

THE b -DOMATIC NUMBER OF A GRAPH

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Abstract

Besides the classical chromatic and achromatic numbers of a graph related to minimum or minimal vertex partitions into independent sets, the b -chromatic number was introduced in 1998 thanks to an alternative definition of the minimality of such partitions. When independent sets are replaced by dominating sets, the parameters corresponding to the chromatic and achromatic numbers are the domatic and adomatic numbers $d(G)$ and $ad(G)$. We introduce the b -domatic number $bd(G)$ as the counterpart of the b -chromatic number by giving an alternative definition of the maximality of a partition into dominating sets. We initiate the study of $bd(G)$ by giving some properties and examples.

Keywords: domatic number, adomatic number, b -domatic number, b -chromatic number, idomatic number, partition.

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1. INTRODUCTION

By analogy with the chromatic number related to vertex partitions of a graph G into independent sets, Cockayne and Hedetniemi studied in 1977 vertex partitions of G into dominating sets and defined the domatic number [6]. Cockayne [4] defined the adomatic number as the counterpart of the achromatic number [11]. In his PhD thesis, Manlove introduced the b -chromatic number by considering a new kind of minimality of a chromatic partition [13, 18]. The purpose of this paper is to extend the analogy and to define the b -domatic number by considering a new kind of maximality of a domatic partition. For this, we need to recall the definition of the chromatic, achromatic and b -chromatic numbers. From now, we write a -chromatic and a -domatic rather than achromatic and adomatic to unify the notation.

We consider finite simple graphs $G = (V, E)$ of order $|V| = n$ and minimum degree δ . A set S of vertices is *independent* if the induced subgraph $G[S]$ has no edge. The independence property is hereditary in the sense that every subset of an independent set is independent. A set S is *dominating* in G if every vertex of $V \setminus S$ has at least one neighbor in S . The domination property is cohereditary in the sense that every superset of a dominating set is dominating. The minimum cardinality of a dominating set is the *domination number* $\gamma(G)$. A dominating set is *divisible* if it contains two disjoint dominating sets of G , *indivisible* otherwise. We denote by $\omega(G)$ the maximum cardinality of a clique of G . A partition \mathcal{P} of G is a partition of its vertex set V . Its cardinality $|\mathcal{P}|$ is the number of classes of \mathcal{P} . A vertex v of G is *colorful* in \mathcal{P} if v has a neighbor in each class of \mathcal{P} different from its own class. We say that a class C of \mathcal{P} is *a-colorful* if there exists an edge of G between C and every other class of \mathcal{P} and *b-colorful* (called colorful in [9]) if it contains a colorful vertex.

A partition \mathcal{P} is *proper*, or *chromatic*, if all its classes are independent. The partition of G into n classes of one vertex each is proper and the most interesting chromatic partitions are those of small cardinality. The *chromatic number* $\chi(G)$ is the minimum cardinality of a proper vertex partition of G . However, proper partitions of large cardinality can also be interesting if they are minimal in some sense. In [10], a proper partition is considered to be minimal, and is called *complete*, if it is not possible to gather two classes C_i and C_j into one new independent class, in other words, if each class is a-colorful. We use the term *a-minimal* to designate this kind of minimality. The maximum cardinality of an a-minimal proper partition is often denoted by $\psi(G)$ and is called the *a-chromatic number* ([11]).

In [18] and [13], a new definition of the minimality of a proper partition was proposed. A proper partition $\{C_1, C_2, \dots, C_p\}$ is minimal if no class C , say C_1 , can be split into $p - 1$ classes C'_2, C'_3, \dots, C'_p (among them $p - 2$ may be empty) such that the smaller partition $\{C_2 \cup C'_2, C_3 \cup C'_3, \dots, C_p \cup C'_p\}$ is still proper, in other words, if each class is b-colorful. We use here the term *b-minimal* to designate this kind of minimality. The *b-chromatic number* of G , often denoted by $b(G)$, is the maximum cardinality of a b-minimal proper partition of G . So far, nearly 50 papers have been written on the b-chromatic number by different authors (see for instance [3, 8, 12, 15, 16]). We denote in this paper the a-chromatic and the b-chromatic numbers of G by $\chi_a(G)$ and $\chi_b(G)$ respectively. Clearly, every minimum chromatic partition is b-minimal and every b-minimal chromatic partition is a-minimal. Therefore every graph G satisfies

$$(1) \quad \chi(G) \leq \chi_b(G) \leq \chi_a(G).$$

The authors of [10] solved the interpolation problem for the a-chromatic number by proving that every graph G admits a-minimal chromatic partitions of any

cardinality between $\chi(G)$ and $\chi_a(G)$. However this is known to be false for the b-chromatic number. Some graphs G admit no b-minimal chromatic partition of cardinality p for some p between $\chi(G)$ and $\chi_b(G)$ [13].

2. DOMATIC PARTITIONS

A *domatic partition* of a graph G is a partition of its vertex set V into dominating sets. The partition $\{V\}$ of cardinality 1 is domatic and the most interesting domatic partitions are those of large cardinality. The *domatic number* $d(G)$ is the maximum cardinality of a domatic partition of G [6]. We note that a partition is domatic if and only if each vertex is dominated by each class different from its own class, i.e., if and only if each vertex is colorful. This implies, by considering a vertex of degree δ , that for every graph G , $d(G) \leq \delta + 1$ as observed in [6]. The graphs such that $d(G) = \delta + 1$ are called *domatically full*. Another immediate property of $d(G)$ is $d(G)\gamma(G) \leq n$ since every class of a domatic partition contains at least $\gamma(G)$ vertices.

As for chromatic partitions, a small domatic partition may be of interest if it is maximal in some sense. In [4], a domatic partition \mathcal{P} is considered to be maximal if no larger domatic partition \mathcal{P}' can be obtained by splitting some class C_i into two new dominating sets C'_i and C''_i , in other words, if each class is an indivisible dominating set of G . We use here the term *a-maximal* to designate this kind of maximality. The minimum cardinality of an a-maximal domatic partition of G is called the *adomatic number* and is denoted by $ad(G)$.

We now define a second kind of maximality as follows. A domatic partition \mathcal{P} is *b-maximal* if no larger domatic partition \mathcal{P}' can be obtained by gathering subsets of some classes of \mathcal{P} to form a new class. More formally, $\mathcal{P} = \{C_1, C_2, \dots, C_p\}$ is b-maximal if there do not exist p subsets C'_i of C_i (among them $p-1$ are possibly empty) such that the partition $\mathcal{P}' = \{C_1 \setminus C'_1, C_2 \setminus C'_2, \dots, C_p \setminus C'_p, C'_1 \cup C'_2 \cup \dots \cup C'_p\}$ is domatic. The construction from \mathcal{P} of such a larger domatic partition \mathcal{P}' is called a *b-operation*. The minimum cardinality of a b-maximal domatic partition of G is called the *b-domatic number* and is denoted by $bd(G)$. If $\delta = 0$, then $\{V\}$ is the unique domatic partition and $1 = ad(G) = bd(G) = d(G)$. In what follows, we consider graphs without isolated vertices. Clearly, every maximum domatic partition is b-maximal and every b-maximal domatic partition is a-maximal. Moreover, Ore [19] observed that the vertex set of every graph without isolated vertex contains two disjoint dominating sets, which implies that $ad(G) \geq 2$. Therefore every graph G with minimum degree $\delta \geq 1$ satisfies

$$(2) \quad 2 \leq ad(G) \leq bd(G) \leq d(G) \leq \delta + 1.$$

In spite of the analogy shown in (1) and (2) between the sequences $\{\chi, \chi_b, \chi_a\}$ and $\{d, bd, ad\}$, we immediately remark a big difference: while the difference between

two chromatic parameters may be arbitrarily large, it is bounded by $\delta - 1$ between two domatic ones.

A $d(G)$ -domatic partition of a graph G is a maximum domatic partition of G , i.e., a domatic partition of cardinality $d(G)$. Similarly, an $ad(G)$ -domatic partition ($bd(G)$ -domatic partition) is an a -maximal (b -maximal) domatic partition of G of cardinality $ad(G)$ ($bd(G)$). In [4], Cockayne posed the interpolation problem for the a -domatic number. This problem was positively solved by Ivancó [14] who proved, as a consequence of a more general result relative to cohereditary properties, that for every graph G there exist vertices partitioned into indivisible dominating sets of all orders between $ad(G)$ and $d(G)$. Previously, Zelinka [22] had given a negative answer based on complete bipartite graphs but there was an error in the proof (cf. Theorem 10). The same interpolation problem can be considered for the b -domatic number. By using the same notation as for the b -chromatic number, we say that a graph G is *b-domatically continuous* if it admits b -maximal domatic partitions of any cardinality between $bd(G)$ and $d(G)$. The *b-domatic spectrum* of a graph G is the set of the cardinalities of all the b -maximal domatic partitions of G . The graphs such that $bd(G) = d(G)$ or $d(G) - 1$ are obviously b -domatically continuous. We will give examples of graphs which are not b -domatically continuous.

We begin with a few easy properties of domatic partitions. Proposition 2 is another formulation of the definition of a b -maximal domatic partition and Propositions 3 to 5 are corollaries of Proposition 2.

Proposition 1. *Every domatic partition \mathcal{P} such that each class \mathcal{C} contains a vertex which is isolated in $G[\mathcal{C}]$ is a -maximal.*

Proof. A dominating set C containing an isolated vertex v in $G[C]$ is indivisible since if we split C into C' containing v and C'' , v is not dominated by C'' . Hence every class of \mathcal{P} is indivisible. ■

Proposition 2. *A domatic partition $\mathcal{P} = \{C_1, C_2, \dots, C_k\}$ is b -maximal if and only if for any choice of minimal dominating sets D_i of G respectively contained in C_i , $V \setminus \bigcup_{i=1}^k D_i$ does not dominate G .*

Proposition 3. *Every domatic partition $\mathcal{P} = \{C_1, C_2, \dots, C_k\}$ such that each class is a minimal dominating set of G is b -maximal.*

Proof. Since each class is a minimal dominating set of G , $D_i = C_i$ for each i and $V \setminus \bigcup_{i=1}^k D_i = \emptyset$ does not dominate G . ■

Proposition 4. *Every a -maximal domatic partition $\mathcal{P} = \{C_1, C_2, \dots, C_k\}$ such that every class C_i , $1 \leq i \leq k-1$, is a minimal dominating set of G is b -maximal.*

Proof. In this case, $D_i = C_i$ for $1 \leq i \leq k-1$ and $V \setminus \bigcup_{i=1}^k D_i = C_k \setminus D_k$ does not dominate G since C_k is indivisible. ■

Proposition 5. *If the graph G admits a b -maximal domatic partition of cardinality k , then there exists a family $\{D_1, D_2, \dots, D_k\}$ of k disjoint minimal dominating sets of G such that $V \setminus \bigcup_{i=1}^k D_i$ is not a dominating set of G .*

The following example shows that the converse of Proposition 5 is not true. Let G be obtained from the complete bipartite graph $K_{p,p}$, $p \geq 3$, with bipartition classes $A = \{a_1, a_2, \dots, a_p\}$, $B = \{b_1, b_2, \dots, b_p\}$ by adding the two edges $a_1 a_2$ and $b_1 b_2$ (cf. Theorem 11 with $q = 2$). The family $\{D_1, D_2\} = \{\{a_2, a_3, \dots, a_p\}, \{a_1, b_p\}\}$ of two disjoint minimal dominating sets is such that $V \setminus (D_1, D_2) = \{b_1, b_2, \dots, b_{p-1}\}$ does not dominate G . But no domatic partition of cardinality 2 is b -maximal. Indeed, let $\mathcal{P} = \{C_1, C_2\}$ with, without loss of generality, $|C_1| \geq |C_2|$ and $|C_1 \cap A| \geq |C_1 \cap B|$ be a domatic partition of $K_{p,p}$. If $|C_1 \cap B| \geq 2$ or $|C_1 \cap B| = 1$ and $A \subseteq C_1$, then, since each set $\{a_i, b_j\}$ is dominating, C_1 is divisible and \mathcal{P} is not b -maximal. If $|C_1 \cap B| \leq 1$ and $|C_1| = p$, a domatic partition of cardinality 3 of $K_{p,p}$ can be obtained by a b -operation on \mathcal{P} . In both cases, \mathcal{P} is not b -maximal.

Proposition 6 below is slightly weaker than the converse of Proposition 5.

Proposition 6. *If there exists a family $\{D_1, D_2, \dots, D_k\}$ of k disjoint minimal dominating sets of G and an index i_0 such that $V \setminus \bigcup_{i=1}^k D_i$ does not dominate G but $V \setminus \bigcup_{i=1, i \neq i_0}^k D_i$ is an indivisible dominating set of G , then G admits a b -maximal domatic partition of cardinality k .*

Proof. This is a consequence of Proposition 4 since $V \setminus \bigcup_{i=1, i \neq i_0}^k D_i$ forms with the minimal dominating sets D_i , $1 \leq i \neq i_0 \leq k$, an a -maximal partition. ■

In [21], Zelinka showed that for all integers a with $2 \leq a \leq n$ and $a \neq n-1$, there exists a connected graph G of order n such that $ad(G) = a$. We show a similar result for $bd(G)$.

Theorem 7. *Let b, n be integers such that $2 \leq b \leq n$. Then there exists a connected graph G of order n such that $bd(G) = b$.*

Proof. If $b = 2$, any connected graph of order n and minimum degree $\delta = 1$ suits since by (2), $2 \leq bd(G) \leq d(G) \leq \delta + 1 = 2$.

If $3 \leq b \leq n$, let G be a complete split graph of parts $X = \{x_1, x_2, \dots, x_{b-1}\}$ and Y with $G[X] = K_{b-1}$ and $G[Y] = \overline{K}_{n-b+1}$ (note that if $b = n$, then $G = K_n$). It is shown in [21] that $ad(G) = b-1$ and each class of every $ad(G)$ -domatic partition contains exactly one vertex of X and possibly at most $n-b$ vertices of Y . Therefore $bd(G) \geq b-1$ and the $ad(G)$ -domatic partitions are not b -maximal since they can be transformed into the larger domatic partition $\{\{x_1\}, \{x_2\}, \dots, \{x_{b-1}\}, Y\}$ by a b -operation. This latter partition is b -maximal by Proposition 3. Hence $bd(G) = b$. Note that this graph satisfies $d(G) = bd(G)$ since $bd(G) \leq d(G) \leq \delta + 1 = b$. ■

When the graph G has several components G_1, \dots, G_k , Chang [7] and Zelinka [21] respectively proved that $d(G) = \min\{d(G_i) \mid 1 \leq i \leq k\}$ and $ad(G) = 2$ if $\delta \geq 1$. The following theorem determines $bd(G)$ when G is disconnected.

Theorem 8. *Let G_1, \dots, G_k be the components of a disconnected graph G without isolated vertices. Then $bd(G) = \min\{bd(G_i) \mid 1 \leq i \leq k\}$.*

Proof. Let \mathcal{P} be a domatic partition of G . The restriction \mathcal{P}_i of \mathcal{P} to the component G_i is a domatic partition of G_i of the same cardinality as \mathcal{P} . Let \mathcal{P}' be another domatic partition of G and \mathcal{P}'_i its restriction to G_i . If for every i between 1 and k , \mathcal{P}'_i is obtained from \mathcal{P}_i by a b -operation in G_i , then \mathcal{P}' is obtained from \mathcal{P} by a b -operation in G . This means that if \mathcal{P}_i is not b -maximal in G_i for every i , then \mathcal{P} is not b -maximal. Therefore, if \mathcal{P} is b -maximal, in particular if \mathcal{P} is a $bd(G)$ -domatic partition, then there exists a component G_i such that \mathcal{P}_i is b -maximal in G_i . Hence $|\mathcal{P}_i| \geq bd(G_i)$ for some i . Since $|\mathcal{P}| = |\mathcal{P}_i|$ and $|\mathcal{P}| = bd(G)$, we get $bd(G) \geq \min\{bd(G_i) \mid 1 \leq i \leq k\}$.

In the other direction, suppose without loss of generality that $bd(G_1) = \min\{bd(G_i) \mid 1 \leq i \leq k\}$ and let \mathcal{P}_i be a $bd(G_i)$ -domatic partition for $1 \leq i \leq k$. By gathering $bd(G_i) - bd(G_1) + 1$ classes of \mathcal{P}_i in a unique new class for each $i \geq 2$, we can form a domatic partition \mathcal{P} of cardinality $bd(G_1)$ of G such that each class of \mathcal{P} contains exactly one class of \mathcal{P}_1 and at least one class of \mathcal{P}_i for $2 \leq i \leq k$. Since the restriction \mathcal{P}_1 of \mathcal{P} to G_1 is b -maximal, \mathcal{P} is b -maximal and $|\mathcal{P}| \geq bd(G)$. Since $|\mathcal{P}| = |\mathcal{P}_1|$, $bd(G) \leq \min\{bd(G_i) \mid 1 \leq i \leq k\}$. Therefore $bd(G) = \min\{bd(G_i) \mid 1 \leq i \leq k\}$. ■

For the b -chromatic number, it is known that under the same hypotheses, $\chi_b(G) \geq \max\{\chi_b(G_i) \mid 1 \leq i \leq k\}$ and the difference $\chi_b(G) - \max\{\chi_b(G_i)\}$ can be arbitrarily large [15], and that $\chi_b(G) \leq \sum_{i=1}^k \chi_b(G_i)$ [1].

3. EXAMPLES

In this section we give some examples of graphs G showing different possibilities for the values of $ad(G), bd(G), d(G)$ between 2 and $\delta + 1$ and for the b -domatic continuity.

By (2), every connected graph of minimum degree $\delta = 1$, and in particular every non-trivial tree, satisfies $ad(G) = bd(G) = d(G) = 2$. The clique K_n satisfies $ad(K_n) = bd(K_n) = d(K_n) = n = \delta + 1$. For the cycles, it is known [6] that $d(C_n) = 2$ if $n \not\equiv 0 \pmod{3}$ and $d(C_n) = 3$ if $n \equiv 0 \pmod{3}$.

Theorem 9. *The cycle C_n satisfies $ad(C_n) = bd(C_n) = 2$ for every $n \geq 4$.*

Proof. Let x_1, x_2, \dots, x_n be the vertices of C_n . The two classes $C_1 = \{x_1, x_3, \dots, x_{n-2}\}$ if n is odd, $C_1 = \{x_1, x_3, x_5, \dots, x_{n-1}\}$ if n is even, and $C_2 = V(C_n) \setminus C_1$ form a domatic partition \mathcal{P} . The class C_1 is a minimal dominating set and since $n \geq 4$, each class \mathcal{C} contains a vertex which is isolated in $G[\mathcal{C}]$. By Propositions 1 and 4, \mathcal{P} is b-maximal. Hence $ad(C_n) = bd(C_n) = 2$. ■

Theorem 7 gave an example of a graph G such that $ad(G) < bd(G) = d(G)$. The following two theorems give examples of graphs G such that $ad(G) = bd(G) < d(G)$ and $ad(G) < bd(G) < d(G)$ respectively.

Theorem 10. Let $K_{p,p'}$ be a complete bipartite graph with $2 \leq p \leq p'$. Then

$$ad(K_{p,p'}) = bd(K_{p,p'}) = 2 \text{ and } d(K_{p,p'}) = p.$$

Moreover, the b-domestic spectrum of $K_{p,p'}$ is $\{2, p\}$.

Proof. Let $V = A \cup B$ with $A = \{a_1, a_2, \dots, a_p\}$, $B = \{b_1, b_2, \dots, b_{p'}\}$ and A, B independent. Every dominating set of G either contains A or B , or intersects A and B . Hence the domatic partition $\mathcal{P} = \{\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_p, b_p, b_{p+1}, \dots, b_{p'}\}\}$ has maximum cardinality and $d(K_{p,p'}) = p$. The domatic partition $\mathcal{P}' = \{A, B\}$ is b-maximal by Proposition 3 since A and B are minimal dominating sets. Therefore $ad(K_{p,p'}) = bd(K_{p,p'}) = 2$.

When $p \geq 4$, we determine the b-domestic spectrum of G . For any integer q with $2 \leq q \leq p - 2$, the domatic partition $\mathcal{P}_q = \{\{a_1, b_1, b_2, \dots, b_q\}, \{a_2, a_3, \dots, a_{q+1}, b_{q+1}\}, \{a_{q+2}, b_{q+2}\}, \{a_{q+3}, b_{q+3}\}, \dots, \{a_p, b_p, b_{p+1}, \dots, b_{p'}\}\}$ of cardinality $p - q + 1$ is a-maximal since every class is indivisible. Hence $K_{p,p'}$ admits an a-maximal domatic partition of any cardinality between $ad(K_{p,p'}) = 2$ and $d(K_{p,p'}) = p$, which conforms to the adomatic interpolation property. However \mathcal{P}_q is not b-maximal since $\{\{a_1, b_1, \dots, b_{q-1}\}, \{a_2, b_q\}, \{a_3, \dots, a_{q+1}, b_{q+1}\}, \{a_{q+2}, b_{q+2}\}, \{a_{q+3}, b_{q+3}\}, \dots, \{a_p, b_p, b_{p+1}, \dots, b_{p'}\}\}$ is a domatic partition of cardinality $p - q + 2$ obtained from \mathcal{P} by a b-operation. To determine for which values of k between 2 and p there exist b-maximal domatic partitions of cardinality k , we apply Proposition 5. The minimal dominating sets of G are A, B and all the sets $\{a_i, b_j\}$ with $1 \leq i \leq p$ and $1 \leq j \leq p'$. For $3 \leq k \leq p$, the families of k disjoint minimal dominating sets of G are of the form $\{D_1, \dots, D_k\}$ with, without loss of generality, $D_i = \{a_i, b_i\}$. The set $V \setminus \bigcup_{i=1}^k D_i$ does not dominate G if and only if $k = p$. By Proposition 5, the b-maximal domatic partitions can only have cardinality 2 or p . Since \mathcal{P} and \mathcal{P}' are such partitions of respective cardinalities p and 2, the b-domestic spectrum of $K_{p,p'}$ is $\{2, p\}$. When $p \geq 4$, the complete bipartite graph is not b-domestically continuous. ■

Theorem 11. Let $G_{p,q}$ be obtained from a complete bipartite graph $K_{p,p}$, $p \geq 4$ with bipartition classes $A = \{a_1, \dots, a_p\}$ and $B = \{b_1, \dots, b_p\}$ by adding the edges

$a_i a_j$ and $b_i b_j$ for $1 \leq i < j \leq q \leq p-1$. Then $d(G_{p,q}) = p$, $ad(G_{p,q}) = 2$ and $bd(G_{p,q}) = q$ if $q \geq 3$, $bd(G_{p,q}) = q+1 = 3$ if $q = 2$. The b -domatic spectrum of $G_{p,q}$ is $\{q, q+1, p\}$ if $q \geq 3$, $\{3, p\}$ if $q = 2$.

Proof. Since $\gamma(G_{p,q}) = 2$, $d(G_{p,q}) \leq 2p/2 = p$. The partition $\mathcal{P} = \{\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_p, b_p\}\}$ of cardinality p is domatic and thus $d(G_{p,q}) = p$. By Proposition 1, the domatic partition $\mathcal{P}' = \{A, B\}$ is a -maximal. Hence $ad(G_{p,q}) = 2$. The minimal dominating sets of $G_{p,q}$ are $A_i = \{a_i, a_{q+1}, a_{q+2}, \dots, a_p\}$ for $1 \leq i \leq q$, $B_i = \{b_i, b_{q+1}, b_{q+2}, \dots, b_p\}$ for $1 \leq i \leq q$ and the sets $\{a_i, b_j\}$, $1 \leq i, j \leq p$. Without loss of generality, the families $\{D_1, D_2, \dots, D_k\}$ of disjoint minimal dominating sets of $G_{p,q}$ such that $V \setminus \bigcup_{i=1}^k D_i$ is not a dominating set of $G_{p,q}$ consist of p sets $\{a_i, b_j\}$, or of A_q, B_q and $q-1$ sets $\{a_r, b_s\}$ with $1 \leq r, s \leq q-1$, or of one of A_q, B_q , say A_q , and $q-1$ sets $\{a_r, b_s\}$ with $1 \leq r \leq q-1$, $1 \leq s \leq p$ and at least one s greater than q . Hence for the family $\{D_1, D_2, \dots, D_k\}$, $k = q, q+1$ or p . By Proposition 5, the b -maximal domatic partitions can only have cardinality $q, q+1$ or p . The domatic partitions \mathcal{P} , $\{A_q, B_q, \{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_{q-1}, b_{q-1}\}\}$ and, if $q \geq 3$, $\{A_q, \{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}, \dots, \{a_{q-2}, b_{q-2}, b_{q-1}, \dots, b_{p-1}\}, \{a_{q-1}, b_p\}\}$ are b -maximal and have respective cardinalities $p, q+1$ and q . Hence if $q \geq 3$, $bd(G_{p,q}) = q$ and the b -domatic spectrum is $\{q, q+1, p\}$. But if $q = 2$, no domatic partition into two classes is b -maximal as shown in the example following Proposition 5. Therefore $bd(G_{p,2}) = 3$ and its b -domatic spectrum is $\{3, p\}$. ■

The previous theorem shows that given two integers p, q with $3 \leq q \leq p-1$, there exist connected graphs G such that $d(G) = p$ and $bd(G) = q$.

4. THE PARTICULAR CASE OF IDOMATIC PARTITIONS

Partitions into independent dominating sets of G are both domatic and chromatic partitions. They were first considered as particular domatic partitions under the term of *indominable partitions* [5] or *idomatic partitions* [21] and their maximum cardinality was called the *idomatic number* of G and denoted by $id(G)$. Now the term idomatic is more usual (cf. for instance [20]). In 2000, they were reintroduced in [9] and afterwards studied by different authors as particular chromatic partitions under the term of *fall colorings*. Their minimum and maximum cardinalities were respectively denoted by $\chi_f(G)$ and $\psi_f(G)$. In the context of this paper, we keep the term idomatic which refers to both independence and domination with emphasis on domination, and we respectively denote their minimum and maximum cardinalities by $\min id(G)$ and $\max id(G)$. Graphs for which idomatic partitions exist are called *idomatic graphs*. As every vertex of an idomatic partition is colorful, since it is true for every domatic partition, it is clear that

such a partition is a b-minimal chromatic partition. As every independent dominating set of G is a minimal dominating set, an idomatic partition is also a b-maximal domatic partition by Proposition 3. Therefore each idomatic graph satisfies, as noticed in [9] for (3),

$$(3) \quad \chi(G) \leq \min id(G) \leq \max id(G) \leq \chi_b(G) \leq \chi_a(G)$$

and

$$(4) \quad ad(G) \leq bd(G) \leq \min id(G) \leq \max id(G) \leq d(G).$$

A graph is *chordal* if each of its cycles of length at least 4 has a chord. A chordal graph is *strongly chordal* if every cycle C of even length at least 6 has an odd chord, that is a chord xy such that the distance $d_C(x, y)$ on C is odd. As a consequence of (3) and (4), we can for instance observe that the idomatic chordal graphs satisfy $\chi(G) = \min id(G) = \max id(G) = d(G) = \delta + 1$ since $\chi(G) = \omega(G) \geq \delta + 1$ for any chordal graph and $d(G) \leq \delta + 1$ for any graph. For strongly chordal graphs, it is proved in [17] that they are idomatic if and only if $\chi(G) = \delta + 1$.

Since each class of an idomatic partition is a maximal independent set of G , we can also note the obvious inequalities

$$\frac{n}{\beta(G)} \leq \min id(G) \leq \max id(G) \leq \frac{n}{i(G)}$$

where $i(G)$ and $\beta(G)$ are the minimum and maximum cardinalities of a maximal independent set of G .

It is known (cf. [5]) that idomatic partitions do not necessarily interpolate. The authors of [9] showed that for any set $S \subseteq \mathbb{N} \setminus \{0, 1\}$, there exist idomatic graphs whose idomatic spectrum contains S (the authors of [2] improved this result by constructing a graph with idomatic spectrum exactly S). This shows that given $S \subseteq \mathbb{N} \setminus \{0, 1\}$, there exist graphs whose b-domestic spectrum contains S .

5. OPEN PROBLEMS

We defined the b-domestic number $bd(G)$ of a graph G by analogy with the b-chromatic number $\chi_b(G)$. It would be interesting to study for $bd(G)$ the same problems as those which are studied for $\chi_b(G)$, in particular the following ones.

- Find bounds on $bd(G)$ in terms of other parameters of G , valid for all graphs or in some classes of graphs.
- Characterize the b-domestic perfect graphs, the graphs for which $bd(H) = d(H)$ for every induced subgraph H of G . This is done in [12] for χ_b .

- Find some classes of b -domatically continuous graphs.
- Determine the complexity status of bd . The χ_b decision problem is NP-complete for arbitrary graphs [13, 18], even for bipartite graphs [16, 18], but polynomial for trees [13, 18].

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