

PRECISE UPPER BOUND FOR THE STRONG  
EDGE CHROMATIC NUMBER  
OF SPARSE PLANAR GRAPHS

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**Abstract**

We prove that every planar graph with maximum degree  $\Delta$  is strong edge  $(2\Delta - 1)$ -colorable if its girth is at least  $40\lfloor \frac{\Delta}{2} \rfloor + 1$ . The bound  $2\Delta - 1$  is reached at any graph that has two adjacent vertices of degree  $\Delta$ .

**Keywords:** planar graph, edge coloring, 2-distance coloring, strong edge-coloring.

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## 1. INTRODUCTION

By a graph we mean a non-oriented graph without loops and multiple edges. By  $V(G)$ ,  $E(G)$ ,  $\Delta(G)$ , and  $g(G)$  we denote the sets of vertices and edges, maximum degree, and girth (i.e., the smallest length of a cycle) of a graph  $G$ , respectively. (We will drop the argument when the graph is clear from context.)

By the celebrated Vizing's Edge Coloring Theorem [29], each simple graph (not necessarily planar) has  $\chi_e \leq \Delta + 1$ , where  $\chi_e$  is its edge chromatic number. Using strong properties of graphs critical w.r.t. edge coloring, Vizing [30] proved that each planar graph with  $\Delta \geq 8$  has  $\chi_e = \Delta$ . Sanders and Zhao [28] and, independently, Zhang [33] proved that  $\chi_e = 7$  if  $\Delta = 7$ .

A lot of research is devoted to the vertex 2-distance coloring of planar graphs.

**Definition.** A coloring  $\varphi : V(G) \rightarrow \{1, 2, \dots, k\}$  of  $G$  is *2-distance* if any two vertices at distance at most two from each other get different colors. The minimum number of colors in 2-distance colorings of  $G$  is its *2-distance chromatic number*, denoted by  $\chi_2(G)$ .

The problem of 2-distance coloring of vertices arises in applications; in particular, it is one of the main models in the mobile phoning. In graph theory there is an old (1977) conjecture of Wegner [31] that  $\chi_2 \leq \lfloor \frac{3}{2}\Delta \rfloor + 1$  for any planar graph with  $\Delta \geq 8$  (see also Jensen and Toft's monograph [24]).

The following upper bounds have been established:  $\lfloor \frac{9\Delta}{5} \rfloor + 2$  for  $\Delta \geq 749$  by Agnarsson and Halldorsson [1] and  $\lceil \frac{9\Delta}{5} \rceil + 1$  for  $\Delta \geq 47$  by Borodin, Broersma, Glebov, and van den Heuvel [3, 4]. Molloy and Salavatipour [25, 26] proved  $\lceil \frac{5\Delta}{3} \rceil + 78$  for all  $\Delta$  and  $\lceil \frac{5\Delta}{3} \rceil + 25$  for  $\Delta \geq 241$ . Havet et. al. [19] gave a proof sketch of  $\frac{3}{2}\Delta(1 + o(1))$ ; a full text can be found in [20].

In [5, 10] we give sufficient conditions (in terms of  $g$  and  $\Delta$ ) for the 2-distance chromatic number of a planar graph to equal the trivial lower bound  $\Delta + 1$ . In particular, we determine the least  $g$  such that  $\chi_2 = \Delta + 1$  if  $\Delta$  is large enough (depending on  $g$ ) to be equal to seven. Constructions of planar graphs with  $g = 6$  and  $\chi_2 = \Delta + 2$  are given in [5, 15].

Dvořák, Král, Nejedlý, and Škrekovski [15] proved that every planar graph with  $\Delta \geq 8821$  and  $g \geq 6$  has  $\chi_2 \leq \Delta + 2$ , and Borodin and Ivanova [6, 7] weakened the restriction on  $\Delta$  to 18.

Borodin, Ivanova, and Neustroeva [11, 12] proved that  $\chi_2 = \Delta + 1$  whenever  $\Delta \geq 31$  for planar graphs of girth six with the additional assumption that each edge is incident with a vertex of degree two.

Ivanova [22] improved the results in [5, 10] for  $\Delta \geq 5$  as follows.

**Theorem 1.** *If  $G$  is a planar graph, then  $\chi_2(G) = \Delta + 1$  in each of the cases:  $\Delta \geq 16$ ,  $g = 7$ ;  $\Delta \geq 10$ ,  $8 \leq g \leq 9$ ;  $\Delta \geq 6$ ,  $10 \leq g \leq 11$ ;  $\Delta = 5$ ,  $g \geq 12$ .*

A lot of attention is paid to coloring graphs with  $\Delta = 3$  (called *subcubic*). For such planar graphs Dvořák, Škrekovski, and Tancer [16] proved that  $\chi_2 = 4$  if  $g \geq 24$  (i.e., they independently obtained a result in [10]) and  $\chi_2 \leq 5$  if  $g \geq 14$ . The second of these results was also obtained by Montassier and Raspaud [27], which was improved by Ivanova and Solov'eva [23] and Havet [18] to  $g \geq 13$  and by Borodin and Ivanova [9] to  $g \geq 12$ . Borodin and Ivanova [8] proved  $\chi_2 = 4$  if  $g \geq 22$ , and Cranston and Kim [14] proved  $\chi_2 \leq 6$  for  $g \geq 9$ .

In 1985, Erdős and Nešetřil introduced the edge analogue of 2-distance coloring into consideration.

**Definition.** An edge coloring  $\varphi : E(G) \rightarrow \{1, 2, \dots, k\}$  of  $G$  is *strong* if any two edges get different colors if they are adjacent (i.e., have a common end vertex) or have a common adjacent edge. The minimum number of colors in strong edge-colorings of  $G$  is its *strong edge chromatic number*, denoted by  $\chi_2^e(G)$ .

They conjectured that  $\chi_2^e \leq \frac{5}{4}\Delta^2$  for  $\Delta$  even and  $\chi_2^e \leq \frac{1}{4}(5\Delta^2 - 2\Delta + 1)$  for  $\Delta$  odd; they gave a construction showing that this number is necessary. Andersen proved this conjecture for the case  $\Delta = 3$  [2]. For  $\Delta = 4$ , the conjectured bound is 20. Horák [21] proved  $\chi_2^e \leq 23$ , which bound was strengthened by Cranston [13] to 22. For other related results, we refer the reader to a brief survey by West [32] and a paper by Faudree *et al.* [17].

Not so much is known about the strong edge chromatic number of planar graphs. It is easy to see that for  $\Delta = 2$  there are graphs with  $\chi_2^e = 4$  and arbitrarily large girth. Indeed, to strong edge color the cycle  $C_{3k}$  it suffices three colors, while for  $C_{3k+1}$  and  $C_{3k+2}$  we need at least four colors, and, moreover,  $C_5$  has  $\chi_2^e(C_5) = 5$ .

Clearly, each graph with two adjacent  $\Delta$ -vertices has  $\chi_2^e \geq 2\Delta - 1$ . The purpose of our paper is to establish a precise upper bound, which is  $2\Delta - 1$ , for the strong edge chromatic number of sufficiently sparse planar graphs.

**Theorem 2.** *Each planar graph  $G$  with maximum degree  $\Delta \geq 3$  and  $g(G) \geq 40\lfloor \frac{\Delta}{2} \rfloor + 1$  has  $\chi_2^e(G) \leq 2\Delta - 1$ .*

**Problem 3.** Give precise upper bound for  $\chi_2^e(G)$  of a planar graph  $G$  in terms of  $g(G)$  and  $\Delta(G)$ .

**Problem 4.** Is every planar graph with large enough girth (depending on  $\Delta$ ) strong edge  $(2\Delta - 1)$ -choosable for each  $\Delta \geq 3$ ?

## 2. PROOF OF THEOREM 2

The main work in the proof is to show that a minimal counterexample cannot contain a long path of  $\Delta$ -vertices, each with  $\Delta - 2$  pendant edges. We first prove

this when  $\Delta = 3$ , and later handle the general case  $\Delta \geq 4$ . To complete the proof by contradiction, we use a short argument based on Euler's formula to show that every planar graph with girth at least  $40\lfloor \frac{\Delta}{2} \rfloor + 1$  must contain such a long path of  $\Delta$ -vertices.

Now we proceed to the formal proof. Among all counterexamples to Theorem 2, we choose a counterexample with the minimum number of  $2^+$ -vertices (i.e., those of degree at least two). To each  $2^+$ -vertex  $v$ , we add  $\Delta - d(v)$  pendant edges. The minimum counterexample  $G$  obtained has vertices only of degree 1 and  $\Delta$ . Without loss of generality, we can assume that  $G$  is connected.

**Lemma 5.**  *$G$  has no  $\Delta$ -vertex adjacent to  $\Delta - 1$  pendant vertices.*

**Proof.** Delete all pendant vertices at such a  $\Delta$ -vertex. Since the graph obtained has fewer  $\Delta$ -vertices, it can be colored, and its coloring can be extended to  $G$  because each uncolored edge has at most  $2\Delta - 2$  restrictions on the choice of color. ■

**Definition.** A  $t$ -caterpillar  $C[v_0, v_{t+1}]$  consists of a path  $v_0v_1 \cdots v_{t+1}$ , where each  $v_i$ ,  $1 \leq i \leq t$ , is incident with  $\Delta - 2$  pendant edges  $e_{i,j}$ ,  $1 \leq j \leq \Delta - 2$  (see Figure 1). The edges incident with  $v_0$  other than  $v_0v_1$  are denoted  $e_{0,j}$ ,  $1 \leq j \leq \Delta - 1$ .

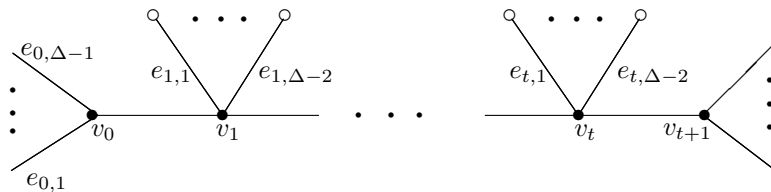


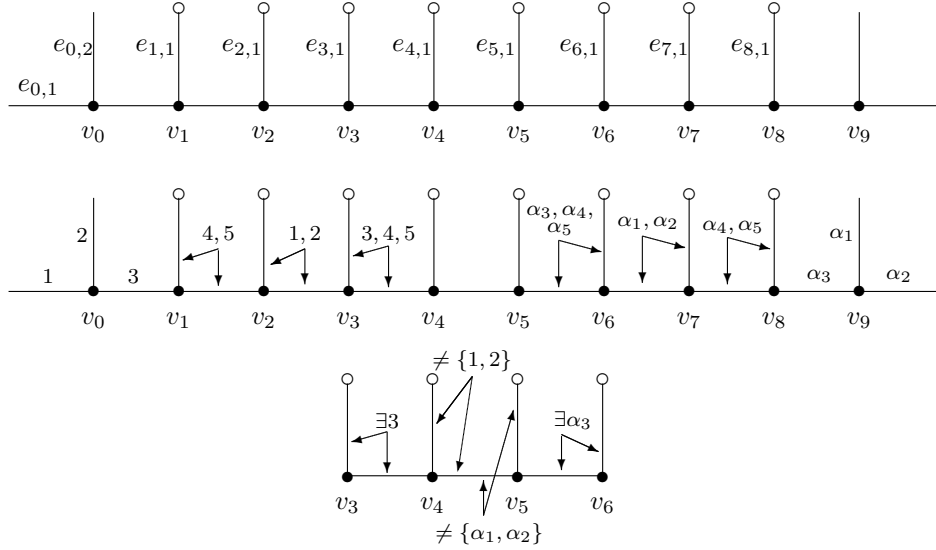
Figure 1. Caterpillar  $C[v_0, v_{t+1}]$ .

## 2.1. Subcubic graphs

**Proposition 6.** *If  $\Delta(G) = 3$ , then  $G$  has no 8-caterpillar.*

**Proof.** Suppose  $G$  contains  $C[v_0, v_9]$  (see Figure 2). We delete  $v_2, \dots, v_7$  and all pendant vertices adjacent to  $v_1, \dots, v_8$ . By the minimality of  $G$ , we have a coloring  $c$  of the graph obtained. Without loss of generality, we can assume that  $c(v_0v_1) = 3$  and  $\{c(e_{0,1}), c(e_{0,2})\} = \{1, 2\}$ . Also, let  $c(v_8v_9) = \alpha_3$ , and denote the colors of the other two edges at  $v_9$  by  $\alpha_1$  and  $\alpha_2$ .

Note that the five edges at any two adjacent  $\Delta$ -vertices should be colored pairwise differently. Hence, in any extension of  $c$  we should have  $\{c(e_{1,1}), c(v_1v_2)\} = \{4, 5\}$ ,  $\{c(e_{2,1}), c(v_2v_3)\} = \{1, 2\}$ , and  $\{c(e_{3,1}), c(v_3v_4)\} \subseteq \{3, 4, 5\}$ . Similar conditions should hold at vertices  $v_8, v_7, v_6$ , and  $v_5$  (see Figure 2).


 Figure 2. Reducing  $C[v_0, v_9]$  for  $\Delta = 3$ .

It is not hard to check that the problem of extending  $c$  to the uncolored edges is reduced to coloring edges  $e_{3,1}, v_3v_4, e_{4,1}, v_4v_5, e_{5,1}, v_5v_6, e_{6,1}$  so that the following conditions (V3)–(V6) are satisfied:

- (V3)  $3 \in \{c(e_{3,1}), c(v_3v_4)\}$ ;
- (V4)  $\{c(e_{4,1}), c(v_4v_5)\} \neq \{1, 2\}$ ;
- (V5)  $\{c(v_4v_5), c(e_{5,1})\} \neq \{\alpha_1, \alpha_2\}$ ;
- (V6)  $\alpha_3 \in \{c(v_5v_6), c(e_{6,1})\}$ .

Indeed, to check (V3) it suffices to note that if  $\{c(e_{3,1}), c(v_3v_4)\} = \{4, 5\}$ , then we have no color for  $v_1v_2$ . Similarly, if  $\{c(e_{4,1}), c(v_4v_5)\} = \{1, 2\}$ , then it is impossible to color  $v_2v_3$ , which proves (V4) (see Figure 2). The same is true for (V5) and (V6).

Note also that if  $e_{3,1}, v_3v_4, e_{4,1}, v_4v_5, e_{5,1}, v_5v_6, e_{6,1}$  are colored according to (V3)–(V6), then we can color the uncolored edges in this order:  $v_2v_3, e_{2,1}, v_1v_2, e_{1,1}, v_6v_7, e_{7,1}, v_7v_8$ , and  $e_{8,1}$ .

Put  $\{\alpha_4, \alpha_5\} = \{1, \dots, 5\} \setminus \{\alpha_1, \alpha_2, \alpha_3\}$ . Coloring the seven “central” uncolored edges is split into three cases: by the symmetry between colors 1 and 2 on the one hand, and between 4 and 5 on the other hand, we can assume that  $\alpha_3 \in \{1, 3, 5\}$ . Note that all the conditions (V3)–(V6) are satisfied in the proofs obtained for each case below.

*Case 1.*  $\alpha_3 = 1$ . Put  $c(e_{6,1}) = c(e_{4,1}) = 1, c(e_{5,1}) = 2$ . If  $3 \in \{\alpha_4, \alpha_5\}$  then we put  $c(e_{3,1}) = c(v_5v_6) = 3$ . Now put  $c(v_4v_5) = 4$  if  $\{\alpha_1, \alpha_2\} = \{2, 5\}$ ; otherwise, we put  $c(v_4v_5) = 5$ . Finally, we put  $c(v_3v_4) \in \{4, 5\} - c(v_4v_5)$ . If  $3 \notin \{\alpha_4, \alpha_5\}$  then we can put  $c(e_{3,1}) = c(v_5v_6) \geq 4, c(v_3v_4) = 3$ , and  $c(v_4v_5) \in \{4, 5\} - c(v_5v_6)$ .

*Case 2.*  $\alpha_3 = 3$ . Put  $c(e_{3,1}) = c(v_5v_6) = 3$  and  $c(e_{5,1}) = 2$ . If  $\{\alpha_4, \alpha_5\} = \{1, 2\}$  then it suffices to put  $c(e_{4,1}) = c(e_{6,1}) = 1$ . Now without loss of generality, we can assume that  $4 \in \{\alpha_4, \alpha_5\}$ ; then we put  $c(v_3v_4) = c(e_{6,1}) = 4$ . Finally, we put  $c(v_4v_5) = 5$  if  $5 \notin \{\alpha_1, \alpha_2\}$  and  $c(v_4v_5) = 1$  otherwise.

*Case 3.*  $\alpha_3 = 5$ . We put  $c(e_{3,1}) = c(v_5v_6) = 5$  and  $c(v_3v_4) = 3$ . If  $3 \in \{\alpha_4, \alpha_5\}$ , then we put  $c(e_{6,1}) = 3$  and further put  $c(v_4v_5) = 4$  if  $4 \notin \{\alpha_1, \alpha_2\}$  and  $c(e_{4,1}) = 4$  otherwise. If  $3 \notin \{\alpha_4, \alpha_5\}$ , i.e.  $3 \in \{\alpha_1, \alpha_2\}$  and  $\{1, 2\} \cap \{\alpha_4, \alpha_5\} \neq \emptyset$ , then we can assume by symmetry that  $1 \in \{\alpha_4, \alpha_5\}$ , and it suffices to put  $c(e_{6,1}) = c(e_{4,1}) = 1$ ,  $c(e_{5,1}) = 2$ , and  $c(v_4v_5) = 4$ . ■

We can rewrite Proposition 6 as follows:

**Lemma 7.** *For  $\Delta = 3$ , suppose that  $c(v_0v_1) = 3$  and  $\{c(e_{0,1}), c(e_{0,2})\} = \{1, 2\}$ ; then for every three colors  $\alpha$ ,  $\beta$ , and  $\gamma$  we can color the caterpillar  $C[v_0, v_8]$  so that  $c(v_8v_9) = \gamma$  and  $\{c(e_{8,1}), c(v_7v_8)\} = \{\alpha, \beta\}$ .*

Informally speaking, we can 5-color the caterpillar  $C[v_0, v_8]$  for arbitrary color assigned to edge  $v_8v_9$  and any two other colors assigned to the pair of edges  $\{e_{8,1}, v_7v_8\}$ . However, we do not claim that we can choose the color of  $e_{8,1}$  as we wish.

## 2.2. Case $\Delta \geq 4$

**Proposition 8.** *If  $\Delta(G) \geq 4$ , then  $G$  has no  $8\lfloor \frac{\Delta}{2} \rfloor$ -caterpillar.*

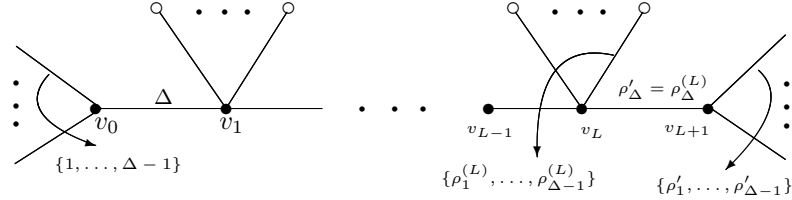
**Proof.** Suppose  $G$  contains  $C[v_0, v_{L+1}]$ , where  $L = 8\lfloor \frac{\Delta}{2} \rfloor$ . We delete  $v_2, \dots, v_{L-1}$  and all pendant vertices adjacent to  $v_1, \dots, v_L$ . By the minimality of  $G$ , we have a coloring  $c$  of the graph obtained.

Without loss of generality, we can assume that  $c(v_0v_1) = \Delta$  and the other  $\Delta - 1$  edges at  $v_0$  are colored with  $1, 2, \dots, \Delta - 1$ . Also, suppose that  $c(vLv_{L+1}) = \rho'_\Delta$  and the other  $\Delta - 1$  edges at  $v_{L+1}$  are colored with  $\rho'_1, \rho'_2, \dots, \rho'_{\Delta-1}$  (see Figure 3).

Let  $\{\rho_1^{(L)}, \dots, \rho_{\Delta-1}^{(L)}\} = \{1, \dots, 2\Delta - 1\} \setminus \{\rho'_1, \dots, \rho'_\Delta\}$ . Then we have  $\{c(v_{L-1}v_L), c(e_{L,1}), \dots, c(e_{L,\Delta-2})\} = \{\rho_1^{(L)}, \dots, \rho_{\Delta-1}^{(L)}\}$ , and we put  $\rho_\Delta^{(L)} = c(vLv_{L+1})$ .

This obvious equivalence makes it possible to split coloring our  $C[v_0, v_{L+1}]$  into manageable pieces of length eight by suitably precoloring neighborhoods of vertices  $v_{8k}$ , where  $1 \leq k \leq L - 1$ , as described in lemmas below.

**Definition.** Colors  $1, \dots, \Delta - 1$  are *minor*, while colors  $\Delta + 1, \dots, 2\Delta - 1$  are *major*.


 Figure 3. Shift of the precoloring from  $v_{L+1}$  to  $v_L$  in Proposition 8.

**Lemma 9.** *Let  $\Delta \geq 4$ . Suppose we have a partial coloring  $c$  of  $C[v_0, v_8]$  such that  $c(v_0v_1) = \Delta$  and  $\{c(e_{0,1}), \dots, c(e_{0,\Delta-1})\} = \{1, \dots, \Delta-1\}$ ; then for any color set  $R = \{\rho_1, \dots, \rho_\Delta\}$  such that at most two of  $\rho_i$ 's are major there is an extension of  $c$  to  $C[v_0, v_8]$  such that  $c(v_8v_9) = \rho_\Delta$  and  $\{c(v_7v_8), c(e_{8,1}), \dots, c(e_{8,\Delta-2})\} = \{\rho_1, \dots, \rho_{\Delta-1}\}$ , except for the case when  $\rho_\Delta$  is minor,  $\Delta \in R$ , and  $R$  contains precisely two major colors (see Figure 4).*

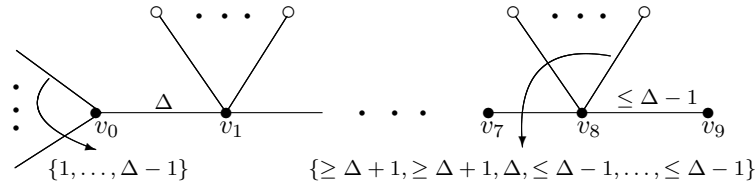


Figure 4. The exception in Lemma 9.

**Proof.** Since  $R$  contains at most two major colors, it follows that it contains at least  $\Delta - 3$  minor colors. Moreover,  $R$  contains  $\Delta - 3$  minor colors different from  $\rho_\Delta$ . Indeed, this is obvious if  $R$  contains at least  $\Delta - 2$  minor colors. So suppose  $R$  contains precisely  $\Delta - 3$  minor colors. It follows from the assumption of Lemma 9 that the other three elements of  $R$  are  $\Delta$  and two major colors. Due to the exception described in the statement of Lemma 9, we have  $\rho_\Delta \geq \Delta$ , as desired.

Without loss of generality, we can assume that  $R$  contains  $\Delta - 3$  minor colors  $R_m = \{\rho_1, \dots, \rho_{\Delta-3}\}$ . We put  $\{c(e_{i,1}), \dots, c(e_{i,\Delta-3})\} = R_m$  for all  $i \in \{2, 4, 6, 8\}$  (see Figure 5). Since there are  $\Delta - 1$  major colors, it follows that there is a set  $R_s$  of  $\Delta - 3$  major colors such that  $R_s \cap R = \emptyset$ . For all  $i \in \{1, 3, 5, 7\}$ , we put  $\{c(e_{i,1}), \dots, c(e_{i,\Delta-3})\} = R_s$ .

Let  $m_1$  and  $m_2$  be the two minor colors avoiding  $R_m$ , and let  $s_1$  and  $s_2$  be the two major colors avoiding  $R_s$ . Note that  $\{\rho_{\Delta-2}, \rho_{\Delta-1}, \rho_\Delta\} \subset \{m_1, m_2, \Delta, s_1, s_2\}$  by our construction. So we are in the situation of Lemma 7 (which deals with the case  $\Delta = 3$ ) with respect to the yet uncolored edges of  $C[v_0, v_8]$ , where  $\{m_1, m_2, \Delta, s_1, s_2\}$  plays the role of  $\{1, 2, 3, 4, 5\}$ . Thus  $c$  can be extended to  $C[v_0, v_8]$  as desired.  $\square$

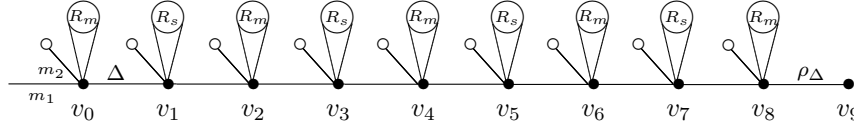


Figure 5. Proof in Lemma 9.

By Lemma 9, we can color any caterpillar  $C[v_{8k}, v_{8k+8}]$  if its end vertices are precolored the same:

**Corollary 10.** *Let  $\Delta \geq 4$ , and let  $k \geq 1$  be an integer. Suppose  $c$  is a partial coloring of  $C[v_{8k}, v_{8k+8}]$  such that  $c(v_{8k}v_{8k+1}) = c(v_{8k+8}v_{8k+9}) = \rho_\Delta$  and  $\{c(v_{8k-1}v_{8k}), c(e_{8k,1}), \dots, c(e_{8k,\Delta-2})\} = \{c(v_{8k+7}v_{8k+8}), c(e_{8k+8,1}), \dots, c(e_{8k+8,\Delta-2})\} = \{\rho_1, \dots, \rho_{\Delta-1}\}$ ; then  $c$  can be extended to  $C[v_{8k}, v_{8k+8}]$ .*

Clearly, the statement of Corollary 10 is equivalent to the special case of Lemma 9 when  $R = \{1, \dots, \Delta\}$ .

Our next lemma easily resolves the exceptional case arising in Lemma 9.

**Lemma 11.** *Let  $\Delta \geq 4$ . Suppose we have a partial coloring  $c$  of  $C[v_0, v_{16}]$  such that  $c(v_0v_1) = \Delta$  and  $\{c(e_{0,1}), \dots, c(e_{0,\Delta-1})\} = \{1, \dots, \Delta-1\}$ ; then for any color set  $R = \{\rho_1, \dots, \rho_\Delta\}$  such that precisely two of  $\rho_i$ 's are major and  $\Delta \in R$  there is an extension of  $c$  to  $C[v_0, v_{16}]$  in which  $c(v_{16}v_{17})$  is some minor color from  $R$  and  $\{c(v_{15}v_{16}), c(e_{16,1}), \dots, c(e_{16,\Delta-2})\} = R - c(v_{16}v_{17})$ .*

**Proof.** Given  $R$ , we define a coloring of edges at the intermediate vertex  $v_8$  as follows:  $c(v_8v_9) = \Delta$  and  $\{c(v_7v_8), c(e_{8,1}), \dots, c(e_{8,\Delta-2})\} = R - \Delta$ . It follows that this coloring can be extended to  $C[v_0, v_8]$  by Lemma 9 and to  $C[v_8, v_{16}]$  by Corollary 10.  $\square$

**Lemma 12.** *Let  $\Delta \geq 4$ , let  $k \geq 1$  be an integer, and we have a color set  $R = \{\rho_1, \dots, \rho_\Delta\}$  such that at least three of  $\rho_i$ 's are major. Suppose  $c$  is a partial coloring of  $C[v_{8k}, v_{8k+8}]$  in which  $c(v_{8k+8}v_{8k+9}) = \rho_\Delta$  and  $\{c(v_{8k+7}v_{8k+8}), c(e_{8k+8,1}), \dots, c(e_{8k+8,\Delta-2})\} = \{\rho_1, \dots, \rho_{\Delta-1}\}$ ; then there exists a color set  $\Lambda = \{\lambda_1, \dots, \lambda_\Delta\}$  such that*

- (1)  $\Lambda$  contains two fewer major colors than  $R$  and contains two more minor colors than  $R$ , and
- (2) *there is an extension of  $c$  to  $C[v_{8k}, v_{8k+8}]$  such that  $c(v_{8k}v_{8k+1}) = \lambda_\Delta = \rho_\Delta$  and  $\{\lambda_1, \dots, \lambda_{\Delta-1}\} = \{c(v_{8k+7}v_{8k+8}), c(e_{8k,1}), \dots, c(e_{8k,\Delta-2})\}$ .*

**Proof.** We first put  $\lambda_\Delta = \rho_\Delta$ . It follows from the assumption on  $R$  that  $R - \rho_\Delta$  contains at least two major colors, say  $\rho_1$  and  $\rho_2$ . Since there are  $\Delta - 1$  minor colors, it follows that  $R - \rho_\Delta$  avoids at least two minor colors, say  $m_1$  and  $m_2$ . We put  $\Lambda = \{m_1, m_2, \rho_3, \dots, \rho_\Delta\}$ .



Now Lemma 9 can be applied to  $C[v_{8k}, v_{8k+8}]$  (it does not matter that  $k$  is not necessarily zero). Indeed, colors  $\{m_1, m_2, \rho_3, \dots, \rho_{\Delta-1}\}$  can be regarded as “minor”, and  $\rho_{\Delta}$  is an analog of color  $\Delta$ . We see that  $R$  contains precisely two “major” colors from the viewpoint of Lemma 9 (i.e., they do not appear in  $\Lambda$ ), namely,  $\rho_1$  and  $\rho_2$ , which are also major in the ordinary sense. Furthermore,  $R$  is not an exception for  $\Lambda$ , since  $\lambda_{\Delta} = \rho_{\Delta}$ .  $\square$

### Finishing the proof of Proposition 8.

We are now able to construct color sets  $R^{(8k)} = \{\rho_1^{(8k)}, \dots, \rho_{\Delta}^{(8k)}\}$ , to be precolorings at vertices  $v_{8k}$  of our  $C[v_0, v_L]$ , for  $k$  from  $\lfloor \frac{\Delta}{2} \rfloor - 1$  to 1 by induction. (Recall that still  $L = 8\lfloor \frac{\Delta}{2} \rfloor$ .)

INDUCTION BASE. We are given a set  $R^{(L)} = \{\rho_1^{(L)}, \dots, \rho_{\Delta}^{(L)}\}$  defined at the beginning of Section 2.2, and we must color the edges incident with  $v_L$  as follows:  $c(v_L v_{L+1}) = \rho_{\Delta}^{(L)}$  and  $\{c(v_{L-1} v_L), c(e_{L,1}), \dots, c(e_{L,\Delta-2})\} = \{\rho_1^{(L)}, \dots, \rho_{\Delta-1}^{(L)}\}$ .

INDUCTION STEP ( $k+1 \rightarrow k$ ). We are given a set  $R^{(8k+8)}$ , and we must color the edges incident with  $v_{8k+8}$  as follows:  $c(v_{8k+8} v_{8k+9}) = \rho_{\Delta}^{(8k+8)}$  and  $\{c(v_{8k+7} v_{8k+8}), c(e_{8k+8,1}), \dots, c(e_{8k+8,\Delta-2})\} = \{\rho_1^{(8k+8)}, \dots, \rho_{\Delta-1}^{(8k+8)}\}$ .

We now construct a set  $R^{(8k)}$  to color the edges incident with  $v_{8k}$  so that  $c(v_{8k} v_{8k+1}) = \rho_{\Delta}^{(8k)}$ ,  $\{c(v_{8k-1} v_{8k}), c(e_{8k,1}), \dots, c(e_{8k,\Delta-2})\} = \{\rho_1^{(8k)}, \dots, \rho_{\Delta-1}^{(8k)}\}$ , and  $C[v_{8k}, v_{8k+8}]$  can be colored.

*Case 1.*  $R^{(8k+8)}$  contains at least three major colors. We apply Lemma 12. The resulting set  $R^{(8k)}$  has two fewer major colors than  $R^{(8k+8)}$ .

*Case 2.*  $R^{(8k+8)}$  contains at most two major colors. Here, we put  $R^{(8k+8)} = R^{(8k)}$ .

So, we have constructed sets  $R^{(8k)}$  for all  $k \geq 1$ . By construction, we can color the caterpillar  $C[v_8, v_L]$  in portions of length eight as described in Lemmas 9, 11, 12, and Corollary 10. We are done if we can color caterpillar  $C[v_0, v_8]$ , so suppose we cannot.

Note that then our  $R^{(8)}$  contains at most two major colors, for if it contains at least three of them, then  $R^{(16)}$  contains at least five (as mentioned in Case 1 above), and so on, and finally,  $R^{(L)}$  contains at least  $2\lfloor \frac{\Delta}{2} \rfloor + 1 \geq \Delta$  major colors, which is impossible.

So,  $R^{(8)}$  is an exceptional set described in Lemma 9, which means that  $\rho_{\Delta}^{(8)}$  is minor,  $R^{(8)}$  contains  $\Delta$  and precisely two major colors. Let us prove that  $R^{(16)} = R^{(8)}$ , i.e.,  $R^{(8)}$  was obtained from  $R^{(16)}$  as in Case 2 above. Indeed, otherwise the argument in the previous paragraph shows that  $R^{(L)}$  contains at least  $2\lfloor \frac{\Delta}{2} \rfloor \geq \Delta - 1$  major colors, which is possible only if  $\Delta$  is odd. However, we should also have  $\rho_{\Delta}^{(L)} = \dots = \rho_{\Delta}^{(8)} \leq \Delta - 1$  (by (2) in Lemma 12) and  $\Delta \in R^{(8k)}$  whenever  $1 \leq k \leq \lfloor \frac{\Delta}{2} \rfloor$  (by (1) in Lemma 12). This implies that  $R^{(L)}$  contains

$\Delta$ , one minor color ( $\rho_{\Delta}^{(L)} = \rho_{\Delta}^{(8)}$ ), and all  $\Delta - 1$  major colors, which is impossible since  $|R^{(L)}| = \Delta$ .

Now we delete this invalid set  $R^{(8)}$  and get in the situation of Lemma 11. Thus, we can color the caterpillar  $C[v_0, v_{16}]$  in addition to the already colored  $C[v_{16}, v_L]$ . This completes the proof of Proposition 8. ■

### Completing the proof of Theorem 2.

So,  $G$  may have only  $\leq (8\lfloor \frac{\Delta}{2} \rfloor - 1)$ -caterpillars. We delete all pendant vertices to obtain graph  $G'$ . By Lemma 5,  $G'$  has no pendant vertices. Now contract all  $k$ -threads, when  $k \geq 1$ , of  $G'$  (i.e., paths consisting of  $k$  vertices of degree 2) to edges.

Euler's formula  $|V| - |E| + |F| = 2$  for the pseudograph  $G^*$  obtained can be rewritten as  $(4|E| - 6|V|) + (2|E| - 6|F|) = -12$ , where  $F$  is a set of faces of  $G^*$ . Hence,

$$\sum_{v \in V} (2d(v) - 6) + \sum_{f \in F} (r(f) - 6) < 0,$$

where  $d(v)$  is the degree of vertex  $v$ , and  $r(f)$  is the size of face  $f$ . Since the minimum degree of  $G^*$  is at least 3, it follows that there is a face  $f$  of size at most 5 in  $G^*$ . Restore all 2-vertices of contracted  $k$ -threads; then each edge of  $f$  becomes a path of at most  $8\lfloor \frac{\Delta}{2} \rfloor$  edges, which implies that  $r(f) \leq 40\lfloor \frac{\Delta}{2} \rfloor$  in  $G$ , contrary to the assumption on  $g(G)$ . Theorem 2 is proved.

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