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# THE MINIMUM SPECTRAL RADIUS OF SIGNLESS LAPLACIAN OF GRAPHS WITH A GIVEN CLIQUE NUMBER

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**Abstract**

In this paper we observe that the minimal signless Laplacian spectral radius is obtained uniquely at the kite graph $PK_{n-\omega,\omega}$ among all connected graphs with $n$ vertices and clique number $\omega$. In addition, we show that the spectral radius $\mu$ of $PK_{m,\omega}$ $(m \geq 1)$ satisfies

$$\frac{1}{2}(2\omega - 1 + \sqrt{4\omega^2 - 12\omega + 17}) \leq \mu \leq 2\omega - 1.$$

More precisely, for $m > 1$, $\mu$ satisfies the equation

$$\mu - \omega - \frac{\omega - 1}{\mu - 2\omega + 3} = a_m \sqrt{\mu^2 - 4\mu} + \frac{1}{t_1},$$

where $a_m = \frac{1}{1-t_1^{2m+3}}$ and $t_1 = \frac{\mu-2+\sqrt{(\mu-2)^2-4}}{2}$. At last the spectral radius $\mu(PK_{\infty,\omega})$ of the infinite graph $PK_{\infty,\omega}$ is also discussed.

**Keywords:** clique number, kite graph, signless Laplacian, spectral radius.

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## 1. Introduction

All graphs considered here are connected and undirected. Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The adjacency matrix of $G$ is the $n \times n$ matrix $A(G) = [a_{ij}]$ given by: $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$, and $a_{ij} = 0$ otherwise. The *signless Laplacian* of $G$ is defined as $Q(G) = D(G) + A(G)$, where $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ with $d_i = d(v_i)$ being the degree of vertex $v_i$ of $G$ $(1 \leq i \leq n)$. The maximum eigenvalue of $Q(G)$ is called the *signless Laplacian spectral radius* of $G$, denote by $\mu(G)$.

The signless Laplacian has received more and more attention since 2005. The papers [5, 6] provide spectral uncertainties with respect to the adjacency matrix, the Laplacian and signless Laplacian of sets of all graphs with order not more than 11. It was found that the spectra of signless Laplacian is more efficient in characterizing graph structure than those of other matrices. An idea was expressed in [5] that, among matrices associated with a graph, the signless Laplacian seems to be the most convenient for use in studying graph properties. Partially it is the reason to make us a believer on the power of signless Laplacian.

Stevanović *et al.* [9] have considered the problem of characterizing the minimum spectral radius of adjacency matrix among all graphs in $G_{n,\omega}$, of connected graphs of order $n$ with a maximum clique size $\omega$. We are interested in the same problem for the signless Laplacian. In this paper we observe that the minimal signless Laplacian spectral radius is obtained uniquely at the kite graph $PK_{n-\omega,\omega}$ among all connected graphs with $n$ vertices and clique number $\omega$. Recall that the kite graph $PK_{m,\omega}$ is a graph on $m+\omega$ vertices obtained from the path $P_m$ and the complete graph $K_\omega$ by adding an edge between an end vertex of $P_m$ and a vertex of $K_\omega$. In addition, we show that the spectral radius $\mu$ of $PK_{m,\omega}$ $(m \geq 1)$ satisfies

$$\frac{1}{2}(2\omega - 1 + \sqrt{4\omega^2 - 12\omega + 17}) \leq \mu \leq 2\omega - 1.$$

More precisely, for $m > 1$, $\mu$ satisfies the equation

$$\mu - \omega - \frac{\omega - 1}{\mu - 2\omega + 3} = a_m\sqrt{\mu^2 - 4\mu} + \frac{1}{t_1},$$

where $a_m = \frac{1}{1-t_1^{2m+3}}$ and $t_1 = \frac{\mu-2+\sqrt{(\mu-2)^2-4}}{2}$. At last the spectral radius $\mu(PK_{\infty,\omega})$ of the infinite graph $PK_{\infty,\omega}$ is also discussed.

## 2. The Extremal Graph in $G_{n,\omega}$

In this section we will characterize the graph(s) in $G_{n,\omega}$ with minimum signless Laplacian spectral radius for given $\omega$. We denote by $P_n$ the path of order $n$. First we list some lemmas which will be used in the sequel.

**Lemma 1** [3]**.** *Let $G(k,l;v)$ $(k,l \geq 0)$ be the graph obtained from a non-trivial connected graph $G$ by attaching pendent paths of lengths $k$ and $l$ at some vertex $v$. If $k \geq l \geq 1$, then $\mu(G(k,l;v)) > \mu(G(k+1,l-1;v))$.*

**Lemma 2** [4]**.** *Let $u$ $v$ be the adjacent vertices of a connected graph $G$, both of degree at least two. Let $G(k,l;u,v)$ $(k,l \geq 0)$ be the graph obtained from $G$ by attaching pendent paths of lengths $k$ and $l$ at $u$ and $v$ respectively. If $k \geq l \geq 1$, then $\mu(G(k,l;u,v)) > \mu(G(k+1,l-1;u,v))$.*

Let $T$ be a tree, $v$ $(d(v) \geq 3)$ be a given vertex of $T$ at which two distinct pendant paths $P = vu_1 \cdots u_k$ and $Q = vv_1 \cdots v_l$ are attached. Then we form a tree $T'$ by removing the paths $P$ and $Q$ and replacing them with a longer path $vu_1 \cdots u_k v_1 \cdots v_l$. We say that $T'$ is a $\pi$-transform of $T$.

**Lemma 3** [8]**.** *Every tree which is not a path contains a vertex of degree at least three at which (at least) two pendant paths are attached. In particular, every tree can be transformed into a path by a sequence of $\pi$-transformations.*

The following result follows directly from the Perron-Frobenius Theorem.

**Lemma 4.** *Let $H$ be a proper subgraph of a connected graph. Then $\mu(H) < \mu(G)$.*

**Theorem 5.** *If $G \in G_{n,\omega}$, $n \geq \omega \geq 2$, then*

$$\mu(G) \geq \mu(PK_{n-\omega,\omega}),$$

*where the equality holds if and only if $G$ is isomorphic to $PK_{n-\omega,\omega}$.*

***Proof.*** Assume $n$ and $\omega$ are two given positive integers such that $n \geq \omega \geq 2$. We have two cases.

*Case* 1. $n = \omega$. In this case obviously $G_{n,\omega}$ consists of a single graph $K_n$, which is also a kite graph $PK_{0,n}$.

*Case* 2. $n > \omega$.

*Case* 2.1. $\omega = 2$. It was known that $P_n$ has the minimal signless Laplacian spectral radius among all connected graphs of order $n$ (see [1]), so is in $G_{n,\omega}$. Note that $PK_{n-2,2} = P_n$ and the assertion follows immediately.

*Case* 2.2. $\omega \geq 3$. Next we will adopt the following methods to transform an arbitrary graph $G \in G_{n,\omega}$ into a kite graph $PK_{n-\omega,\omega}$ by a series of transformations in which the signless Laplacian spectral radius decreases at each step.

i) Denote by $K$ the maximum clique in $G$. Delete from $G$ (in an arbitrary order) any edge not in $K$ which belongs to a cycle as long as they exist. The resulting graph is denoted by $G_1$. Note that $G_1$ consists of the clique $K$ with a number of rooted trees attached to clique vertices. It is known from Lemma 4 that the spectral radius of $G_1$ has been strictly decreased. In addition, $G_1$ still belong to $G_{n,\omega}$.

ii) Let $T$ be a rooted tree of $G_1$, attached to a clique vertex. If $T$ is not a path, then the tree $T$ can be transformed into a path by applying a sequence of $\pi$-transformations by Lemma 3, in which the signless Laplacian spectral radius decreases every time by Lemma 4. Finally, we reach a graph $G_2$ in which every rooted tree, attached to a vertex of $K$, becomes a path.

iii) Suppose that $G_2$ consist of clique $K$ and the paths $P_{k_1}, P_{k_2}, \dots, P_{k_s}$ attached to $s$ distinct vertices of $K$. Without loss of generality, let $P_{k_1}$ be one of the longest paths among $P_{k_1}, P_{k_2}, \dots, P_{k_s}$. By repeatedly using Lemma 2 to paths $P_{k_1}$ and $P_{k_i}$, $2 \le i \le m$, we may decrease the signless Laplacian spectral radius of $G_2$ until the attached paths $P_{k_2}, \dots, P_{k_m}$ disappear, and we finally arrive to the kite graph $PK_{n-\omega,\omega}$.

Since the signless Laplacian spectral radius of $G$ has been decreased strictly at each step, we may conclude that the kite graph $PK_{n-\omega,\omega}$ has minimum spectral radius in $G_{n,\omega}$. From the above process, if a graph $G \in G_{n,\omega}$ is not isomorphic to a kite graph $PK_{n-\omega,\omega}$, then $\mu(PK_{n-\omega,\omega}) < \mu(G)$. Therefore it follows that the result holds. ∎

## 3. ESTIMATING THE VALUE OF $\mu(PK_{m,\omega})$

Though the values of $\mu(PK_{m,\omega})$, $m \ge 1$ are not straightforward to obtain, we may get a small interval to which the spectral radius of kite graphs $\mu(PK_{m,\omega})$ belongs.

**Lemma 6** [2]. *Let $G$ be a graph on $n$ vertices with vertex degrees $d_1, d_2, \dots, d_n$. Then*

$$\min(d_i + d_j) \le \mu \le \max(d_i + d_j),$$

*where $(i, j)$ runs over all pairs of adjacent vertices of $G$. If $G$ is connected, then equality holds in either of these inequalities if and only if $G$ is regular or semiregular bipartite.*

Since $K_\omega$ is a proper subgraph of $PK_{m,\omega}$ and by Lemma 4, we have $\mu(PK_{m,\omega}) > \mu(K_\omega) = 2\omega - 2$. Meantime by Lemma 6 it follows immediately that

$$2\omega - 2 \le \mu(PK_{m,\omega}) \le 2\omega - 1. \tag{1}$$

Recall that the kite graph $PK_{m,\omega}$ may be obtained by joining a vertex of a complete graph $K_\omega$ to an end vertex of a path $P_m$. Let $x$ be a principal eigenvector of $PK_{m,\omega}$. By symmetry, all vertices but the vertex to which the path $P_m$ is joined has the same value, name it $x_{-1}$, with respect to $x$. Let $x_0, x_1, x_2, \dots x_m$ be the components of $x$ along the path vertices.

(a) $m = 1$. In this case the eigenvalue equation reads

$$\begin{aligned}\mu x_{-1} &= (\omega-2)x_{-1} + x_0 + (\omega-1)x_{-1},\\ \mu x_0 &= (\omega-1)x_{-1} + x_1 + \omega x_0,\\ \mu x_1 &= x_0 + x_1.\end{aligned}$$

These equations, taken together, imply that

$$(\mu^2 + (1-2\omega)\mu + 2\omega - 4)(\mu + 1 - \omega) = 0,$$

whose maximum root is $\frac{1}{2}(2\omega - 1 + \sqrt{4\omega^2 - 12\omega + 17})$, which is also the value of $\mu(PK_{1,\omega})$.

(b) $m > 1$. The eigenvalue equation $Q(PK_{m,\omega})x = \mu x$ yields the following recurrence equation

$$(\mu - 2)x_i = x_{i-1} + x_{i+1}, \tag{2}$$

whose characteristic equation has roots

$$t_{1,2} = \frac{\mu - 2 \pm \sqrt{(\mu-2)^2 - 4}}{2}, \quad 0 < t_2 < 1 < t_1.$$

The eigenvalue equation written for component $x_m$ yields the following boundary condition

$$(\mu - 1)x_m = x_{m-1}. \tag{3}$$

We may use the recurrence equation (2) to formally extend the sequence $x_0, x_1, \ldots, x_m$ with new terms $x_{m+1}, x_{m+2}, \ldots$, where terms $x_{m+1}, x_{m+2}$ and so on are imaginary, so that it represents a particular solution of (2). Thus, an imaginary equation $(\mu - 2)x_m = x_{m-1} + x_{m+1}$, together with the real boundary condition (3), implies that $(\mu - 1)x_m = x_{m-1} = (\mu - 2)x_m - x_{m+1}$, namely,

$$x_m = -x_{m+1}. \tag{4}$$

According to the well known result in combinatorics on solving linear recurrence equation, there exist constants $a_m$ and $b_m$ such that for $i \geq 0$ it holds

$$x_i = a_m t_1^i + b_m t_2^i. \tag{5}$$

Applying (5) to (4), we have $a_m t_1^m + b_m t_2^m = x_m = -x_{m+1} = -(a_m t_1^{m+1} + b_m t_2^{m+1})$, and by simplifying it becomes

$$a_m t_1^{m+1}(t_1 + 1) = -b_m t_2^{m+1}(t_2 + 1). \tag{6}$$

Also, note that,

$$x_0 = a_m + b_m. \tag{7}$$

With appropriate scaling to $x$, we may suppose that $x_0 = 1$. The values of $a_m$ and $b_m$ may be obtained by solving (6) and (7),

$$a_m = \frac{1}{1 - t_1^{2m+3}}, \quad b_m = \frac{-t_1^{2m+3}}{1 - t_1^{2m+3}}.$$

Consider the eigenvalue equation written for component $x_{-1}$ and $x_0$,

$$\mu x_{-1} = (\omega - 2)x_{-1} + x_0 + (\omega - 1)x_{-1}, \tag{8}$$

$$\mu x_0 = (\omega - 1)x_{-1} + x_1 + \omega x_0. \tag{9}$$

From (8), $(\mu - 2\omega + 3)x_{-1} = x_0 = 1$; and from (9), $\mu - \omega = (\omega - 1)x_{-1} + x_1$. Combining these two equations, we have

$$\mu - \omega - \frac{\omega - 1}{\mu - 2\omega + 3} = x_1. \tag{10}$$

Note that $x_1 = a_m t_1 + b_m t_2 = a_m t_1 + (1 - a_m)t_2 = a_m(t_1 - t_2) + t_2 = a_m\sqrt{\mu^2 - 4\mu} + \frac{1}{t_1}$. Therefore we come to the following main result.

**Theorem 7.** *The signless Laplacian spectral radius $\mu$ of the kite graph $PK_{m,\omega}$ $(m \geq 1)$ satisfies*

$$\frac{1}{2}(2\omega - 1 + \sqrt{4\omega^2 - 12\omega + 17}) \leq \mu < 2\omega - 1.$$

*More precisely, for $m > 1$, $\mu$ satisfies the equation*

$$\mu - \omega - \frac{\omega - 1}{\mu - 2\omega + 3} = a_m\sqrt{\mu^2 - 4\mu} + \frac{1}{t_1}, \tag{11}$$

*where $a_m = \frac{1}{1 - t_1^{2m+3}}$ and $t_1 = \frac{\mu - 2 + \sqrt{(\mu-2)^2 - 4}}{2}$.*

## 4. Some Remarks on $\mu(PK_{\infty,\omega})$

Let $PK_{\infty,\omega}$ denote the infinite kite graph which consist of a clique $K_\omega$, to one vertex of which an infinite path is attached. Note that $PK_{\infty,\omega}$ is an infinite locally finite graph. More details on the spectra of infinite graphs may be found in [7]. Since $PK_{m,\omega}$ is a proper subgraph of $PK_{m+1,\omega}$ and so by Lemma 4, the sequence $\mu(PK_{m,\omega})_{m \geq 0}$ is strictly increasing and bounded from above by $2\omega - 1$, which implies that $\lim_{m\to\infty}\mu(PK_{m,\omega})$ exists with a value between $2\omega-2$ and $2\omega-1$.

Consider the equation (11). Note that $\lim_{m\to\infty} a_m = 0$ since $t_1 > 1$. Then the equation (11) becomes

$$\mu - 2\omega + 2 - \frac{2(\omega-1)}{\mu - 2\omega + 3} = -\sqrt{\mu^2 - 4\mu}.$$

Thus we conclude that $\mu(PK_{\infty,\omega})$ is the maximal root of the equation

$$x - 2\omega + 2 - \frac{2(\omega-1)}{x - 2\omega + 3} = -\sqrt{x^2 - 4x}.$$

While solving this cubic equation explicitly is possible, the obtained solution is cumbersome.

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