# SUPERMAGIC GRAPHS HAVING A SATURATED VERTEX ${ }^{1}$ 

Jaroslav Ivančo and Tatiana Polláková<br>Institute of Mathematics,<br>P.J. Šafárik University Jesenná 5, 04154 Košice, Slovakia<br>e-mail: jaroslav.ivanco@upjs.sk<br>tatiana.pollakova@student.upjs.sk


#### Abstract

A graph is called supermagic if it admits a labeling of the edges by pairwise different consecutive integers such that the sum of the labels of the edges incident with a vertex is independent of the particular vertex. In this paper we establish some conditions for graphs with a saturated vertex to be supermagic. Inter alia we show that complete multipartite graphs $K_{1, n, n}$ and $K_{1,2, \ldots, 2}$ are supermagic.


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## 1. INTRODUCTION

We consider finite undirected graphs without loops, multiple edges and isolated vertices. If $G$ is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and the edge set of $G$, respectively. Cardinalities of these sets are called the order and the size of $G$.

Let a graph $G$ and a mapping $f$ from $E(G)$ into positive integers be given. The index-mapping of $f$ is the mapping $f^{*}$ from $V(G)$ into positive integers defined by

$$
f^{*}(v)=\sum_{e \in E(G)} \eta(v, e) f(e) \quad \text { for every } v \in V(G)
$$

[^0]where $\eta(v, e)$ is equal to 1 when $e$ is an edge incident with a vertex $v$, and 0 otherwise. An injective mapping $f$ from $E(G)$ to positive integers is called a magic labeling of $G$ for an index $\lambda$ if its index-mapping $f^{*}$ satisfies
$$
f^{*}(v)=\lambda \quad \text { for all } v \in V(G)
$$

A magic labeling $f$ of $G$ is called a supermagic labeling of $G$ if the set $\{f(e)$ : $e \in E(G)\}$ consists of consecutive positive integers. We say that a graph $G$ is supermagic (magic) whenever there exists a supermagic (magic) labeling of $G$.

The concept of magic graphs was introduced by Sedláček [9]. Supermagic graphs were introduced by Stewart [11]. There is by now a considerable number of papers published on magic and supermagic graphs; we single out $[7,10,6,4]$ as being more particularly relevant to the present paper, and refer the reader to [2] for comprehensive references.

Let $G \cup H$ denote the disjoint union of graphs $G$ and $H$. The join $G \oplus H$ of the disjoint graphs $G$ and $H$ is the graph $G \cup H$ together with all edges joining vertices of $V(G)$ and vertices of $V(H)$. The vertex $v$ of a graph $H$ is called saturated vertex, if it is adjacent to every other vertex. The graph $H$ with a saturated vertex $v$ is isomorphic to $(H-v) \oplus K_{1}$. So the graph with a saturated vertex is also denoted by $G \oplus K_{1}$.

Magic graphs with a saturated vertex were characterized in [10]. In the paper there are also given some conditions for the existence of supermagic graphs $G \oplus K_{1}$. In [7] there are given other sufficient conditions for existence of such graphs. Similar problems are solved in [6].

In this paper we will deal with supermagic graphs $G \oplus K_{1}$ for regular graphs $G$.

## 2. Vertex-magic Total Labelings

The notion of a vertex-magic total labeling was introduced in [8]. A bijective mapping $g: V(G) \cup E(G) \rightarrow\{1,2, \ldots,|V(G)|+|E(G)|\}$ is called a vertex-magic total labeling of a graph $G$ if there is a constant $h$ such that

$$
g(v)+\sum_{e \in E(G)} \eta(v, e) g(e)=h \quad \text { for every } v \in V(G)
$$

that is, $g(v)+g^{*}(v)=h$ for every vertex $v$. The constant $h$ is called the magic constant for $g$.

For regular graphs $G$, the supermagic graphs $G \oplus K_{1}$ can be characterized in the following way.

Theorem 1. Let $G$ be a d-regular graph of order n. The graph $G \oplus K_{1}$ is supermagic if and only if $G$ admits a vertex-magic total labeling with magic con-
stant $h$ such that $(n-d-1)$ is a divisor of the non-negative integer $(n+1) h-$ $n \frac{d+2}{2}\left(n \frac{d+2}{2}+1\right)$.

Proof. Let $v$ be a saturated vertex of $H=G \oplus K_{1}$.
Suppose that $f: E(H) \rightarrow\{a, a+1, \ldots, a+|E(H)|-1\}$ is a supermagic labeling of $H$ for an index $\lambda$. As $|E(H)|=n(d+2) / 2$, the index $\lambda$ satisfies

$$
\lambda=\frac{2}{|V(H)|} \sum_{e \in E(H)} f(e)=\frac{1}{n+1}\left(2 a+n \frac{d+2}{2}-1\right) n \frac{d+2}{2} .
$$

Consider the mapping $g$ defined by

$$
g(x)= \begin{cases}1+f(x)-a & \text { for } x \in E(G) \\ 1+f(v x)-a & \text { for } x \in V(G)\end{cases}
$$

Evidently, $g$ is a bijection from $V(G) \cup E(G)$ onto $\{1,2, \ldots,|V(G)|+|E(G)|\}$ and $g(w)+g^{*}(w)=\lambda-(a-1)(d+1)=h$ for any vertex $w \in V(G)$. Therefore $g$ is a vertex-magic total labeling of $G$. Moreover, we have

$$
\begin{aligned}
(n+1) h-n \frac{d+2}{2}\left(n \frac{d+2}{2}+1\right) & =(n+1)\left(\frac{1}{n+1}\left(2 a+n \frac{d+2}{2}-1\right) n \frac{d+2}{2}-(a-1)(d+1)\right) \\
& -n \frac{d+2}{2}\left(n \frac{d+2}{2}+1\right) \\
& =a n(d+2)-n \frac{d+2}{2}-(a-1)(d+1)(n+1)-n \frac{d+2}{2} \\
& =(a-1) n(d+2)-(a-1)(d+1)(n+1) \\
& =(a-1)(n-d-1) \geq 0 .
\end{aligned}
$$

Thus, $(n-d-1)$ is a divisor of $(n+1) h-n \frac{d+2}{2}\left(n \frac{d+2}{2}+1\right)$.
Now, let us assume that $g$ is a vertex-magic total labeling of $G$ with magic constant $h$ such that ( $n-d-1$ ) is a divisor of the non-negative integer $(n+1) h-$ $n \frac{d+2}{2}\left(n \frac{d+2}{2}+1\right)$. Then there is a non-negative integer $\kappa$ such that

$$
(n+1) h-n \frac{d+2}{2}\left(n \frac{d+2}{2}+1\right)=\kappa(n-d-1) .
$$

Consider the mapping $f$ given by

$$
f(e)= \begin{cases}\kappa+g(e) & \text { for } e \in E(G) \\ \kappa+g(w) & \text { for } e=v w\end{cases}
$$

Clearly, $f$ is a bijection from $E(H)$ onto $\{\kappa+1, \kappa+2, \ldots, \kappa+|V(G)|+|E(G)|\}$. Moreover, $f^{*}(v)=\kappa|V(G)|+\sum_{w \in V(G)} g(w)$ and $f^{*}(w)=h+\kappa(d+1)$ for any vertex $w \in V(G)$. In this case we obtain

$$
\begin{aligned}
\sum_{w \in V(G)} g(w)+|V(G)| h & =2 \sum_{x \in V(G) \cup E(G)} g(x) \\
& =(|V(G)|+|E(G)|)(|V(G)|+|E(G)|+1) \\
& =n \frac{d+2}{2}\left(n \frac{d+2}{2}+1\right) .
\end{aligned}
$$

This implies

$$
\begin{aligned}
\sum_{w \in V(G)} g(w) & =n \frac{d+2}{2}\left(n \frac{d+2}{2}+1\right)-n h \\
& =n \frac{d+2}{2}\left(n \frac{d+2}{2}+1\right)-(n+1) h+h \\
& =h-\kappa(n-d-1) .
\end{aligned}
$$

Hence

$$
f^{*}(v)=\kappa n+h-\kappa(n-d-1)=h+\kappa(d+1),
$$

which means that $f$ is a supermagic labeling of $H$.
Using known results on vertex-magic total labelings of regular graphs of odd order the previous theorem implies the same assertions that was proved in [7] by other methods. Therefore, we apply Theorem 1 to regular graphs of even order and we have immediately

Corollary 2. Let $G$ be a $2(k-1)$-regular graph of order $2 k$. The graph $G \oplus K_{1}$ is supermagic if and only if $G$ admits a vertex-magic total labeling with magic constant $h$ such that $(2 k+1) h \geq 2 k^{2}\left(2 k^{2}+1\right)$.

Corollary 3. Let $G$ be a $k$-regular graph of order $2 k$. The graph $G \oplus K_{1}$ is supermagic if and only if $G$ admits a vertex-magic total labeling with magic constant $h$ such that the non-negative integer $(2 k+1) h-k(k+2)(k+1)^{2}$ is an integral multiple of $(k-1)$.

Note that for 2-regular graphs a vertex-magic total labeling corresponds to an edge-magic total labeling introduced by Kotzig and Rosa. Using this correspondence we can rewrite the following known result for cycles (see [12]).

Proposition 1. The cycle $C_{2 k}$ has a vertex-magic total labeling with magic constant $h=7 k+1$.

Now we are able to prove the following assertion.
Theorem 4. Let $G$ be a d-regular graph of order $2 k$. If $G$ contains a Hamilton cycle $C$ such that $G-E(C)$ is supermagic then $G$ admits a vertex-magic total labeling with magic constant $h=7 k+1+\frac{1}{2}(d-2)(k d+6 k+1)$.

Proof. Put $H=G-E(C)$. By the assumption there is a supermagic labeling $f: E(H) \rightarrow\{1,2, \ldots,|E(H)|\}$. Since $|E(H)|=k(d-2)$, the index of $f$ satisfies

$$
\lambda=\frac{(|E(H)|+1)|E(H)|}{|V(G)|}=\frac{1}{2}(k(d-2)+1)(d-2) .
$$

Similarly, according to Proposition 1 there exists a vertex-magic total labeling $g: V(C) \cup E(C) \rightarrow\{1,2, \ldots, 4 k\}$ with magic constant $h=7 k+1$. Consider
the mapping $\varphi$ defined by

$$
\varphi(x)= \begin{cases}g(x) & \text { for } x \in V(C) \cup E(C) \\ 4 k+f(x) & \text { for } x \in E(H)\end{cases}
$$

Clearly, $\varphi$ is a bijection from $V(G) \cup E(G)$ onto $\{1,2, \ldots,|V(G)|+|E(G)|\}$. Accordingly

$$
\begin{aligned}
\varphi(w)+\varphi^{*}(w) & =h+\lambda+4 k(d-2) \\
& =7 k+1+\frac{1}{2}(k(d-2)+1)(d-2)+4 k(d-2) \\
& =7 k+1+\frac{1}{2}(d-2)(k d+6 k+1)
\end{aligned}
$$

for any vertex $w \in V(G)$. Therefore $\varphi$ is a desired vertex-magic total labeling of $G$.

## 3. Complete Multipartite Graphs

A complete $k$-partite graph is a graph whose vertices can be partitioned into $k \geq 2$ disjoint classes $V_{1}, \ldots, V_{k}$ such that two vertices are adjacent if and only if they belong to distinct classes. If $\left|V_{i}\right|=n_{i}$ for all $i=1, \ldots, k$, then the complete $k$-partite graph is denoted by $K_{n_{1}, \ldots, n_{k}}$. If $n_{i}=n$ for all $i=1, \ldots, k$, then the complete $k$-partite graph is regular of degree $(k-1) n$ and is denoted by $K_{k[n]}$. Similarly, if $n_{i}=n$ for all $i=1, \ldots, k$ and $n_{k+1}=p$ then the complete $(k+1)$ partite graph is denoted by $K_{p, k[n]}$.

In this section we characterize supermagic graphs $K_{1,2[n]}=K_{n, n} \oplus K_{1}$ and $K_{1, k[2]}=K_{k[2]} \oplus K_{1}$. Let us recall some notions and assertions, which we shall use in the next.

A $k$-factor (or only a factor) of a graph is defined to be its $k$-regular spanning subgraph.

Proposition 2 [3]. If $G$ is a graph decomposable into pairwise edge-disjoint supermagic factors, then $G$ is supermagic.

For any graph $G$ we define a graph $G^{\bowtie}$ by $V\left(G^{\bowtie}\right)=\bigcup_{v \in V(G)}\left\{v^{0}, v^{1}\right\}$ and $E\left(G^{\bowtie}\right)=\bigcup_{v u \in E(G)}\left\{v^{0} u^{1}, v^{1} u^{0}\right\} \cup \bigcup_{v \in V(G)}\left\{v^{0} v^{1}\right\}$. In [7] the following result is proved.

Proposition 3 [7]. Let $G$ be a $2 r$-regular graph of odd order. If $G$ is Hamiltonian, then $G^{\bowtie}$ is a supermagic graph.

In [4] the following assertions are proved.
Proposition 4 [4]. Let $G$ be a $4 k$-regular bipartite graph which can be decomposed into two edge-disjoint connected $2 k$-factors. Then $G$ is a supermagic graph.

Proposition 5 [4]. Let $G$ be a 6-regular bipartite graph of order $2 n$ which can be decomposed into three edge-disjoint 2-factors where the first is isomorphic to $2 C_{n}$ and the others are Hamilton cycles. Then $G$ is a supermagic graph.

For $X, Y \subseteq V(G)$ the subgraph of a graph $G$ induced by $\{u v \in E(G): u \in X, v \in$ $Y\}$ is denoted by $G(X, Y)$. The complement of a graph $G$ is denoted by $\bar{G}$. In [5] the following result is proved.

Proposition 6 [5]. Let $G$ be a d-regular bipartite graph of order $2 n$ with parts $U_{1}$ and $U_{2}$. If $n \geq 5$ and $d$ are odd and $\bar{G}\left(U_{1}, U_{2}\right)$ is a Hamiltonian graph, then the complement of $G$ is a supermagic graph.

Let $n, m$ and $a_{1}<\cdots<a_{m} \leq\lfloor n / 2\rfloor$ be positive integers. A graph with the vertex set $\left\{v_{0}, \ldots, v_{n-1}\right\}$ and the edge set $\left\{v_{i} v_{i+a_{j}}: 0 \leq i<n, 1 \leq j \leq m\right\}$, the indices being taken modulo $n$, is called a circulant graph and it is denoted by $C_{n}\left(a_{1}, \ldots, a_{m}\right)$. It is easy to see that the circulant graph $C_{n}\left(a_{1}, \ldots, a_{m}\right)$ is a regular graph of degree $r$, where $r=2 m-1$ when $a_{m}=n / 2$, and $r=2 m$ otherwise. In [1] the following result is proved.

Proposition 7 [1]. Any circulant graph of degree $8 k$ is supermagic.

Now we are able to prove the following assertions.

Theorem 5. Let $C$ be a Hamilton cycle of the complete bipartite graph $K_{n, n}$, where $n \geq 5$. Then $K_{n, n}-E(C)$ is a supermagic graph.

Proof. Consider the following cases.
Case A. Let $n \equiv 1(\bmod 2)$. It is not difficult to check that the graph $\left(\bar{C}_{n}\right)^{\bowtie}$ is isomorphic to $K_{n, n}-E(C)$. The graph $\bar{C}_{n}$ is Hamiltonian, $(n-3)$-regular and so, by Proposition $3, K_{n, n}-E(C)$ is a supermagic graph.

Case B. Let $n \equiv 0(\bmod 2)$. Put $k=\frac{n}{2}$. Suppose that $\left\{v_{0}, v_{1}, \ldots, v_{n-1}\right\}$ and $\left\{u_{0}, u_{1}, \ldots, u_{n-1}\right\}$ are parts of $K_{n, n}$. The subgraph $C^{j}, 0 \leq j \leq k-1$, induced by $\bigcup_{i=0}^{n-1}\left\{v_{i} u_{i+2 j}, u_{i+2 j} v_{i+1}\right\}$ (indices are taken modulo $n$ ) is a Hamilton cycle of $K_{n, n}$. Moreover, $C^{0}, C^{1}, \ldots, C^{k-1}$ form a decomposition of $K_{n, n}$ into pairwise edge-disjoint cycles.

If $k$ is odd, then there is an integer $r \geq 1$ such that $k=2 r+1$. In this case the graph $K_{n, n}-E\left(C^{0}\right)$ is regular of degree $4 r$ and the sets $\bigcup_{j=1}^{r} E\left(C^{j}\right)$, $\bigcup_{j=r+1}^{2 r} E\left(C^{j}\right)$ form its decomposition into two edge-disjoint connected $2 r$-factors. Thus, according to Proposition $4, K_{n, n}-E\left(C^{0}\right)$ is a supermagic graph.

A supermagic labeling of $K_{8,8}-E\left(C^{3}\right)$ is described below by giving the labels
of edges in the following matrix.

|  | $u_{0}$ | $u_{1}$ | $u_{2}$ | $u_{3}$ | $u_{4}$ | $u_{5}$ | $u_{6}$ | $u_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{0}$ | 8 | 39 | 16 | 11 | 25 | - | - | 48 |
| $v_{1}$ | 41 | 7 | 33 | 13 | 24 | 29 | - | - |
| $v_{2}$ | - | 42 | 6 | 36 | 17 | 20 | 26 | - |
| $v_{3}$ | - | - | 43 | 5 | 32 | 9 | 23 | 35 |
| $v_{4}$ | 27 | - | - | 44 | 4 | 40 | 18 | 14 |
| $v_{5}$ | 22 | 34 | - | - | 45 | 3 | 31 | 12 |
| $v_{6}$ | 19 | 15 | 28 | - | - | 46 | 2 | 37 |
| $v_{7}$ | 30 | 10 | 21 | 38 | - | - | 47 | 1 |

Let $T_{1}$ and $T_{2}$ be subgraphs of $K_{12,12}$ induced by $E\left(C^{0}\right) \cup E\left(C^{1}\right) \cup E\left(C^{2}\right)$ and $E\left(C^{3}\right) \cup E\left(C^{4}\right)$, respectively. A supermagic labeling of $T_{1}$ is described below by giving the labels of edges in the following matrix.

|  | $u_{0}$ | $u_{1}$ | $u_{2}$ | $u_{3}$ | $u_{4}$ | $u_{5}$ | $u_{6}$ | $u_{7}$ | $u_{8}$ | $u_{9}$ | $u_{10}$ | $u_{11}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{0}$ | 12 | 58 | 24 | 16 | 37 | - | - | - | - | - | - | 72 |
| $v_{1}$ | 61 | 11 | 49 | 19 | 36 | 43 | - | - | - | - | - | - |
| $v_{2}$ | - | 62 | 10 | 54 | 25 | 30 | 38 | - | - | - | - | - |
| $v_{3}$ | - | - | 63 | 9 | 48 | 13 | 35 | 51 | - | - | - | - |
| $v_{4}$ | - | - | - | 64 | 8 | 60 | 26 | 22 | 39 | - | - | - |
| $v_{5}$ | - | - | - | - | 65 | 7 | 47 | 14 | 34 | 52 | - | - |
| $v_{6}$ | - | - | - | - | - | 66 | 6 | 59 | 27 | 21 | 40 | - |
| $v_{7}$ | - | - | - | - | - | - | 67 | 5 | 46 | 18 | 33 | 50 |
| $v_{8}$ | 41 | - | - | - | - | - | - | 68 | 4 | 53 | 28 | 23 |
| $v_{9}$ | 32 | 53 | - | - | - | - | - | - | 69 | 3 | 45 | 17 |
| $v_{10}$ | 29 | 20 | 42 | - | - | - | - | - | - | 70 | 2 | 56 |
| $v_{11}$ | 44 | 15 | 31 | 57 | - | - | - | - | - | - | 71 | 1 |

By Proposition $4, T_{2}$ is a supermagic graph because it is decomposable into two Hamilton cycles. Therefore, according to Proposition 2, the graph $K_{12,12}-E\left(C^{5}\right)$ is supermagic.

For even $k \geq 8$ there is an integer $r \geq 4$ such that $k=2 r$. Let $H_{1}$ be a subgraph of $K_{n, n}$ induced by $\left\{v_{0} u_{k-1}, v_{k} u_{n-1}\right\} \cup\left(E\left(C^{0}\right)-\left\{u_{k-1} v_{k}, u_{n-1} v_{0}\right\}\right)$. It is easy to see that $H_{1}$ is isomorphic to $2 C_{n}$. Similarly, the subgraph $H_{2}$ induced by $\left(E\left(C^{0}\right) \cup E\left(C^{r}\right)\right)-E\left(H_{1}\right)$ is a 2 -factor of $K_{n, n}$. According to Proposition 5, the subgraph $G_{1}$ of $K_{n, n}$ induced by $E\left(H_{1}\right) \cup E\left(C^{1}\right) \cup E\left(C^{r+1}\right)$ is supermagic. Similarly, the subgraph $G_{2}$ induced by $\bigcup_{j=2}^{r-1} E\left(C^{j}\right) \cup \bigcup_{j=r+2}^{2 r-2} E\left(C^{j}\right) \cup E\left(H_{2}\right)$ is a $4(r-2)$-regular bipartite graph decomposable into two edge-disjoint connected $2(r-2)$-factors. By Proposition $4, G_{2}$ is a supermagic graph. As $G_{1}$ and $G_{2}$ are edge-disjoint factors of $K_{n, n}-E\left(C^{2 r-1}\right)$, according to Proposition 2, the graph $K_{n, n}-E\left(C^{2 r-1}\right)$ is supermagic.

Corollary 6. The complete tripartite graph $K_{1,2[n]}$ is supermagic if and only if $n \geq 2$.

Proof. According to Proposition 1, the graph $C_{4}=K_{2,2}$ has a vertex-total labeling with magic constant 15. Vertex-magic total labelings of $K_{3,3}$ and $K_{4,4}$ with magic constants 36 and 70 are described below by giving the labels of vertices and edges in the following matrices.

|  |  | $u_{0}$ | $u_{1}$ | $u_{2}$ |  |  | $u_{0}$ | $u_{1}$ | $u_{2}$ | $u_{3}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | 4 | 3 | 5 |  |  | 1 | 4 | 6 | 9 |
| $v_{0}$ | 1 | 7 | 15 | 13 | $v_{0}$ | 2 | 24 | 22 | 14 | 8 |
| $v_{1}$ | 2 | 14 | 12 | 8 | $v_{1}$ | 3 | 23 | 13 | 12 | 19 |
| $v_{2}$ | 9 | 11 | 6 | 10 | $v_{2}$ | 5 | 15 | 11 | 21 | 18 |
|  |  |  |  | $v_{3}$ | 10 | 7 | 20 | 17 | 16 |  |

According to Theorems 4 and 5 the graph $K_{n, n}$, for $n \geq 5$, admits a vertex -magic total labeling with magic constant

$$
h=7 n+1+\frac{1}{2}(n-2)\left(n^{2}+6 n+1\right)=\frac{1}{2} n(n+1)(n+3)
$$

Therefore, for every $n \geq 2$, the graph $K_{n, n}$ has a vertex-magic total labeling with magic constant $h=\frac{1}{2} n(n+1)(n+3)$. Moreover,

$$
(2 n+1) h-n(n+2)(n+1)^{2}=\frac{n(n+1)}{2}(n-1)>0
$$

Thus, by Corollary $3, K_{n, n} \oplus K_{1}=K_{1,2[n]}$ is a supermagic graph.
The opposite implication is obvious.

Theorem 7. The complement of the circulant graph $C_{2 n}(1, n)$ is supermagic for any integer $n \geq 4$.

Proof. Clearly, the circulant graph $C_{2 n}(2,3, \ldots, n-1)$ is a complement of $C_{2 n}(1, n)$. Consider the following cases.

Case A. Let $n \equiv 1(\bmod 2)$. In this case $C_{2 n}(1, n)$ is a 3 -regular bipartite graph with parts $U_{1}=\left\{v_{0}, v_{2}, \ldots, v_{2 n-2}\right\}$ and $U_{2}=\left\{v_{1}, v_{3}, \ldots, v_{2 n-1}\right\}$. Moreover, $C_{2 n}(n-2)$ is a Hamilton cycle of $\overline{C_{2 n}(1, n)}\left(U_{1}, U_{2}\right)$. Therefore, by Proposition 6 , the complement of $C_{2 n}(1, n)$ is a supermagic graph.

Case B. Let $n \equiv 2(\bmod 4)$. Then there is an integer $k$ such that $n=4 k+2$. In this case $C_{2 n}(2,3, \ldots, n-1)$ is a circulant graph of degree $2(n-2)=8 k$. According to Proposition 7, the complement of $C_{2 n}(1, n)$ is supermagic.

Case C. Let $n \equiv 0(\bmod 4)$. Then there is an integer $k$ such that $n=4 k$. If $k=1$, then a supermagic labeling of $C_{8}(2,3)$ is described below by giving the
labels of edges in the following matrix.

|  | $v_{0}$ | $v_{1}$ | $v_{2}$ | $v_{3}$ | $v_{4}$ | $v_{5}$ | $v_{6}$ | $v_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{0}$ | - | - | 2 | 10 | - | 9 | 13 | - |
| $v_{1}$ | - | - | - | 16 | 4 | - | 6 | 8 |
| $v_{2}$ | 2 | - | - | - | 5 | 15 | - | 12 |
| $v_{3}$ | 10 | 16 | - | - | - | 7 | 1 | - |
| $v_{4}$ | - | 4 | 5 | - | - | - | 14 | 11 |
| $v_{5}$ | 9 | - | 15 | 7 | - | - | - | 3 |
| $v_{6}$ | 13 | 6 | - | 1 | 14 | - | - | - |
| $v_{7}$ | - | 8 | 12 | - | 11 | 3 | - | - |

If $k \geq 2$, then the graph $C_{8 k}(2,3, \ldots, 4 k-1)$ is decomposable into factors $F_{1}=$ $C_{8 k}(2 k-1,4 k-1)$ and $F_{2}=C_{8 k}(2, \ldots, 2 k-2,2 k, \ldots, 4 k-2)$. The factor $F_{1}$ is a 4-regular bipartite graph which can be decomposed into Hamilton cycles $C_{8 k}(2 k-1)$ and $C_{8 k}(4 k-1)$. According to Proposition 4, $F_{1}$ is a supermagic graph. Similarly, $F_{2}$ is a circulant graph of degree $8(k-1)$ and by Proposition 7 , it is also supermagic. Finally, by Proposition 2, the complement of $C_{8 k}(1,4 k)$ is a supermagic graph.

Corollary 8. The complete multipartite graph $K_{1, n[2]}$ is supermagic if and only if $n \geq 2$.

Proof. According to Corollary 6, the graph $K_{1,2[2]}$ is a supermagic graph.
A vertex-magic total labeling of $K_{3[2]}$ with magic constant 53 is described below by giving the labels of vertices and edges in the following matrix.

|  |  | $v_{0}$ | $v_{1}$ | $v_{2}$ | $v_{3}$ | $v_{4}$ | $v_{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | 1 | 2 | 3 | 7 | 6 | 5 |
| $v_{0}$ | 1 | - | 11 | 10 | - | 18 | 13 |
| $v_{1}$ | 2 | 11 | - | 9 | 16 | - | 15 |
| $v_{2}$ | 3 | 10 | 9 | - | 14 | 17 | - |
| $v_{3}$ | 7 | - | 16 | 14 | - | 4 | 12 |
| $v_{4}$ | 6 | 18 | - | 17 | 4 | - | 8 |
| $v_{5}$ | 5 | 13 | 15 | - | 12 | 8 | - |

Since $K_{n[2]}=C_{2 n}(1,2, \ldots, n-1)$, the complement of $C_{2 n}(1, n)$ is isomorphic to $K_{n[2]}-E\left(C_{2 n}(1)\right)$. Thus, according to Theorems 4 and 7 , the graph $K_{n[2]}$, for $n \geq 4$, admits a vertex-magic total labeling with magic constant

$$
h=7 n+1+\frac{1}{2}(2 n-4)(n(2 n-2)+6 n+1)=2 n^{3}-1 .
$$

Therefore, for every $n \geq 3$, the graph $K_{n[2]}$ has a vertex-magic total labeling with magic constant $h=2 n^{3}-1$. Moreover, for $n \geq 3$, we have

$$
(2 n+1) h=4 n^{4}+2 n\left(n^{2}-1\right)-1>4 n^{4}+2 n^{2}=2 n^{2}\left(2 n^{2}+1\right)
$$

Thus, by Corollary $2, K_{n[2]} \oplus K_{1}=K_{1, n[2]}$ is a supermagic graph.
The opposite implication is obvious.

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