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$L(2,1)$-LABELINGS OF SOME FAMILIES OF ORIENTED PLANAR GRAPHS

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Abstract

In this paper we determine, or give lower and upper bounds on, the 2-dipath and oriented $L(2,1)$-span of the family of planar graphs, planar graphs with girth $5, 11, 16$, partial k-trees, outerplanar graphs and cacti.

Keywords: 2-dipath $L(2,1)$-labeling, oriented $L(2,1)$-labeling, homomorphism, planar graph, girth, partial k-tree, outerplanar graph, cactus.

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1. Introduction

To distinguish close and very close transmitters in a wireless communication system, Griggs and Yeh [6] proposed a variation of the Frequency Assignment Problem (or simply FAP) by introducing the $L(2,1)$-labeling which was generalized by Georges and Mauro [4] as follows.

For any two positive integers p and q, a *k-$L(p,q)$-labeling* of a graph G is a mapping ℓ from the vertex set $V(G)$ to the set $\{0,1,\ldots,k\}$ such that

(i) $|\ell(u)-\ell(v)| \geq p$ if u and v are at distance 1 in G,

(ii) $|\ell(u)-\ell(v)| \geq q$ if u and v are at distance 2 in G.

The *$L(p,q)$-span* $\lambda_{p,q}(G)$ of a graph G is defined as $\min\{k \mid G$ has a k-$L(p,q)$-labeling $\}$. For a family $\mathcal{F}$ of graphs, $\lambda_{p,q}(\mathcal{F}) = \max\{\lambda_{p,q}(H) \mid H \in \mathcal{F}\}$.

A common feature of graph theoretic models for FAP is that communication is assumed to be possible in both directions (duplex) between two radio transmitters and, therefore, these models are based on undirected graphs. But in reality,

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to model FAP on directed or oriented graphs could be interesting as pointed by Aardal *et al.* [1] in their survey.

An *oriented graph* is a directed graph with no cycle of length 1 or 2. By replacing each edge of a simple graph G with an arc (ordered pair of vertices) we obtain an oriented graph $\vec{G}$; $\vec{G}$ is an orientation of G and G is the underlying graph of $\vec{G}$. We denote by $V(\vec{G})$ and $A(\vec{G})$ respectively the set of vertices and arcs of $\vec{G}$. Similarly, $V(G)$ and $E(G)$ denote respectively the set of vertices and edges of G. An arc (x, y) (where x and y are vertices) is denoted by $\vec{xy}$. A path obtained by two consiqutive arcs $\vec{xy}$ and $\vec{yz}$ is called a 2-*dipath*. In this paper every undirected graph is a simple graph and every directed graph is an oriented graph, unless otherwise stated.

There are two different oriented versions of $L(p, q)$-labeling, namely 2-dipath $L(p, q)$-labeling, introduced by Chang *et al.* [3], and oriented $L(p, q)$-labeling, introduced by Gonçalves, Raspaud and Shalu [5].

A 2-*dipath* k-$L(p, q)$-*labeling* of an oriented graph $\vec{G}$ is a mapping ℓ from the vertex set $V(\vec{G})$ to the set $\{0, 1, \ldots, k\}$ such that

(i)$|\ell(u) - \ell(v)| \geq p$ if u and v are adjacent in $\vec{G}$,

(ii) $|\ell(u) - \ell(v)| \geq q$ if u and v are connected by a 2-dipath in $\vec{G}$.

The 2-*dipath span* $\vec{\lambda}_{p,q}(\vec{G})$ of an oriented graph $\vec{G}$ is defined as $\min\{k \mid \vec{G}$ has a 2-dipath k-$L(p, q)$-labeling $\}$. The 2-*dipath span* $\vec{\lambda}_{p,q}(G)$ of an undirected graph G is defined as $\max\{\vec{\lambda}_{p,q}(\vec{G}) \mid \vec{G}$ is an orientation of $G\}$. The 2-*dipath span* $\vec{\lambda}_{p,q}(\mathcal{F})$ of a family $\mathcal{F}$ of (oriented or undirected) graphs is defined as $\max\{\vec{\lambda}_{p,q}(H) \mid H \in \mathcal{F}\}$.

An *oriented* k-$L(p, q)$-*labeling* of an oriented graph $\vec{G}$ is a mapping ℓ from the vertex set $V(\vec{G})$ to the set $\{0, 1, \ldots, k\}$ such that

(i) ℓ is a 2-dipath k-$L(p, q)$-labeling of G,

(ii) if $\vec{xy}$ and $\vec{uv}$ are two arcs in $\vec{G}$ then, $\ell(x) = \ell(v)$ implies $\ell(y) \neq \ell(u)$.

The *oriented spans* $\lambda^o_{p,q}(\vec{G})$, $\lambda^o_{p,q}(G)$ and $\lambda^o_{p,q}(\mathcal{F})$ are defined similarly as 2-dipath spans.

From the definitions, the following is immediate:

Lemma 1. *For every* (*undirected or oriented*) *graph* G *and every* $p, q > 0$, $\vec{\lambda}_{p,q}(G) \leq \lambda^o_{p,q}(G)$.

An *oriented* k-*coloring* of an oriented graph $\vec{G}$ is a mapping f from the vertex set $V(\vec{G})$ to the set $\{0, 1, \ldots, k-1\}$ such that,

(i) $f(u) \neq f(v)$ whenever u and v are adjacent,

(ii) if $\vec{xy}$ and $\vec{uv}$ are two arcs in $\vec{G}$, then $f(x) = f(v)$ implies $f(y) \neq f(u)$.

The *oriented chromatic number* $\vec{\chi}(\vec{G})$ of an oriented graph $\vec{G}$ is defined as $\min\{k \mid \vec{G}$ has an oriented k-coloring$\}$. The *oriented chromatic number* $\vec{\chi}(G)$ of an undirected graph G is defined as $\max\{\vec{\chi}(\vec{G}) \mid \vec{G}$ is an orientation of $G\}$. The *oriented chromatic number* $\vec{\chi}(\mathcal{F})$ of a family $\mathcal{F}$ of (oriented or undirected) graphs is defined as $\max\{\vec{\chi}(H) \mid H \in \mathcal{F}\}$.

The additional condition in oriented $L(p,q)$-labeling ensures that any oriented $L(p,q)$-labeling is an oriented coloring [11]. Note that any oriented k-$L(p,q)$-labeling is an oriented $(k+1)$-coloring but a 2-dipath k-$L(p,q)$-labeling is not necessarily an oriented $(k+1)$-coloring.

The most frequently studied $L(p,q)$-labeling, other than the ones that correspond to chromatic numbers, is for $(p,q)=(2,1)$ (both undirected and oriented versions). In this paper, we mainly focus on studying 2-dipath and oriented $L(2,1)$-span of some families of planar graphs. For the family $\mathcal{P}$ of planar graphs and for the family $\mathcal{P}_g$ of planar graphs with girth at least g, where the *girth* of a graph is the size of its smallest cycle, for $g=5,11$ and 16, we will prove the following result in Section 3.

Theorem 2. (a) $18 \le \vec{\lambda}_{2,1}(\mathcal{P}) \le \lambda^o_{2,1}(\mathcal{P}) \le 83$.

(b) $6 \le \vec{\lambda}_{2,1}(\mathcal{P}_5) \le \lambda^o_{2,1}(\mathcal{P}_5) \le 22$.

(c) $4 \le \vec{\lambda}_{2,1}(\mathcal{P}_{11}) \le \lambda^o_{2,1}(\mathcal{P}_{11}) \le 10$.

(d) $4 \le \vec{\lambda}_{2,1}(\mathcal{P}_{16}) \le \lambda^o_{2,1}(\mathcal{P}_{16}) \le 7$.

Theorem 2(b) disproves the conjecture $\vec{\lambda}_{2,1}(\mathcal{P}_5) \le 5$ proposed by Calamoneri and Sinaimeri [2] and Theorem 2(c,d) improve the previous bounds $\vec{\lambda}_{2,1}(\mathcal{P}_{11}) \le 12$ and $\vec{\lambda}_{2,1}(\mathcal{P}_{16}) \le 8$ given by the same authors [2]. For the family $\mathcal{O}$ of outerplanar graphs, we prove in Section 5 the following:

Theorem 3. $9 \le \vec{\lambda}_{2,1}(\mathcal{O}) \le \lambda^o_{2,1}(\mathcal{O}) \le 10$.

As we were not able to provide exact results for the family of outerplanar graphs, we also consider a planar superfamily and a planar subfamily of it, namely the family $\mathcal{T}_2$ of partial 2-trees and the family $\mathcal{C}$ of cacti. For both these families we managed to give exact results. In fact, in Section 4 we prove the following general result for the family $\mathcal{T}_k$ of partial k-trees:

Theorem 4. (a) $\vec{\lambda}_{2,1}(\mathcal{T}_2) = \lambda^o_{2,1}(\mathcal{T}_2) = 10$.

(b) $\vec{\lambda}_{2,1}(\mathcal{T}_3) \le \lambda^o_{2,1}(\mathcal{T}_3) \le 22$.

(c) $\vec{\lambda}_{2,1}(\mathcal{T}_k) \le \lambda^o_{2,1}(\mathcal{T}_k) \le (k+1)(2^k+1)-2$.

In [2] Calamoneri and Sinaimeri proved that $6 \le \vec{\lambda}_{2,1}(\mathcal{C}) \le 8$. We improve this result as follows (proof in Section 6):

Theorem 5. $\vec{\lambda}_{2,1}(\mathcal{C}) = \lambda^o_{2,1}(\mathcal{C}) = 7$.

The precise definitions of these families are given in the beginning of their respective sections. In Section 2, we mainly define, state and prove some results which we will use for the main proofs in the following sections. In particular, we prove a general upper bound for the 2-dipath and oriented $L(p,q)$-span of multipartite graphs, which in some cases is tight.

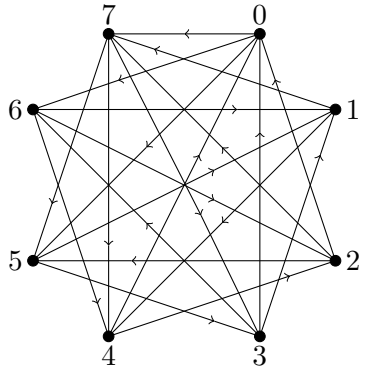

Figure 1. $\vec{B}$ is a 4-nice graph.

2. Preliminaries

The set of all adjacent vertices of a vertex v in a graph is called its set of *neighbors* and is denoted by $N(v)$. For oriented graphs, if there is an arc from u to v, then u is an *in-neighbor* of v and v is an *out-neighbor* of u. The sets of all in-neighbors and out-neighbors of v are denoted by $N^-(v)$ and $N^+(v)$ respectively.

An *oriented clique* or simply *oclique* is an oriented graph whose any two distinct vertices are either adjacent or connected by a 2-dipath. Ocliques are therefore precisely those oriented graphs $\vec{G}$ for which $\vec{\chi}(\vec{G}) = |V(\vec{G})|$.

A *homomorphism* f of an oriented graph $\vec{G}$ to an oriented graph $\vec{H}$ is a mapping $f : V(\vec{G}) \longrightarrow V(\vec{H})$ such that $\vec{xy} \in A(\vec{G})$ implies $\vec{f(x)f(y)} \in A(\vec{H})$.

From these definitions, we easily get the following:

Lemma 6. *If there is a homomorphism $f : \vec{G} \longrightarrow \vec{H}$, then $\vec{\lambda}_{p,q}(\vec{G}) \leq \vec{\lambda}_{p,q}(\vec{H})$ and $\lambda^o_{p,q}(\vec{G}) \leq \lambda^o_{p,q}(\vec{H})$, for every $p, q > 0$. In particular, $\vec{G} \subseteq \vec{H}$ implies $\vec{\lambda}_{p,q}(\vec{G}) \leq \vec{\lambda}_{p,q}(\vec{H})$ and $\lambda^o_{p,q}(\vec{G}) \leq \lambda^o_{p,q}(\vec{H})$, for all $p, q > 0$.*

Now we prove a general upper bound on oriented $L(p, q)$-span of multipartite graphs.

Theorem 7. *For every k-partite oriented graph $\vec{G}$, where $k \geq 3$, we have*

$$\vec{\lambda}_{p,q}(\vec{G}) \leq \lambda^o_{p,q}(\vec{G}) \leq |V(\vec{G})|q + k(\max(p, q) - q) - \max(p, q).$$

In particular for $p \geq q$, both the equalities hold if $\vec{G}$ is a complete k-partite oclique.

Proof. Let, $K = K_{n_1,n_2,\ldots,n_k}$ be the complete k-partite graph with the parts being $V_1, V_2, \ldots V_k$ with $|V_i| = n_i$ for all $i = 1, 2, \ldots, k$. Also, let the vertices of V_i be denoted by $\{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$.

Let $\vec{K}$ be any orientation of K. Now, consider the labeling L of $\vec{K}$ given by $L(v_{ij}) = \left(\sum_{t<i}(n_t-1)q\right) + (j-1)q + (i-1)\max(p,q)$, for $i = 1,2,\ldots,k$ and $j = 1,2,\ldots,n_i$.

For any i, v_{ir} and v_{is} $(r \neq s)$ cannot be connected by an arc but can be connected by a 2-dipath. While for any v_{ir} and v_{js}, $i \neq j$, can be connected by either an arc or a 2-dipath.

Then we have,

$$\begin{aligned}|L(v_{ir}) - L(v_{is})| &= \Big|\Big[\left(\textstyle\sum_{t<i}(n_t-1)q\right) + (r-1)q + (i-1)\max(p,q)\Big] \\ &\quad - \Big[(\textstyle\sum_{t<i}(n_t-1)q) + (s-1)q + (i-1)\max(p,q)\Big]\Big| \\ &= |\ \Big[\left(\textstyle\sum_{t<i}(n_t-1)q\right) - (\textstyle\sum_{t<i}(n_t-1)q)\Big] \\ &\quad +[(r-1)q - (s-1)q] \\ &\quad +[(i-1)\max(p,q) - (i-1)\max(p,q)]| \\ &= |(r-s)q| \geq q, \text{ for } r \neq s\end{aligned}$$

and

$$\begin{aligned}|L(v_{ir}) - L(v_{js})| &= \Big|\Big[\left(\textstyle\sum_{t<i}(n_t-1)q\right) + (r-1)q + (i-1)\max(p,q)\Big] \\ &\quad - \Big[\left(\textstyle\sum_{t<j}(n_t-1)q\right) + (s-1)q + (j-1)\max(p,q)\Big]\Big| \\ &= |(\textstyle\sum_{j<t<i}(n_t-1)q) + (n_j-1)q + (r-1)q - (s-1)q \\ &\quad +(i-j)\max(p,q)| \\ &\quad (\text{without loss of generality, assume } i > j) \\ &= \Big|\left(\textstyle\sum_{j<t<i}(n_t-1)q\right) + (n_j-s)q + (r-1)q \\ &\quad + (i-j)\max(p,q)|\ \ (n_j \geq s \text{ as } v_{js} \in V_j) \\ &\geq \max(p,q).\end{aligned}$$

As all vertices have different labels, L is an oriented coloring of $\vec{K}$.

Hence we have,

$$\begin{aligned}\lambda^o_{p,q}(K) \leq \vec{\lambda}_{p,q}(K) &\leq \textstyle\sum_{t=1}^{k-1}(n_t-1)q + (n_k-1)q + (k-1)\max(p,q) \\ &= |V(K)|q + k(\max(p,q) - q) - \max(p,q).\end{aligned}$$

Now as any oriented k-partite graph $\vec{G}$ is a subgraph of some orientation of the complete k-partite graph K, using Lemma 1 and Lemma 6 the theorem follows.

In particular, if $\vec{K}$ is an oclique, then any two vertices are at distance at most 2. Moreover, if $\vec{K}$ is also an orientation of the complete k-partite graph, then any two vertices from different parts, are adjacent. Hence both the equalities hold for $p \geq q$. ■

For any prime $p \equiv 3 (\text{mod } 4)$ and for any positive integer n the *Paley tournament* P_q of order $q = p^n$ is the oriented graph with set of vertices $\{0,1,2,\ldots,q-1\}$ and set of arcs $\{\vec{xy} \mid y - x \text{ (mod } p) \text{ is a non-zero square}\}$.

As -1 (mod p) is not a square , either $(x-y)$ or $(y-x)$ (but not both) is a square for all $x, y \in F_q$. Hence P_q is a tournament.

The *Tromp graph* [11] T_{2q+2} of order $(2q+2)$ is the oriented graph with set of vertices $V(T_{2q+2}) = \{0,1,\ldots,(q-1)\} \cup \{0',1',\ldots,(q-1)'\} \cup \{v,v'\}$ and set of

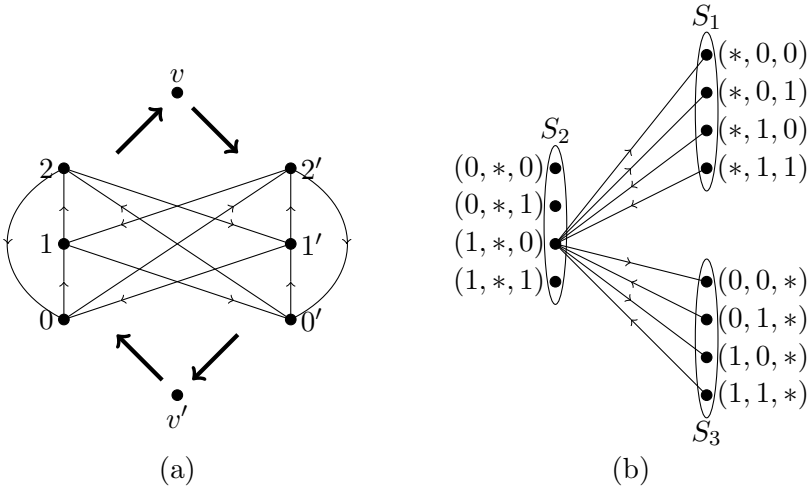

Figure 2. (a) Tromp graph T_8 (thick arrows refer to the three arcs between v or v' AND $\{1,2,3\}$ or $\{1',2',3'\}$). (b) Adjacency of a vertex of the Zielonka graph Z_3.

arcs $A(T_{2q+2}) = \{\vec{ij}, \vec{i'j'}, \vec{j'i}, \vec{ji'} \mid i,j \in \{0,1,\ldots,(q-1)\}$ and $(j-i) \pmod p$ is a non-zero square $\} \cup \{\vec{iv}, \vec{vi'}, \vec{i'v'}, \vec{v'i} \mid i \in \{0,1,\ldots,(q-1)\}\}$. Intuitively, in T_{2q+2} there are two vertices v, v' such that $N^+(v) = N^-(v')$ and $N^+(v) = N^-(v')$ with each of the sets $N^+(v)$ and $N^-(v)$ inducing a Paley tournament P_q. Also, if $\vec{ij}$ is an arc in the P_q induced by $N^+(v)$ and $\vec{i'j'}$ is the corresponding arc of the P_q induced by $N^-(v)$, then we also have the arcs $\vec{ji'}$ and $\vec{j'i}$. Note that, T_{2q+2} is a complete $(q+1)$-partite oclique with all parts of size two. For further details about this graph, see Marshall's paper [8]. For example, the graph depicted in Figure 2(a) is the Tromp graph T_8.

For any positive integer k the *Zielonka graph* [11] Z_k of order $k \times 2^{k-1}$ is the oriented graph, with set of vertices $V(Z_k) = \bigcup_{i=1,2,\ldots,k} S_i$, where $S_i = \{x = (x^1,\ldots,x^k) \,|x^j \in \{0,1\}$ for $j \neq i$ and $x^i = *\}$ and set of arcs $A(Z_k) = \{\vec{xy} \mid x = (x^1,\ldots,x^k) \in S_i, y = (y^1,\ldots,y^k) \in S_j$ and either $x^j = y^i$ and $i < j$ or $x^j \neq y^i$ and $i > j\}$. Note that Z_k is a complete k-partite oclique with all parts of size 2^{k-1}. For example, we depicted the adjacency of a vertex of the Zielonka graph Z_3 in Figure 2(b).

So, by Theorem 7 we have the following:

Corollary 8. $\vec{\lambda}_{2,1}(T_{2q+2}) = \lambda^o_{2,1}(T_{2q+2}) = 3q+1$.

Corollary 9. $\vec{\lambda}_{2,1}(Z_k) = \lambda^o_{2,1}(Z_k) = k(2^{k-1}+1) - 2$.

In this paper, we shall use the following notion from [7]. A *pattern* Q of length k is a word $Q = q_0 q_1 \cdots q_{k-1}$ with $q_i \in \{+,-\}$ for every i, $0 \leq i \leq k-1$. A Q-*walk*

in a digraph $\vec{G}$ is a walk $P = x_0x_1 \cdots x_k$ such that for every i, $0 \leq i \leq k-1$, $x_ix_{i+1} \in A(\vec{G})$ if $q_i = +$ and $x_{i+1}x_i \in A(G)$ otherwise. For $X \subseteq V(\vec{G})$ we denote by $N_Q(X)$ the set of all vertices y such that there exists a Q-walk going from some vertex $x \in X$ to y. We then say that a digraph $\vec{G}$ is *k-nice* if for every pattern Q of length k and for every vertex $x \in V(\vec{G})$ we have $N_Q(\{x\}) = V(\vec{G})$. In other words, a digraph is k-nice if for all pairs of vertices x, y (allowing $x = y$) there is a k-walk from x to y for each of the 2^k possible oriented patterns. Observe that if a digraph G is k-nice for some k, then it is k'-nice for every $k' \geq k$. For example, the graph $\vec{B}$ (Figure 1) is a 4-nice graph.

The *girth* of a graph is the length of its shortest cycle (by the girth of an oriented graph we will mean the girth of its underlying graph). We denote the family of planar graphs by $\mathcal{P}$ and the family of planar graphs with girth at least g by $\mathcal{P}_g$.

Now we state a theorem from [7].

Theorem 10 (Hell *et al.* 1997)**.** *Let N_k be a k-nice oriented graph, $k \geq 3$. Every oriented graph whose underlying graph is in $\mathcal{P}_{5k-4}$ admits a homomorphism to N_k.*

3. Planar Graphs

A *planar graph* is a graph that can be drawn in the plane in such a way that no two edges cross each other. Now we prove Theorem 2.

Proof of Theorem 2. (a) Raspaud and Sopena [10] showed that every oriented planar graph admits a homomorphism to the Zielonka graph Z_5. Hence, using Lemma 6 and Corollary 9, we get the upper bound.

For proving the lower bound, assume $\vec{O}^*$ is an outerplanar graph with 2-dipath span at least 9 and also contains an outerplanar oclique of order 7. Now, take a 2-dipath $\vec{Q}$ along with six disjoint copies of $\vec{O}^*$. Then, connect (different) two copies of $\vec{O}^*$ with each of the vertices of $\vec{Q}$ by adding arcs. We choose orientations of the new arcs in such a way that, for each vertex $v \in V(\vec{Q})$, the graph induced by $N^\alpha(v)$ contains $\vec{O}^*$ as a subgraph for $\alpha = +, -$. For each vertex $\in N^\alpha(v)$ add two dipaths to the graph and connect it with v and w in such a way that both $N^\alpha(w)$ and $N^\beta(v) \cap N^\beta(w)$ each contains a 2-dipath for $\{\alpha, \beta\} = \{+, -\}$. Name this graph $\vec{R}$. Notice that $\vec{R}$ is a planar graph.

Now let f be a 2-dipath k-$L(2,1)$-labeling of $\vec{R}$ for some k. Then there will be a vertex $v \in V(\vec{Q})$ such that $f(v) \neq 0, k$. This means $f(t) \notin \{f(v)-1, f(v), f(v)+1\}$ for $t \in N(v)$. Also, we know that $f(x) \neq f(y)$ for $x \in N^+(v)$ and $y \in N^-(v)$. Moreover, as $\vec{O}^*$ is a subgraph of both $N^+(v)$ and $N^-(v)$ we need at least 16 labels other than $\{f(v)-1, f(v), f(v)+1\}$ to label $N(v)$. Hence, $k \geq 18$.

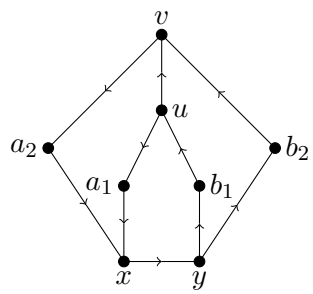

Figure 3. $\vec{F}$ is an oriented planar graph with girth 5.

We will construct such an outerplanar graph $\vec{O}^*$ in the proof of Theorem 3 (5). That will complete the proof of Theorem 2(a).

(b) We know that every planar graph of girth at least 5 admits a homomorphism to the Tromp graph T_{16} [9]. Then, using Lemma 6 and Corollary 8, we get the upper bound.

To prove the lower bound, we first show that it is impossible to have a 2-dipath 5-$L(2,1)$-labeling f of the graph $\vec{F}$, depicted in Figure 3, with $\{f(x), f(y)\} = \{3,5\}$.

Notice that, if $f(x) = 3$ and $f(y) = 5$ then $f(a_1) \in \{0,1\}$ and $f(b_1) \in \{0,1,2\}$. This implies $f(u) = 4$. Similarly, we have $f(v) = 4$ which is not possible as u, v are adjacent. The case $f(x) = 5$ and $f(y) = 3$ is similar.

The oriented planar graph $\vec{E}$, depicted in Figure 4, has girth 5. Moreover, the vertices $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ will get pairwise different labels for any 2-dipath $L(2,1)$-labeling since they are pairwise connected by a 2-dipath. Consider a 2-dipath 5-$L(2,1)$-labeling g of $\vec{E}$ such that $g(x_6) = 0$. Then we have $\{g(x_1), g(x_2), g(x_3), g(x_4), g(x_5)\} = \{1,2,3,4,5\}$. Hence there exists an arc $\vec{wz} \in A(\vec{G}[\{x_1, x_2, x_3, x_4, x_5\}])$ such that $\{f(w), f(z)\} = \{3,5\}$.

Now on each of the five vertices $x \in \{x_1, x_2, x_3, x_4, x_5\}$ of $\vec{E}$, we glue a copy of $\vec{E}$ by identifying x with the vertex x_6 of the copy. Call this graph $\vec{G}$.

Note that $\vec{G}$ is a planar graph with girth 5 such that, for any 2-dipath 5-$L(2,1)$-labeling ℓ of $\vec{G}$, there is $\vec{wz} \in A(\vec{G})$ with $\{\ell(w), \ell(z)\} = \{3,5\}$. We can then construct a new graph $\vec{H}$ by identifying each arc $\vec{ab}$ of $\vec{G}$ with the arc $\vec{xy}$ of $\vec{F}$. Clearly, $\vec{H}$ is also a planar graph with girth 5 which does not have any 2-dipath 5-$L(2,1)$-labeling. Hence we get the lower bound which completes the proof.

(c), (d) It is easy to observe that the directed path of length 5 has 2-dipath $L(2,1)$-span 4. Now, given any girth g, there is an oriented planar graph with girth g containing the directed path of length 5 as a subgraph. Also, one can check that the Tromp graph T_8 is 3-nice, and that the graph $\vec{B}$ (Figure 1) is 4-

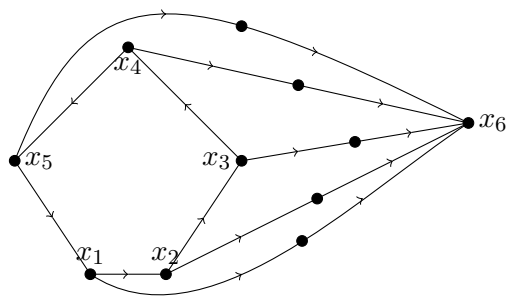

Figure 4. $\vec{E}$ is an oriented planar graph with girth 5.

nice (we have verified both using computer). Then, using Theorem 10, Lemma 6, Lemma 1 and Corollary 8, we have the results. ■

4. Partial k-trees

A *k-tree* is a graph obtained from the complete graph K_k on k vertices by adding zero or more vertices, one by one, in such a way that each newly added vertex has exactly k neighbors that form a clique. A subgraph of a k-tree is a *partial k-tree* We denote the family of partial k-trees by $\mathcal{T}_k$. Partial k-trees have been extensively studied in the last past years, since they often lead to polynomial algorithms for problems which are known to be NP-complete in the general case [14]. The notion of a 1-tree obviously corresponds to the usual notion of a tree. The family of outerplanar graphs is strictly contained in the family of partial 2-trees. It is easy to see that every partial 2-tree is a planar graph. The following Lemmas will be useful for proving Theorem 4.

To prove the following lemma we use the same technique as the one used to prove that every oriented outerplanar graph has oriented chromatic number at most 7 in [12].

Lemma 11. *Every oriented partial 2-tree $\vec{D}$ admits a homomorphism to the Tromp graph T_8.*

Proof. It is possible to check that for every $u, v \in V(T_8)$ and every $\alpha, \beta \in \{+, -\}$, there exists $w_{\alpha\beta} \in N^{\alpha}(u) \cap N^{\beta}(v)$.

Let $\vec{G}$ be a minimal (with respect to the number of vertices) counterexample to the lemma. Without loss of generality, we may assume that $\vec{G}$ is a 2-tree. Since $\vec{G}$ is a 2-tree, $\vec{G}$ must have a vertex x of degree 2. Let $N(x) = \{x_1, x_2\}$.

Now, by removing the vertex x from $\vec{G}$ and adding an arc between x_1 and x_2 (if there was not already one), we get a 2-tree that admits a homomorphism to T_8 (because of the minimality of $\vec{G}$). Using the property of T_8 stated in the begining of the proof, clearly this homomorphism can be extended to a homomorphism of $\vec{G}$ to T_8, a contradiction. ■

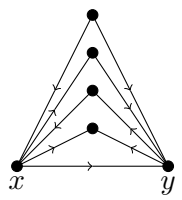

Figure 5. The oriented 2-tree D_1.

Lemma 12. *There exists an oriented* 2*-tree* D_{13} *for which* $\vec{\lambda}_{2,1}(D_{13}) \geq 10$.

Proof. First, we will describe a family of oriented 2-trees by induction. We start with the oriented 2-tree D_1 (Figure 5). By induction, we construct a graph D_{i+1} by gluing D_1 on each arc of D_i by identifying that arc with the arc $\vec{xy}$ of D_1. Note that every so obtained graph D_i is a 2-tree.

Assume that f is a 9-$L(2,1)$-labeling of D_{13}.

Step 0: Notice that in each copy of D_1, all the vertices should get different labels and for any vertex $v \in N(x) \cap N(y)$ we have, $|f(t) - f(v)| \geq 2$ for $t = x, y$.

Step 1: If we restrict f to D_1 then there is a vertex $v_1 \in N(x) \cap N(y)$ such that $f(v_1) \notin \{0, 9\}$. Similarly, if we restrict f to D_2, we can find a vertex $v_2 \in N(x) \cap N(v_1)$ such that $f(v_2) \notin \{0, 9\}$.

Step 2: Now, if we restrict f to D_3, we can find a $v_3 \in N(v_1) \cap N(v_2)$ such that $f(v_3) \notin \{0, 9\}$. So we have $\{f(v_1), f(v_2), f(v_3)\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$ with no two of $\{f(v_1), f(v_2), f(v_3)\}$ being consecutive numbers, since $\{v_1, v_2, v_3\}$ are pairwise adjacent vertices. Hence there exists $i, j \in \{1, 2, 3\}$ such that $\{f(v_i) - 1, f(v_i), f(v_i) + 1\} \cap \{f(v_j) - 1, f(v_j), f(v_j) + 1\} = \emptyset$ and $f(v_i) < f(v_j)$.

Step 3: In D_4 there exists $v_{\alpha\beta} \in N^{\alpha}(v_i) \cap N^{\beta}(v_j)$ for all $\alpha, \beta \in \{+, -\}$. Notice that the vertices $\{v_{++}, v_{+-}, v_{-+}, v_{--}\}$ will be labeled by the four remaining labels different from the labels $\{f(v_i) - 1, f(v_i), f(v_i) + 1, f(v_j) - 1, f(v_j), f(v_j) + 1\}$.

Step 4: Now we want to show that there is a vertex in D_5 that receives label 1 or 8. If $f(t) \in \{1, 8\}$ for some $t \in \{v_i, v_j, v_{++}, v_{+-}, v_{-+}, v_{--}\}$, we are done.

If not, then we can conclude that $f(v_i) = 2$, $f(v_j) = 7$ since any other possible choice of labels (other than 1 or 8) for v_i, v_j will force at least one of the labels among $\{f(v_{++}), f(v_{+-}), f(v_{-+}), f(v_{--})\}$ to be equal to 1 or 8. This

will imply $\{f(v_{++}), f(v_{+-}), f(v_{-+}), f(v_{--})\} = \{0,4,5,9\}$. Choose v_4 from the set $\{v_{++}, v_{+-}, v_{-+}, v_{--}\}$ such that $f(v_4) = 5$. Then in D_5, there is a vertex $v_5 \in N(v_i) \cap N(v_4)$ with $f(v_5) = 8$.

Hence in D_5, there exists a vertex v_6 with $f(v_6) \in \{1,8\}$.

Step 5: Now we want to show that there is a vertex in D_7 that receives label 1. If $f(v_6) = 1$, we are done.

If not, then $f(v_6) = 8$. This implies that, in D_6, there exists $t \in N(v_6)$ such that $f(t) \in \{1,4,5\}$, since we need to use at least five distinct labels from $\{0,1,2,3,4,5,6\}$ to label all vertices of $N(v_6)$. If $f(t) = 1$, we are done. Otherwise, in D_7, we can find some $s \in N(v_6) \cap N(t)$ such that $f(s) = 1$.

Hence in D_7 we can find a vertex a with $f(a) = 1$.

Step 6: Now we want to show that in D_9 there is a vertex $b \in N(a)$ with $f(b) = 8$.

Now, in D_8, there are at least five vertices in $N(a)$ which receive pairwise different labels. Therefore, for some $t \in N(a)$, we will have $f(t) \in \{8,4,5\}$. If $f(t) = 8$, we are done. Otherwise, in D_9, we can find $s \in N(a) \cap N(t)$ with $f(s) = 8$.

Hence, in D_9, there is a pair of adjacent vertices a and b with $f(a) = 1$ and $f(b) = 8$.

Step 7: Therefore, in D_{10}, there will be a copy of D_1 with vertices $\{a,b\}$ corresponding to the vertices $\{x,y\}$ of D_1 (as in Figure 5).

Step 8: Now notice that, in D_{12}, there are $u_i \in N(a)$ such that $f(u_i) = i$ for all $i \in \{3,4,\ldots,9\}$. Hence, in D_{13}, there are $u_i^{\alpha\beta} \in N^{\alpha}(a) \cap N^{\beta}(u_i)$ for all $\alpha, \beta \in \{+,-\}$.

Step 9: Note that it is not possible to have $p \in N^+(a)$ and $q \in N^-(a)$ with $f(p) = f(q)$. Hence the function $F_a(i) = \alpha$ if $t \in N^{\alpha}(a)$ and $f(t) = i$ for $i \in \{3,4,\ldots,9\}$ and $\alpha \in \{+,-\}$, is well defined. Intuitively, the function F_a is a function indicating whether a label is used for an in-neighbor of a or for an out-neighbor of a.

Step 10: Note that for each $i \in \{3,4,\ldots,9\}$, $F_a(f(u_i^{++})) = F_a(f(u_i^{+-})) = +$ and $F_a(f(u_i^{-+})) = F_a(f(u_i^{--})) = -$.

Also, notice that $\{f(u_i^{\alpha\beta}) \mid \alpha, \beta \in \{+,-\}\} = \{3,4,\ldots,9\} \setminus \{i-1, i, i+1\}$ for each $i \in \{4,5,6,7,8\}$.

We will use the two above observations repeatedly in the following.

Step 11: Let $\{\gamma, \overline{\gamma}\} = \{+,-\}$. Without loss of generality, assume that $F_a(3) = \gamma$.

Claim. $F_a(6) = \gamma$.

Proof. If possible, let $F_a(6) = \overline{\gamma}$. Now $\{f(u_8^{\alpha\beta}) \mid \alpha, \beta \in \{+,-\}\} = \{3,4,5,6\}$. So two of $F_a(3), F_a(4), F_a(5), F_a(6)$ will be $+$ and the other two will be $-$. But we already have $F_a(3) = \gamma$ and $F_a(6) = \overline{\gamma}$. Hence, $\{F_a(4), F_a(5)\} = \{\gamma, \overline{\gamma}\}$.

Similarly, we have $\{f(u_7^{\alpha\beta}) \mid \alpha, \beta \in \{+,-\}\} = \{3,4,5,9\}$. This will force $F_a(9) = \overline{\gamma}$. After that we have $\{f(u_4^{\alpha\beta}) \mid \alpha, \beta \in \{+,-\}\} = \{6,7,8,9\}$ which forces $F_a(7) = F_a(8) = \gamma$. Now we also have $\{f(u_5^{\alpha\beta}) \mid \alpha, \beta \in \{+,-\}\} = \{3,7,8,9\}$. But $F_a(3) = F_a(7) = F_a(8) = \gamma$, a contradiction. Hence, $F_a(6) = \gamma$. □

Step 12: Now $\{f(u_8^{\alpha\beta}) \mid \alpha, \beta \in \{+,-\}\} = \{3,4,5,6\}$ implies $F_a(4) = F_a(5) = \overline{\gamma}$. Similarly, $\{f(u_7^{\alpha\beta}) \mid \alpha, \beta \in \{+,-\}\} = \{3,4,5,9\}$ implies $F_a(9) = \gamma$. Lastly $\{f(u_4^{\alpha\beta}) \mid \alpha, \beta \in \{+,-\}\} = \{6,7,8,9\}$ implies $F_a(7) = F_a(8) = \overline{\gamma}$.

Hence, we got the full description of F_a (depending on the value of γ).

Step 13: Similarly, we can define a function F_b (one can imitate the previous steps, or just use symmetry). As $f(a) = 1$, $f(b) = 8$ and $F_a(8) = \overline{\gamma}$, we have $F_b(1) = \gamma$. Now, by symmetry we get $F_b(1) = F_b(2) = F_b(4) = F_b(5) = \gamma$ and $F_b(0) = F_b(3) = F_b(6) = \overline{\gamma}$.

Step 14: Therefore, $F_a(l) \neq F_b(l)$ for all such l on which both the functions are defined. But we have $F_a(f(u_8^{++})) = F_b(f(u_8^{++})) = +$. This is a contradiction.

Hence, we are done. ■

We are now able to prove Theorem 4.

Proof of Theorem 4. (a) The proof follows by Lemmas 6, 11, 12 and Corollary 8.

(b), (c) From [11] we know that any partial 3-tree admits a homomorphism to the tromp graph T_{16} and that any partial k-tree admits a homomorphism to the Zielonka graph Z_{k+1}. Hence the proof follows using Lemma 6 and Corollaries 8 and 9. ■

5. Outerplanar Graphs

A graph G is outerplanar if the graph formed from G by adding a new vertex, with edges connecting it to all the other vertices is a planar graph. We denote the family of outerplanar graphs by $\mathcal{O}$. Now we prove Theorem 3.

Proof of Theorem 3. Every outerplanar graph is also a partial 2-tree. So, the upper bound follows from Theorem 4.

To prove the lower bound, we will construct an oriented outerplanar graph $\vec{O}^*$ with $\vec{\lambda}_{2,1}(\vec{O}^*) \geq 9$. This will complete the proof.

First, we show that the outerplanar graph $\vec{O}$ (Figure 6) has no 2-dipath 8-$L(2,1)$-labeling if v gets label 1.

Let f be a 2-dipath 8-$L(2,1)$-labeling of $\vec{O}$ such that $f(v) = 1$. This implies $f(t) \notin \{0,1,2\}$ for $t \in \{x_1, x_2, \ldots, x_8, y_1, \ldots, y_8\}$ and $f(x_i) \neq f(y_j)$ for any $i, j = 1, 2, \ldots, 8$.

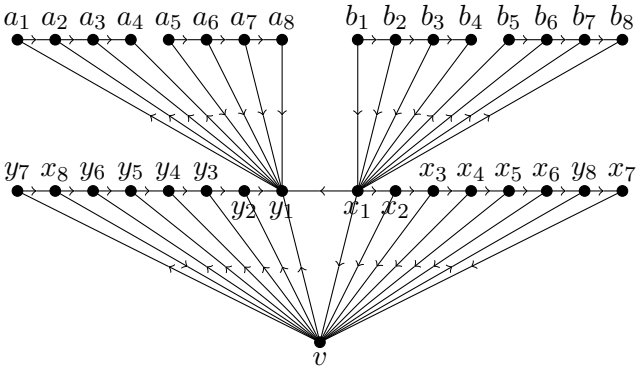

Figure 6. The oriented outerplanar graph $\vec{O}$.

Clearly, we need at least 3 distinct labels for each of the sets $\{x_i \mid i = 1, \ldots, 8\}$ and $\{y_i \mid i = 1, \ldots, 8\}$. Also, if we use exactly 3 labels for either of these sets, then those 3 labels should have pairwise difference at least 2.

To satisfy the above conditions, by symmetry, we may assume without loss of generality that we use labels $\{3, 5, 7\}$ for $\{x_1, \ldots, x_8\}$ and $\{4, 6, 8\}$ for $\{y_1, \ldots, y_8\}$.

Now, with these assumptions, the following conditions are forced:

(a) $f(b_i) \notin \{f(x_1) - 1, f(x_1), f(x_1) + 1\}$ for $i = 1, 2, \ldots, 8$.

(b) $f(b_i) \neq f(b_j)$ for $i \in \{1, 2, 3, 4\}$ and $j \in \{5, 6, 7, 8\}$.

(c) $f(b_i) \notin \{1, f(x_2), f(y_1)\}$ for $i = 1, 2, 3, 4$.

(d) $f(x_i) = f(x_{i+3})$ for all $i = 1, 2, 3$.

(e) we need at least 3 distinct labels for either of the sets $\{b_1, \ldots, b_4\}$ and $\{b_5, \ldots, b_8\}$ for $i = 1, \ldots, 8$. Also, if we use exactly 3 labels for either of these sets, then those 3 labels should have mutual difference at least 2.

We have three cases to consider.

Case 1. If $f(x_1) = 7$, then $f(y_1) = 4$ and $f(x_2) = 3$ or 5. Then, $\{f(b_1), f(b_2), f(b_3), f(b_4)\} = \{0, 2, 5\}$ (by *(a), (c), (e)*). This implies $\{f(b_5), f(b_6), f(b_7), f(b_8)\} = \{1, 3, 4\}$ (by (a), (b)) which contradicts (e).

Case 2. If $f(x_1) = 5$, then $f(y_1) = 8$ and $f(x_2) = 3$ or 7. Then, $\{f(b_1), f(b_2), f(b_3), f(b_4)\} = \{0, 2, 7\}$ (by (a), (c), (e)). Hence $f(x_2) = 3$. This implies $f(x_3) = 7$. Therefore, $f(x_6) = 7$ (by (d)).
Now, the only possibility is to have $f(y_8) = 4$ which will force $f(x_7) = 7$ since $f(x_7) \in \{3, 5, 7\}$. But x_6 and x_7 cannot have same labels since they are connected by a 2-dipath through y_8. This is a contradiction.

Case 3. If $f(x_1) = 3$, then $f(y_1) = 6$ or 8 and $f(x_2) = 5$ or 7. Then, $\{f(b_1), f(b_2), f(b_3), f(b_4)\} = \{0, 5, 8\}$ (by (a), (c), (e)). This implies $\{f(b_5), f(b_6), f(b_7), f(b_8)\} = \{1, 6, 7\}$ (by (a), (b)) which contradicts (e).

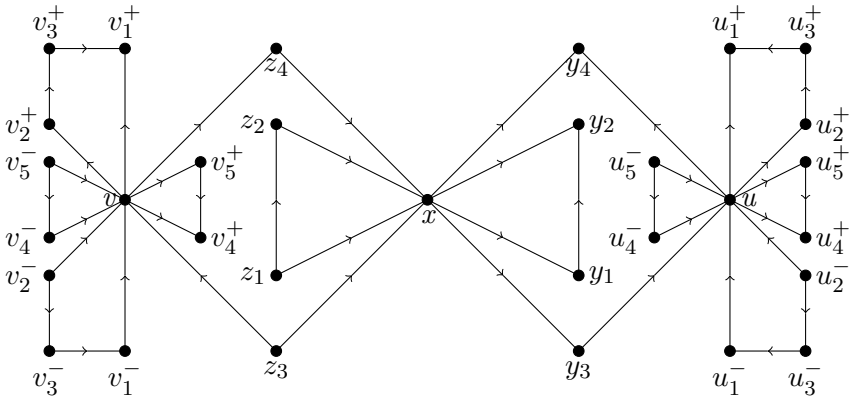

Figure 7. The oriented cactus $\vec{H}$.

Hence, we do not have a 8-$L(2,1)$-labeling f of $\vec{O}$ such that $f(v) = 1$. By symmetry, we can say that we do not have a 8-$L(2,1)$-labeling f of $\vec{O}$ such that $f(v) = 7$.

Now define $S = V(\vec{O}) \setminus \{x_2, x_7, x_8, y_2, y_7, y_8\}$ and let $\vec{G} = \vec{O}[S]$.

Notice that if we try to 2-dipath 8-$L(2,1)$-label $\vec{G}$, then we need to use 3 different labels for the vertices v, x_1 and y_1. One of these three vertices should have a label $l \notin \{0, 8\}$. To label the neighbors of that vertex, we clearly need at least 6 labels other than $l-1, l$ and $l+1$. So, we have to use all the remaining 6 labels and whatever the value of l may be, we necessarily use label 1 or 7 to 2-dipath 8-$L(2,1)$-label $\vec{G}$.

Now, we construct a new graph $\vec{O}^*$ by gluing a copy of $\vec{O}$ on each vertex of $\vec{G}$ by identifying that vertex of $\vec{G}$ with the vertex v of $\vec{O}$.

Note that $\vec{O}^*$ is an outerplanar graph that cannot be 2-dipath 8-$L(2,1)$-labelled, which proves the theorem. ■

6. Cacti

A *cactus* is a connected graph in which any two cycles can have at most one vertex in common. We denote the family of cacti by $\mathcal{C}$. Now we prove Theorem 5. The following lemmas will be useful for proving Theorem 5.

Lemma 13. *There exists an oriented cactus $\vec{C}$ with $\vec{\lambda}_{2,1}(C) \geq 7$.*

Proof. Let $\vec{H}$ be the oriented cactus depicted in Figure 7. We first show that there is no 6-$L(2,1)$-labeling f of $\vec{H}$ with $f(x) = 2$. Assume to the contrary that such a labeling f exists.

The assumption implies that $f(t) \notin \{1,2,3\}$ for $t \in \{z_1, z_2, z_3, z_4, y_1, y_2, y_3, y_4\}$. Also we have, $f(z_i) \neq f(y_j)$ for $i, j = 1,2,3,4$ and for $t \in \{y, z\}$, $|f(t_1) - f(t_2)| \geq 2$ and $f(t_3) \neq f(t_4)$.

This will force either $\{f(z_3), f(z_4)\} = \{0,5\}$ or $\{f(y_3), f(y_4)\} = \{0,5\}$ (as $\{f(z_3), f(z_4)\} = \{0,6\}$ will force $f(y_3), f(y_4) \in \{4,5\}$, which is not possible). But these two cases are symmetric. So, without loss of generality, we can assume $\{f(z_3), f(z_4)\} = \{0,5\}$.
Again, by symmetry, we can assume $f(z_3) = 0$ and $f(z_4) = 5$. This will force $f(v) = 3$. Then, $f(t) \notin \{2,3,4\}$ for $t \in \{v_1^+, v_2^+, v_4^+, v_5^+, v_1^-, v_2^-, v_4^-, v_5^-\}$.

Similarly as before, we have $f(v_i^+) \neq f(v_j^-)$ for $i, j = 1,2,4,5$ and for $t \in \{v^+, v^-\}$, $|f(t_4) - f(t_5)| \geq 2$ and $f(t_1) \neq f(t_2)$. Moreover, $f(v_i^+) \neq f(z_3) = 0$ and $f(v_i^-) \neq f(z_4) = 5$ for $i = 1,2,4,5$. This forces $\{f(v_1^+), f(v_2^+)\} = \{5,1\}$. Then no label is available for v_3^+, a contradiction. Hence, we do not have a 6-$L(2,1)$-labeling f of $\vec{H}$ such that $f(x) = 2$.

Let $\vec{G}$ be a graph obtained by gluing a copy of the induced subgraph $\vec{H}[x, y_1, y_2, z_1, z_2]$ on each vertex of the directed 5-cycle $\vec{C_5}$ by identifying each vertex of $\vec{C_5}$ with the vertex x of $\vec{H}[x, y_1, y_2, z_1, z_2]$. Clearly, $\vec{G}$ is a cactus.

Now, if we 2-dipath 6-$L(2,1)$-label $\vec{G}$, we need to use at least 5 labels for the $\vec{C_5}$ inside it. If 2 is not among those 5 labels, then at least one of $\{4,5\}$ is among those 5 labels. Now, the $\vec{H}[x, y_1, y_2, z_1, z_2]$ glued with the vertex that got label 4 (or 5) clearly must use label 2. Hence, for any 2-dipath 6-$L(2,1)$-labeling of the cactus $\vec{G}$, we need to use 2 as one of the labels.

Now, we construct a new graph $\vec{C}$ by gluing a copy of $\vec{H}$ on each vertex of $\vec{G}$ by identifying that vertex of $\vec{G}$ with the vertex x of $\vec{H}$. Note that $\vec{C}$ is a cactus that cannot be 2-dipath 6-$L(2,1)$-labelled. This completes the proof. ■

Let $\vec{B}$ be the oriented graph depicted in Figure 1. Then we have:

Lemma 14. *Let $\vec{O}$ be an oriented cycle. Given any $x \in V(\vec{O})$ and $y \in V(\vec{B})$, there exists a homomorphism $h : \vec{O} \longrightarrow \vec{B}$ such that $h(x) = y$.*

Proof. We know that $\vec{B}$ is 4-nice. Hence it is enough to show that for any oriented 3-cycle $\vec{T}$ and given any $x \in V(\vec{T})$ and $y \in V(\vec{B})$, there exists a homomorphism $h : \vec{O} \longrightarrow \vec{B}$ such that $h(x) = y$. In other words, we need to show that for each $y \in V(\vec{B})$, the 3-cycles in Figure 8 are subgraphs of $\vec{B}$, which can easily be checked. ■

Lemma 15. *Every oriented catus $\vec{C}$ admits a homomorphism to $\vec{B}$.*

Proof. Let $\vec{G}$ be a minimal counterexample to Lemma 15.

If there is a degree one vertex v in $\vec{G}$ such that $v \in N^+(u)$ (or $v \in N^-(u)$) for some $u \in V(\vec{G})$, then $\vec{G}[V(\vec{G}) \setminus \{v\}]$ is also a cactus. As $\vec{G}$ is a minimal counterexample, there is a homomorphism f from $\vec{G}[V(\vec{G} \setminus \{v\}]$ to $\vec{B}$. Now, since

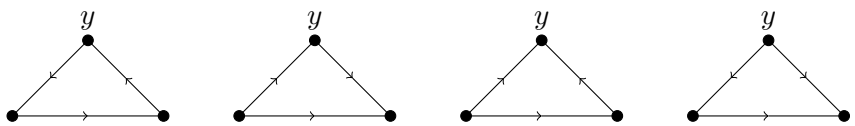

Figure 8. Four different oriented 3-cycles with respect to the vertex y.

all the vertices of $\vec{B}$ have at least one in-neighbor and one out-neighbor, we can extend the homomorphism f to a homomorphism of $\vec{G}$ to $\vec{B}$ by mapping v to any vertex $x \in N^+(f(u))$ (or $x \in N^-(f(u))$). This is a contradiction. Hence there cannot be a degree one vertex in $\vec{G}$.

No vertex of degree one in $\vec{G}$ implies at least one cycle $\vec{C} \subseteq \vec{G}$ such that exactly one vertex z of the cycle $\vec{C}$ has degree greater than 2 (since, by Lemma 14, $\vec{G}$ cannot be a cycle).

Now, $\vec{G}[V(\vec{G}) \setminus \{V(\vec{C}) \setminus \{z\}\}]$ is a cactus and, since $\vec{G}$ is a minimal counterexample, there is a homomorphism f from $\vec{G}[V(\vec{G}) \setminus \{V(\vec{C}) \setminus \{z\}\}]$ to $\vec{B}$. By Lemma 14, we can extend f to a homomorphism of $\vec{G}$ to $\vec{B}$, a contradiction. This completes the proof. ■

We are now able to prove Theorem 5.

***Proof of Theorem* 5.** The proof follows from Lemmas 13, 15, 6 and the fact that $\lambda^o_{2,1}(\vec{B}) = 7$ (from Figure 1). ■

7. Conclusion

In this paper we studied 2-dipath and oriented $L(2,1)$-span of some planar families of graphs. For the family $\mathcal{P}$ of planar graphs we have $17 \leq \vec{\chi}(\mathcal{P}) \leq 80$ where the lower bound is due to Marshall [8] and the upper bound is due to Raspaud and Sopena [10]. In this paper we proved $18 \leq \vec{\lambda}_{2,1}(\mathcal{P}) \leq \lambda^o_{2,1}(\mathcal{P}) \leq 83$. We proved the upper bound using Raspaud and Sopena's result [10]. But for the lower bound, our proof is independent from Marshall's one [8]. Indeed, using Marshall's result one can only prove $17 \leq \lambda^o_{2,1}(\mathcal{P})$.

For the family $\mathcal{O}$ of outerplanar graphs we have $9 \leq \vec{\lambda}_{2,1}(\mathcal{O}) \leq \lambda^o_{2,1}(\mathcal{O}) \leq 10$. Now, according to this paper, improvements of the form $\vec{\lambda}_{2,1}(\mathcal{O}) = 10$ (or $\lambda^o_{2,1}(\mathcal{O}) = 10$) in the above result will imply $20 \leq \vec{\lambda}_{2,1}(\mathcal{P})$ (or $20 \leq \lambda^o_{2,1}(\mathcal{P})$).

We know that there exists an oriented graph on $|\vec{\chi}(\mathcal{P})|$ vertices to which every oriented planar graph admits a homomorphism [13]. This paper tells us that 18 labels $(1, 2, \ldots, 17)$ are not enough to 2-dipath $L(2,1)$-label such a graph. We hope one might be able to use this (and other things, especially the configura-

tions used in Marshall's paper [8]) to improve the lower bound for the oriented chromatic number $\vec{\chi}(\mathcal{P})$ of the family of planar graphs.

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