n \in \Delta_{14}$.

Proof. By Lemma 20 and Theorem 21, we only have to show that $K_{36}, K_{49} \in \Delta_{14}$.

To see that $K_{36} \in \Delta_{14}$, start with an S_{14}-decomposition of K_{28} (this is possible by Corollary 15), add the cycles $(v_0, v_{28}, v_1, v_{29}, v_2, v_{30}, v_3, v_{31}, v_4, v_{32}, v_5, v_{33}, v_6, v_{34})$, and $(v_7, v_{28}, v_8, v_{29}, v_9, v_{30}, v_{10}, v_{31}, v_{11}, v_{32}, v_{12}, v_{33}, v_{13}, v_{34})$, with the following stars:

$(v_{28}; v_2, v_3, v_4, v_5, v_6, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{29}, v_{30}, v_{31}, v_{35})$,
$(v_{28}; v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27})$,
$(v_{29}; v_0, v_3, v_4, v_5, v_6, v_7, v_{10}, v_{11}, v_{12}, v_{13}, v_{30}, v_{31}, v_{32}, v_{35})$,
$(v_{29}; v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27})$,
$(v_{30}; v_0, v_1, v_4, v_5, v_6, v_7, v_8, v_{11}, v_{12}, v_{13}, v_{31}, v_{32}, v_{33}, v_{35})$,
$(v_{30}; v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27})$,
$(v_{31}; v_0, v_1, v_2, v_5, v_6, v_7, v_8, v_9, v_{12}, v_{13}, v_{32}, v_{33}, v_{34}, v_{35})$,
$(v_{31}; v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27})$,
$(v_{32}; v_0, v_1, v_2, v_3, v_6, v_7, v_8, v_9, v_{10}, v_{13}, v_{28}, v_{33}, v_{34}, v_{35})$,
$(v_{32}; v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27})$,
$(v_{33}; v_0, v_1, v_2, v_3, v_4, v_7, v_8, v_9, v_{10}, v_{11}, v_{28}, v_{29}, v_{34}, v_{35})$,
$(v_{33}; v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27})$,
$(v_{34}; v_1, v_2, v_3, v_4, v_5, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{28}, v_{29}, v_{30}, v_{35})$,
$(v_{34}; v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27})$,
$(v_{35}; v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13})$,
$(v_{35}; v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27})$.

We know that $K_{49} \in \Delta_{14}$ by Corollary 24. ■

We conclude by stating that Lemma 20 and Theorem 21 can be used to obtain more results for different values of m.

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