# ALMOST-RAINBOW EDGE-COLORINGS OF SOME SMALL SUBGRAPHS 

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#### Abstract

Let $f(n, p, q)$ be the minimum number of colors necessary to color the edges of $K_{n}$ so that every $K_{p}$ is at least $q$-colored. We improve current bounds on these nearly "anti-Ramsey" numbers, first studied by Erdős and Gyárfás. We show that $f(n, 5,9) \geq \frac{7}{4} n-3$, slightly improving the bound of Axenovich. We make small improvements on bounds of Erdős and Gyárfás by showing $\frac{5}{6} n+1 \leq f(n, 4,5)$ and for all even $n \not \equiv 1(\bmod 3), f(n, 4,5) \leq n-$ 1. For a complete bipartite graph $G=K_{n, n}$, we show an $n$-color construction to color the edges of $G$ so that every $C_{4} \subseteq G$ is colored by at least three colors. This improves the best known upper bound of Axenovich, Füredi, and Mubayi.


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## 1. Introduction

### 1.1. Definitions

For basic graph theoretic notation and definition see Diestel [3]. All graphs $G$ are undirected with the vertex set $V$ and edge set $E$. We use $|G|$ for $|V|$ and $\|G\|$ for
$|E| . K_{n}$ denotes the complete graph on $n$ vertices and $K_{n, m}$ the bipartite graph with $n$ vertices and $m$ vertices in the first and second part, respectively. For any edge $(u, v)$, let $\mathcal{C}(u, v)$ be the color on that edge, and for any vertex $v$, let $\mathcal{C}(v)$ be the set of colors on the edges incident to $v$. We say that an edge-coloring is proper if every pair of incident edges are of different colors. If vertices $u, v$ are adjacent, we write $u \sim v$.

### 1.2. Coloring edges

Given a graph $G$ of order $n$ and integers $p, q$ so that $2 \leq p \leq n$ and $1 \leq q \leq\binom{ p}{2}$, call an edge-coloring $(p, q)$ if every $K_{p} \subseteq K_{n}$ receives at least $q$ colors on its edges. Let $f(n, p, q)$ be the minimum colors in a $(p, q)$ coloring of $K_{n}$. This generalization of classical Ramsey functions was first mentioned by Erdős in [4] and later studied by Erdős and Gyárfás in [5]. Further, define $\phi(n, p, q)$ to be the minimum colors in a proper $(p, q)$ coloring of $K_{n}$.

Extending the definition, for any graph $G$, call an edge coloring $(H, q)$ if every subgraph $H \subseteq G$ receives at least q colors on its edges. Let $f(G, H, q)$ be the minimum colors in an $(H, q)$ coloring of the edges of $G$. We say that a coloring of $H$ is almost-rainbow if $q=\|H\|-1$, that is, one color is repeated once.

For an extended survey regarding bounds on rainbow colorings, see [7].
Using the Local Lemma, the authors in [5] were able to produce bounds for $f(n, p, q)$, with several difficult cases unresolved. Among those were $f(n, 4,3)$, $f(n, 4,4), f(n, 4,5)$, and $f(n, 5,9)$. In these cases they showed that $f(n, 4,3) \leq$ $c \sqrt{n}, c \sqrt{n} \leq f(n, 4,4) \leq c n^{\frac{2}{3}}, \frac{5 n-1}{6} \leq f(n, 4,5) \leq n$, and $\frac{4}{3} n \leq f(n, 5,9) \leq$ $c n^{\frac{3}{2}}$. The authors further mentioned that in this branch of generalized Ramsey theory, finding the orders of magnitude of $f(n, 4,4)$ and $f(n, 5,9)$ are "the most interesting open problems, at least to show that the latter is non-linear". The authors then stated the linearity of said function as Problem 1.

As for $f(n, 4,5)$, the authors showed that $\frac{5(n-1)}{6} \leq f(n, 4,5)$ with an upper bound of $n$ for odd $n$ and $n-1$ for even $n$ if $n-1$ is prime.

In [9], Mubayi showed that

$$
f(n, 4,3) \leq e^{O(\sqrt{\log n})}
$$

and in [8] Kostochka and Mubayi showed that for some constant c,

$$
f(n, 4,3) \geq \frac{c \log n}{\log \log \log n}
$$

Fox and Sudakov in [6], further improved the lower bound to $\frac{\log n}{4000}$.
As for the other case, in [1], Axenovich showed that for some constant c,

$$
\frac{1+\sqrt{5}}{2} n \leq f(n, 5,9) \leq 2 n^{1+\frac{c}{\sqrt{\log n}}}
$$

In that same paper, she remarked that Tóth had communicated to her that the lower bound can be improved to $2 n-6$, however, the result has remained unpublished for over ten years.

In Section 2, we show

$$
f(n, 5,9) \geq \frac{7}{4} n-3 .
$$

In Section 3, we make minimal improvements in the work of [5], showing $\frac{5}{6}(n-$ 1) $+1 \leq f(n, 4,5) \leq n-1$ for even $n$ not congruent to one mod three.

In [2], the authors showed that $f\left(K_{n, n}, C_{4}, 3\right) \geq \frac{2}{3} n, f\left(K_{n, n}, C_{4}, 3\right) \leq n$ for odd $n \geq 5$, and $f\left(K_{n, n}, C_{4}, 3\right) \leq n+1$ for even $n \geq 5$.

In Section 4, we show

$$
f\left(K_{n, n}, C_{4}, 3\right) \leq n, \text { for all } n \geq 3
$$

We believe that this upper-bound is the best possible.

## 2. Almost-Rainbow Five-cliques

### 2.1. The main tool

Let $f(G)$ be the minimum number of colors needed to color the edges of $G$ so that every path or cycle with four edges is at least three-colored.

Let $\phi(G)$ be defined as $f(G)$ above, except replace "color" by "properly color". By arguments from [1] it is easy to see that $f(n, 5,9) \leq \phi(n, 5,9)=\phi\left(K_{n}\right)$.
Lemma 1. $\phi\left(K_{2, n}\right)=\left\lceil\frac{3}{2} n\right\rceil$.
Proof. Suppose the edges of $G=K_{2, n}$ are properly colored so that every path of length four receives at least three colors. Call the vertices in the first part of $G, u$ and $v$. Choose a color $a \in \mathcal{C}(u) \cap \mathcal{C}(v)$ so that for some vertices $x, y$ in the second part of $G, a=\mathcal{C}(u, x)=\mathcal{C}(v, y)$. Note that there exist colors $b, c$ so that $b=\mathcal{C}(u, y), c=\mathcal{C}(v, x)$, and $b, c \in(\mathcal{C}(u) \cup \mathcal{C}(v)) \backslash(\mathcal{C}(u) \cap \mathcal{C}(v))$. Since there are two colors for every one in $\mathcal{C}(u) \cap \mathcal{C}(v)$, we can say that

$$
\begin{equation*}
|\mathcal{C}(u) \cap \mathcal{C}(v)| \leq\left\lfloor\frac{1}{2}|(\mathcal{C}(u) \cup \mathcal{C}(v)) \backslash(\mathcal{C}(u) \cap \mathcal{C}(v))|\right\rfloor \tag{1}
\end{equation*}
$$

Applying this inequality to the principle of inclusion-exclusion, we write

$$
|\mathcal{C}(u) \cup \mathcal{C}(v)|=|\mathcal{C}(u)|+|\mathcal{C}(v)|-|\mathcal{C}(u) \cap \mathcal{C}(v)| \geq 2 n-\frac{1}{3}|\mathcal{C}(u) \cup \mathcal{C}(v)|
$$

Solving for the union we get

$$
\begin{equation*}
|\mathcal{C}(u) \cup \mathcal{C}(v)| \geq \frac{3}{2} n \tag{2}
\end{equation*}
$$

For the upper bound, we construct an edge-coloring of $G=K_{2, n}$ with $\left\lceil\frac{3}{2} n\right\rceil$ colors. Label the vertices of the first part of $G, u, v$ and the second part $\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$. Let $r=\left\lceil\frac{n}{2}\right\rceil$. Color the edges $\left(v_{1}, u\right),\left(v_{2}, u\right), \ldots,\left(v_{r}, u\right)$ by the colors $1, \ldots, r$. If $n$ is even, color the edges $\left(v_{n}, v\right),\left(v_{n-1}, v\right), \ldots,\left(v_{n-r+1}, v\right)$ from the set of colors $\{1, \ldots, r\}$. If $n$ is odd, color the edges $\left(v_{n}, v\right),\left(v_{n-1}, v\right), \ldots,\left(v_{n-r+2}, v\right)$ by some of the colors from the set $\{1, \ldots, r\}$. Color the remaining edges distinctly by all the colors not previously used. Let $i$ and $j$ be such that $\mathcal{C}\left(u, v_{i}\right)=\mathcal{C}\left(v, u_{j}\right)$. Notice that for any $k \in\{1, \ldots, n\},\left\{\mathcal{C}\left(u, v_{i}\right), \mathcal{C}\left(u, v_{j}\right), \mathcal{C}\left(v, v_{i}\right), \mathcal{C}\left(u, v_{k}\right)\right\}$ are pairwise distinct. Hence every 4 -path receives at least three colors.

### 2.2. A small improvement

Theorem 2. $f(n, 5,9) \geq \frac{7}{4} n-3$.
Proof. Consider a $(5,9)$ edge-coloring of $G=K_{n}$ using $s$ colors. Using the argument of Axenovich [1], we first assume that the coloring is not proper, so there exist incident edges $\left(v_{1}, v_{2}\right)$ and $\left(v_{1}, v_{3}\right)$ of the same color. For the coloring to remain $(5,9)$, all edges of $G \backslash\left\{\left(v_{1}, v_{2}\right),\left(v_{1}, v_{3}\right)\right\}$ incident to $\left\{v_{1}, v_{2}, v_{3}\right\}$ must be of different colors and not $\mathcal{C}\left(v_{1}, v_{2}\right)$ or $\mathcal{C}\left(v_{2}, v_{3}\right)$. Therefore, $s \geq 3 n-7 \geq \frac{7}{4} n-3$ for $n \geq 5$.

Next we assume the coloring is proper. By the pigeonhole principle there exists a color, call it $a$, used on at least $\binom{n}{2} / s$ edges. Let $A$ be the set of vertices adjacent to edges colored $a$ and choose vertices $u, v \in A$ so that $c(u, v)=a$.

We say that an edge is in $A$ if both vertices adjacent to that edge are in $A$. Notice that the number of colors on the edges in $A$ adjacent to $u \geq 2\binom{n}{2} / s-1$, the same for $v$, and $c(u, v)$ is counted both times. Let $H$ be the complete bipartite graph with vertices $\{u, v\}$ in the first part and the vertices of $G \backslash A$ in the second part. Let the edge coloring of $H$ be induced by the edge coloring of $G$. For any $x \in A$ and $y \in G, \mathcal{C}(u, x) \neq \mathcal{C}(v, y)$, else we produce a two-colored four-edge path. The same reasoning holds for $y \in A$ and $x \in G$. This implies that the colors on the edges of $H$ are distinct from the colors previously counted. Hence we apply Lemma 1 to $H$ to obtain

$$
\begin{equation*}
s \geq 2 \frac{\binom{n}{2}}{s}-1+2 \frac{\binom{n}{2}}{s}-1-1+\frac{3}{2}\left(n-2 \frac{\binom{n}{2}}{s}\right) . \tag{3}
\end{equation*}
$$

Solving for s we obtain the result.

## 3. Almost-rainbow Four-cliques

We obtain a marginal improvement for the lower bound on $f(n, 4,5)$ and extend the even case of the upper bound from [5] to all complete graphs with orders not congruent to one modulo three.

Theorem 3. (i) $\frac{5}{6}(n-1)+1 \leq f(n, 4,5)$.
(ii) $f(n, 4,5) \leq n-1$ for even $n \not \equiv 1(\bmod 3)$.

Proof. Given a $(4,5)$ coloring of the edges of $G=K_{n}$, for a fixed vertex $u$, let $P_{u}$ denote the set of edges incident to $u$, whose colors are repeated on other edges incident to $u$. Let $S_{u}$ denote the set of edges with non-repeated colors, incident to $u$. Let $T_{u}$ denote the set of edges incident to edges from $P_{u}$ of the same color.


Notice that

1. $\mathcal{C}\left(P_{u}\right) \cap \mathcal{C}\left(S_{u}\right)=\emptyset$ by definition.
2. $\mathcal{C}\left(P_{u}\right) \cap \mathcal{C}\left(T_{u}\right)=\emptyset$ else we obtain an induced four-colored $K_{4}$ on the edges $p \in P_{u}$ and $t \in T_{u}$ that share the same same color and the edges $p_{1}, p_{2} \in P_{u}$ that share the same color and are incident to $t$ ( $p$ may be equal to $p_{1}$, depending on the coloring).
3. $\mathcal{C}\left(S_{u}\right) \cap \mathcal{C}\left(T_{u}\right)=\emptyset$ else we obtain an induced four colored $K_{4}$ on the the edge $s \in S_{u}$ and $t \in T_{u}$ of the same color and the two edges of $P_{u}$ with the same color, which are incident to $t$.
4. For any vertex $v$ distinct from $u$, if $(u, v) \in P_{u}$ so that $\mathcal{C}(u, v)=\mathcal{C}(u, w)$ for some $w$, then $(u, v) \notin P_{v}$ and $(v, w) \notin P_{v}$.
5. For any vertex $v$ distinct from $u, T_{u} \cap T_{v}=\emptyset$.

Notice that

$$
2 \sum_{u \in V(G)}\left|T_{u}\right|=\sum_{u \in V(G)}\left|P_{u}\right|
$$

so that

$$
\sum_{u}\left|T_{u}\right|+\sum_{u}\left|P_{u}\right|=3 \sum_{u}\left|T_{u}\right|=3 \frac{1}{n} \sum_{u}\left|T_{u}\right| \times n \leq\binom{ n}{2}
$$

by the above claim 5 , and we obtain

$$
\frac{1}{n} \sum_{u}\left|T_{u}\right| \leq \frac{n-1}{6} .
$$

By the pigeonhole principle, choose a vertex $u$ so that $\left|T_{u}\right| \leq \frac{n-1}{6}$. Notice that $n-1=\operatorname{deg} u=\left|S_{u}\right|+\left|P_{u}\right| \leq\left|S_{u}\right|+\frac{n-1}{3}$, so that

$$
\left|S_{u}\right| \geq \frac{2}{3}(n-1)
$$

Summing up the colors of edges incident to $u$ we get

$$
|\mathcal{C}(u)|=\left|S_{u}\right|+\frac{1}{2}\left|P_{u}\right| \geq \frac{2}{3}(n-1)+\frac{1}{6}(n-1)=\frac{5}{6}(n-1) .
$$

However, $\mathcal{C}\left(T_{u}\right)$ must be nonempty and distinct from the colors counted above, hence

$$
|\mathcal{C}(u)| \geq \frac{5}{6}(n-1)+1 .
$$

For the upper bound we color the edges of $K_{n}$ by a classical proper coloring (see [10] for example) and show that such a coloring is $(4,5)$.

For odd $n$, we $n$-color the edges of $K_{n}$ by drawing the vertices in the form of a regular $n$-gon and coloring the consecutive edges around the boundary in order with colors 1 to $n$. Next we color every edge parallel to a boundary edge by the same color as that boundary edge. Call the resulting labeled graph $G_{n}$. Notice that every $K_{4} \subseteq G_{n}$ with a pair of parallel edges is a non-rectangular trapezoid. Hence the coloring is $(4,5)$.

For even $n$, choose a $K_{n-1}$ subgraph and color it as above, obtaining $G_{n-1}$. Next construct the graph $w \times G_{n-1}$, joining the above graph to a vertex $w$. Since for any vertex $u$ of $G_{n-1}$, there are only $n-2$ incident edges, some color is missing. Apply this color to the edge $(u, w)$ and continue likewise for all vertices of $G_{n-1}$. Call the resulting labeled graph $G_{n}^{*}$.

For vertices $x, y, z \in G_{n}^{*}$ with so that $(x, y)$ and $(y, z)$ are boundary edges, we say that $y$ is opposite an edge $e$ if the line bisecting angle $u v w$ is the perpendicular bisector of $e$. Notice that the edges opposite to $y$ share the same color, which is not used on any edge incident to $y$. By the above observation, $G_{n-1} \subseteq G_{n}^{*}$ is $(4,5)$ colored, hence it is enough to show that for $w$ as chosen above in the definition of $G_{n}^{*}$ and any other distinct vertices $x, y, z$ of $G_{n}^{*}$, the induced subgraph receives
at most one repeated color. Choose any vertex $v \in G_{n}^{*}$. For $i=1, \ldots, n-2$ label the vertices with counterclockwise distance $i$ from $v, u_{i}$, where arithmetic of label indices is performed modulo $n-1$. Notice that the only edges that share the color $\mathcal{C}(w, v)$ are $\left(u_{1}, u_{-1}\right),\left(u_{2}, u_{-2}\right), \ldots,\left(u_{n-2}, u_{-(n-2)}\right)$. For $i=1, \ldots, \frac{n-2}{2}$, if $\mathcal{C}\left(u_{i}, w\right)=\mathcal{C}\left(u_{-i}, v\right)$, then for any edge $e$ opposite $u_{i}, \mathcal{C}(e)=\mathcal{C}\left(u_{-i}, v\right)$. However, this means that

$$
\begin{aligned}
& \mathcal{C}\left(u_{i-1}, u_{i+1}\right)=\mathcal{C}(e)=\mathcal{C}\left(u_{-i}, v\right) \Leftrightarrow v u_{2 k}=v u_{-k} \\
& \Leftrightarrow 3 k \equiv 0(\bmod (n-1)) \Leftrightarrow n \equiv 1(\bmod 3) .
\end{aligned}
$$

## 4. Almost-rainbow Four-cycles

We show the improved upper bound for the bipartite problem, when the two parts of $G$ are of equal size.

## Theorem 4.

$$
f\left(K_{n, n}, G_{4}, 3\right) \leq n, \text { for all } n \geq 3
$$

### 4.1. The coloring

We will explore the matrix

$$
G=\left(\begin{array}{ccccccccc}
1 & 2 & 3 & \ldots & r & \ldots & c+1 & \ldots & n \\
3 & 1 & 2 & \ldots & r-1 & \ldots & c & \ldots & n-1 \\
v_{3} & n-1 & 1 & \ldots & r-2 & \ldots & c-1 & \ldots & n-2 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
v_{n+1-r} & r+1 & r+2 & \ldots & 1 & \ldots & r+c & \ldots & r \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
v_{n-1} & 3 & 4 & \ldots & r+1 & \ldots & c+2 & \ldots & 2 \\
n-2 & u_{2} & u_{3} & \ldots & u_{r} & \ldots & u_{c+1} & \ldots & 1
\end{array}\right) .
$$

The values of $v_{i}$ and $u_{i}$ will be defined shortly.
Let permutation $\sigma$ be the $n-1$ cycle $(12 \cdots n-1)$. That is, $\sigma$ sends $i$ to $i+1(\bmod n-1)$. For a natural number $m$ we shall write $m(\bmod (n-1))$ for its representative in $\{1,2, \ldots, n-1\}$. For each $r$ we defined $\sigma^{(r)}$ by the rule $\sigma^{(r)}(c) \equiv r+c(\bmod (n-1))$. Let us start with the matrix

$$
C=\left(\begin{array}{cccccc}
2 & 3 & \ldots & c+1 & \ldots & n \\
\sigma^{0}(1) & \sigma^{0}(2) & \ldots & \sigma^{0}(c) & \ldots & \sigma^{0}(n-1) \\
\sigma^{n-2}(1) & \sigma^{n-2}(2) & \ldots & \sigma^{n-2}(c) & \ldots & \sigma^{n-2}(n-1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\sigma^{r}(1) & \sigma^{r}(2) & \ldots & \sigma^{r}(c) & \ldots & \sigma^{r}(n-1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\sigma^{2}(1) & \sigma^{2}(2) & \ldots & \sigma^{2}(c) & \ldots & \sigma^{2}(n-1)
\end{array}\right) .
$$

We define the matrix $G$ by adding the first column $V=\left\{v_{1}, \ldots, v_{n-1}, v u\right\}$ and the last row $U=\left\{v u, u_{2}, \ldots, u_{n}\right\}$ to the matrix $C$.

$$
G=\left(\begin{array}{ccccccc}
v_{1} & 2 & 3 & \ldots & c+1 & \ldots & n \\
v_{2} & \sigma^{0}(1) & \sigma^{0}(2) & \ldots & \sigma^{0}(c) & \ldots & \sigma^{0}(n-1) \\
v_{3} & \sigma^{n-2}(1) & \sigma^{n-2}(2) & \ldots & \sigma^{n-2}(c) & \ldots & \sigma^{n-2}(n-1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
v_{n+1-r} & \sigma^{r}(1) & \sigma^{r}(2) & . & \sigma^{r}(c) & . & \sigma^{r}(n-1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
v_{n-1} & \sigma^{2}(1) & \sigma^{2}(2) & \ldots & \sigma^{2}(c) & \ldots & \sigma^{2}(n-1) \\
v u & u_{2} & u_{3} & \ldots & u_{c+1} & \ldots & u_{n}
\end{array}\right) .
$$

The entries of $G$ will be defined as follows: for every 4 -tuple ( $i, j ; l, m$ ) with $1 \leq i<j \leq n$ and $1 \leq l<m \leq n$ the ( $2 \times 2$ ) matrix

$$
G(i, j ; l, m)=\left(\begin{array}{cc}
a_{i l} & a_{i m} \\
a_{j l} & a_{j m}
\end{array}\right) .
$$

We consider the colorings for the edges $V$ and $U$ in three types of even $n(\bmod 6)$.
Type 1: Matrix $G_{1}=G$ for $n \equiv 2(\bmod 6) ; \quad[n=2+6 k, k \geq 1]$

$$
\begin{aligned}
& a_{i, 1}= \begin{cases}1, & i=1, \\
3, & i=2, \\
n, & 3 \leq i \leq \frac{n}{2}+1, \\
2(i-1)-n, & \frac{n}{2}+2 \leq i \leq n-1, \\
n-2, & i=n .\end{cases} \\
& a_{n, l}= \begin{cases}n-2 l, & 1 \leq l \leq \frac{n}{2}-1, \\
n, & \frac{n}{2} \leq l \leq n-2, \\
n-1, & l=n-1, \\
1, & l=n .\end{cases}
\end{aligned}
$$

Type 2: Matrix $G_{2}=G$ for $n \equiv 6(\bmod 6) ; \quad[n=6+6 k, k \geq 1]$

We define $Y$ as $\frac{n}{2}-2$ for even $k$, and as $\frac{n}{2}+1$ for odd $k$.

$$
\begin{aligned}
& a_{i, 1}= \begin{cases}1, & i=1 \\
3, & i=2 \\
n, & 3 \leq i \leq \frac{n}{2}+1 \\
Y, & i=\frac{n}{2}+2 \\
2(i-2)-n, & \frac{n}{2}+3 \leq i \leq n-1 \\
n-2, & i=n\end{cases} \\
& a_{n, l}= \begin{cases}n-2, & l=1 \\
n-2(l+1), & 2 \leq l \leq \frac{n}{2}-2 \\
Y, & l=\frac{n}{2}-1 \\
n, & \frac{n}{2} \leq l \leq n-2 \\
n-1, & l=n-1 \\
1, & l=n\end{cases}
\end{aligned}
$$

Exception for $n=6$; $[k=0]$ the first row $V=\{1,5,6,6,4\}$, the last column $U=\{3,6,6,6,5,1\}$.
Type 3: Matrix $G_{3}=G$ for $n \equiv 4(\bmod 6) ; \quad[n=4+6 k, k \geq 4]$
The regularity starts with $n>22$.

$$
\begin{aligned}
& a_{i, 1}= \begin{cases}1, & i=1, \\
3, & i=2 \\
n, & 3 \leq i \leq \frac{n}{2}+1, \\
n-9, & i=\frac{n}{2}+2 \\
2(i-2)-n, & \frac{n}{2}+3 \leq i \leq \frac{5 n+4}{6} \\
2(i-1)-n, & \frac{5 n+10}{6} \leq i \leq n-1, \\
n-2, & i=n\end{cases} \\
& a_{n, l}= \begin{cases}n-2 l, & 1 \leq l \leq \frac{n-4}{6}, \\
n-2(l+1), & \frac{n+2}{6} \leq l \leq \frac{n}{2}-2 \\
n-9, & l=\frac{n}{2}-1, \\
n, & \frac{n}{2} \leq l \leq n-2 \\
n-1, & l=n-1, \\
1, & l=n\end{cases}
\end{aligned}
$$

## Exceptions:

For $n=10$ we replace $(n-9)$ with $(n-8)$.
For $n=16$ we replace $(n-9)$ with $(n-11)$.
For $n=22$ we replace $(n-9)$ with $(n-5)$ and the definitions:

$$
\begin{aligned}
& a_{i, 1}= \begin{cases}2(i-2)-n, & \frac{n}{2}+3 \leq i \leq \frac{5 n-2}{6} \\
2(i-1)-n, & \frac{5 n+4}{6} \leq i \leq n-1\end{cases} \\
& a_{n, l}= \begin{cases}n-2 l, & 1 \leq l \leq \frac{n-10}{6} \\
n-2(l+1), & \frac{n-4}{6} \leq l \leq \frac{n}{2}-2\end{cases}
\end{aligned}
$$

### 4.2. Sketch of proof

First, we show that every 4-cycle defined in the basic coloring (matrix entries $a_{i j}$ where $1<i \leq n, 1 \leq j<n$ ) is almost rainbow. That is, given $i<j$ and $l<m$ we show that $a_{i, l}, a_{j, l}, a_{i, m}, a_{j, m}$ contains at least three distinct elements in the basic coloring.

Step 1. We start with the matrix $C$ and look at two occurrences, which are identical for each of the types of even $n(\bmod 6)$ specified above.

Case 1. We take the submatrix of $G(i, j ; l, m)$ with $2 \leq l<m \leq n, 2 \leq i<$ $j<n$, and let $s=(n+1)-i, t=(n+1)-j$. A typical $(2 \times 2)$ submatrix in this case has the form:

$$
\left(\begin{array}{cc}
\sigma^{s}(l-1) & \sigma^{s}(m-1) \\
\sigma^{t}(l-1) & \sigma^{t}(m-1)
\end{array}\right)
$$

We wish to show there are three distinct elements: $\sigma^{s}(l-1) \neq \sigma^{t}(l-1)$, $\sigma^{t}(l-1) \neq \sigma^{t}(m-1), \sigma^{s}(l-1) \neq \sigma^{t}(m-1)$.
Suppose $\sigma^{(s)}(l-1) \equiv \sigma^{(t)}(l-1) \Rightarrow s \equiv t(\bmod n-1)$, which is a contradiction. Suppose $\sigma^{(t)}(l-1) \equiv \sigma^{(t)}(m-1) \Rightarrow l \equiv m(\bmod n-1)$, which is a contradiction. Suppose $\sigma^{(s)}(l-1) \equiv \sigma^{(t)}(m-1) \Rightarrow s+l \equiv t+m(\bmod n-1)$, and assume there are three distinct elements: $\sigma^{t}(l-1) \neq \sigma^{t}(m-1), \sigma^{s}(m-1) \neq \sigma^{t}(m-1)$, $\sigma^{s}(m-1) \neq \sigma^{t}(l-1)$. Follow the argument above the first two inequalities are correct. Suppose $\sigma^{s}(m-1) \equiv \sigma^{t}(l-1) \Rightarrow s+m \equiv t+l(\bmod n-1)$. Subtracting equations $s+l \equiv t+m$ and $s+m \equiv t+l \Rightarrow l \equiv m(\bmod n-1)$, which is a contradiction. One of the following two sets has three distinct elements: $\left\{\sigma^{s}(l-1), \sigma^{t}(l-1), \sigma^{t}(m-1)\right\}$ or $\left\{\sigma^{t}(l-1), \sigma^{t}(m-1), \sigma^{s}(m-1)\right\}$.

Case 2. We take the submatrix of $G(i, j ; l, m)$ with $2 \leq l<m \leq n, i=1$, $1<j<n$, and let $r=(n+1)-j$. A typical $(2 \times 2)$ submatrix has the form:

$$
\left(\begin{array}{cc}
l & m \\
\sigma^{r}(l-1) & \sigma^{r}(m-1)
\end{array}\right)
$$

We wish to show there are three distinct elements: $l \neq m, m \neq \sigma^{r}(m-1)$, $l \neq \sigma^{r}(m-1)$. Suppose $m \equiv \sigma^{(r)}(m-1) \Rightarrow r \equiv 1(\bmod n-1)$, which is a contradiction. Suppose $l \equiv \sigma^{(r)}(m-1) \Rightarrow l \equiv r+m-1(\bmod n-1)$, and assume there are three distinct elements: $\sigma^{r}(l-1) \neq \sigma^{r}(m-1), m \neq \sigma^{r}(m-1), m \neq$ $\sigma^{r}(l-1)$. The first two inequalities are correct. Suppose $m \equiv \sigma^{r}(l-1) \Rightarrow m \equiv$ $r+l-1(\bmod n-1)$. Subtracting equations $l \equiv r+m-1$ and $m \equiv r+l-1 \Rightarrow r=1$ $(\bmod n-1)$, which is a contradiction. One of the following two sets has three distinct elements: $\left\{l, m, \sigma^{r}(m-1)\right\}$ or $\left\{\sigma^{r}(l-1), \sigma^{r}(m-1), m\right\}$.

Step 2. For matrix $G(i, j ; l, m)$ with $i=1, j=n$ and $2 \leq l<m \leq n$ we look at five cases and consider every matrix type defined above of even $n(\bmod 6)$.

Case 1. We take $G(i, j ; l, m)$ with $2 \leq l<m \leq \frac{n}{2}-1, i=1, j=n$.
Subcase 1.1. Consider $G_{1}$,

$$
\left(\begin{array}{cc}
l & m \\
n-2 l & n-2 m
\end{array}\right) .
$$

We wish to show there are three distinct entries: $l \neq n-2 l, n-2 l \neq n-2 m$, $l \neq n-2 m$. Suppose $l=n-2 l \Rightarrow 3 l=n$ and since $n=2+6 k$ this is a contradiction. Suppose $n-2 l=n-2 m \Rightarrow l=m$, which is a contradiction. Suppose $l=n-2 m$ and we wish to show there are three distinct elements: $l \neq m, l \neq n-2 l, m \neq n-2 l$. As shown above the first inequality is correct. Suppose $m=n-2 l$ and since $l=n-2 m \Rightarrow l=m$, which is a contradiction. One of the following two sets has three distinct elements: $\{l, n-2 l, n-2 m\}$ or $\{l, m, n-2 l\}$.

Subcase 1.2. Consider $G_{2}$.

1. $G_{2}$ with $2 \leq l<m \leq \frac{n}{2}-2, i=1, j=n$,

$$
\left(\begin{array}{cc}
l & m \\
n-2(l+1) & n-2(m+1)
\end{array}\right) .
$$

We wish to show there are three distinct entries: $l \neq n-2(m+1), l \neq n-2(l+1)$, $n-2(l+1) \neq n-2(m+1)$. Suppose $l=n-2 l-2 \Rightarrow 3 l=n-2$ and since $n=6+6 k$ this is a contradiction. Suppose $n-2 l=n-2 m \Rightarrow l=m$, which is a contradiction. Suppose $l=n-2(m+1)$ and we wish to show there are three distinct elements: $l \neq m, l \neq n-2(l+1), m \neq n-2(l+1)$. As shown above the first inequality is correct. Suppose $m=n-2(l+1)$ and since $l=n-2(m+1) \Rightarrow l=m$, which is a contradiction. One of the following two sets has three distinct elements: $\{l, n-2(m+1), n-2(l+1)\}$ or $\{l, m, n-2(l+1)\}$.
2. $G_{2}$ with $2 \leq l \leq \frac{n}{2}-2, m=\frac{n}{2}-1, i=1, j=n$,

$$
\left(\begin{array}{cc}
l & m \\
n-2(l+1) & Y
\end{array}\right) .
$$

If $K$ is even $\Rightarrow m=\frac{n}{2}-1, Y=\frac{n}{2}-2 \Rightarrow Y=m-1$.
We wish to show there are three distinct entries: $l \neq m, m \neq m-1, l \neq m-1$. Assume $l=m-1$ and we wish to show there are three distinct entries: $m \neq m-1$, $m \neq n-2(l+1), m-1 \neq n-2(l+1)$. Suppose $m=n-2(l+1)$ and since $l=m-1$ and $m=\frac{n}{2}-1 \Rightarrow n=6$, which is a contradiction. Suppose $m-1=n-2(l+1)$ and since $m=\frac{n}{2}-1$ and $l=m-1 \Rightarrow n=8$, which is a contradiction. One of the following two sets has three distinct elements: $\{l, m, Y\}$ or $\{m, Y, n-2(l+1)\}$.
If $K$ is odd $\Rightarrow m=\frac{n}{2}-1, Y=\frac{n}{2}+1 \Rightarrow Y=m+2$. Three distinct elements are $\{l, m, Y\}$.

Subcase 1.3. Consider $G_{3}$.

1. $G_{3}$ with $i=1, j=n$ and $\left(2 \leq l<m \leq \frac{n-4}{6}\right.$ or $\left.\frac{n+2}{6} \leq l<m \leq \frac{n}{2}-2\right)$. The argument is similar to above one with $G_{1}$. One of the following two sets has three distinct elements: $\{l, n-2 l, n-2 m\}$ or $\{l, m, n-2 l\}$.
2. $G_{3}$ with $2 \leq l \leq \frac{n-4}{6}, \frac{n+2}{6} \leq m \leq \frac{n}{2}-2, i=1, j=n$,

$$
\left(\begin{array}{cc}
l & m \\
n-2 l & n-2(m+1)
\end{array}\right)
$$

We wish to show there are three distinct entries: $l \neq n-2 l, n-2 l \neq n-2(m+1)$, $l \neq n-2(m+1)$. Suppose $l=n-2 l \Rightarrow 3 l=n$ and since $n=4+6 k$ this is a contradiction. Suppose $n-2 l=n-2(m+1) \Rightarrow l=m+1$, which is a contradiction. Suppose $l=n-2(m+1)$ and we wish to show there are three distinct elements: $l \neq m, l \neq n-2 l, m \neq n-2 l$. The first two inequalities are correct. Suppose $m=n-2 l$ and since $l=n-2(m+1) \Rightarrow m=l-2$, which is a contradiction. One of the following two sets has three distinct elements: $\{l, n-2 l, n-2(m+1)\}$ or $\{l, m, n-2 l\}$.
3. $G_{3}$ with $2 \leq l \leq \frac{n-4}{6}, m=\frac{n}{2}-1, i=1, j=n$,

$$
\left(\begin{array}{cc}
l & m \\
n-2 l & n-9
\end{array}\right)
$$

We wish to show there are three distinct entries: $l \neq m, m \neq n-9, l \neq n-9$. Suppose $l=n-9$ and since $l<\frac{n-4}{6} \Rightarrow n-9<\frac{n-4}{6} \Rightarrow n<10$, which is a contradiction. Suppose $m=n-9 \Rightarrow \frac{n}{2}-1=n-9 \Rightarrow n=16$, which is a contradiction. There are three distinct elements $\{l, m, n-9\}$.
4. $G_{3}$ with $\frac{n+2}{6} \leq l \leq \frac{n}{2}-2, m=\frac{n}{2}-1, i=1, j=n$,

$$
\left(\begin{array}{cc}
l & m \\
n-2(l+1) & n-9
\end{array}\right)
$$

We wish to show there are three distinct entries: $l \neq m, m \neq n-9, l \neq n-9$. Suppose $l=n-9$ and since $l<\frac{n}{2}-2 \Rightarrow n-9<\frac{n}{2}-2 \Rightarrow n<14$, which is a contradiction. Suppose $m=n-9 \Rightarrow \frac{n}{2}-1=n-9 \Rightarrow n=16$, which is a contradiction. There are three distinct elements $\{l, m, n-9\}$.

Case 2. For the submatrix $G(i, j ; l, m)$ with $i=1, j=n$ and $\left(\frac{n}{2} \leq l<m \leq\right.$ $n-2$ or $\left.2 \leq l \leq \frac{n}{2}-1, \frac{n}{2} \leq m \leq n-2\right)$ three distinct elements are $\{l, m, n\}$.

Case 3. We take the submatrix $G(i, j ; l, m)$ with $2 \leq l \leq \frac{n}{2}-1, m=n-1$, $i=1, j=n$.

Subcase 3.1. Consider $G_{1}$,

$$
\left(\begin{array}{cc}
l & m \\
n-2 l & n-1
\end{array}\right) .
$$

We wish to show there are three distinct entries: $l \neq n-1, n-2 l \neq n-1$, $l \neq n-2 l$. Suppose $n-2 l=n-1 \Rightarrow l=\frac{1}{2}$, which is a contradiction. Suppose $l=n-2 l \Rightarrow 3 l=n$ and since $n=2+6 k$ this is a contradiction. There are three distinct elements $\{l, n-1, n-2 l\}$.

Subcase 3.2. Consider $G_{2}$.

1. $G_{2}$ with $2 \leq l \leq \frac{n}{2}-2, m=n-1, i=1, j=n$,

$$
\left(\begin{array}{cc}
l & m \\
n-2(l+1) & n-1
\end{array}\right) .
$$

We wish to show there are three distinct entries: $l \neq n-1, n-2(l+1) \neq n-1$, $l \neq n-2(l+1)$. Suppose $n-2(l+1)=n-1 \Rightarrow l=-\frac{1}{2}$, which is a contradiction. Suppose $l=n-2(l+1) \Rightarrow 3 l=n-2$ and since $n=6+6 k$ this is a contradiction. There are three distinct elements $\{l, n-1, n-2(l+1)\}$.
2. $G_{2}$ with $l=\frac{n}{2}-1, m=n-1, i=1, j=n$,

$$
\left(\begin{array}{cc}
l & m \\
Y & n-1
\end{array}\right)
$$

If $K$ is even $\Rightarrow Y=\frac{n}{2}-2$. If $n-1=Y \Rightarrow n-1=\frac{n}{2}-2 \Rightarrow n=-2$, which is a contradiction. Three distinct entries are $\{l, Y, n-1\}$.

If $K$ is $o d d \Rightarrow Y=\frac{n}{2}+1$, and three distinct entries are $\{l, Y, n-1\}$.
Subcase 3.3. Consider $G_{3}$.

1. For $G_{3}$ with $l \leq \frac{n-4}{6}, m=n-1, i=1, j=n$ three distinct elements are $\{l, n-1, n-2 l\}$ (similar to $G_{1}$.)
2. $G_{3}$ with $\frac{n+2}{6} \leq l<n-1, m=n-1, i=1, j=n$,

$$
\left(\begin{array}{cc}
l & m \\
n-2(l+1) & n-1
\end{array}\right) .
$$

We wish to show there are three distinct entries: $l \neq n-1, n-2(l+1) \neq n-1$, $l \neq n-2(l+1)$. Suppose $n-2(l+1)=n-1 \Rightarrow l=-\frac{1}{2}$, which is a contradiction. Suppose $l=n-2(l+1) \Rightarrow 3 l=n-2$ and since $n=4+6 k$ this is a contradiction. Three distinct elements are $\{l, n-1, n-2(l+1)\}$.
3. For $G_{3}$ with $l=\frac{n}{2}-1, m=n-1, i=1, j=n$ three distinct entries are $\{l, n-9, n-1)\}$.

Case 4. For the submatrix $G(i, j ; l, m)$ with $\frac{n}{2} \leq l \leq n-2, m=n-1, i=1$, $j=n$ three distinct elements are $\{l, n, n-1\}$.

Case 5. For the submatrix $G(i, j ; l, m)$ with $l=n-1, m=n, i=1, j=n$ three distinct elements are $\{n-1, n, 1\}$.
The argument for other steps is similar. To see the details please view the appendix to this article on ArXiv at http://arxiv.org/ or contact the first author.

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