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PRECISE UPPER BOUND FOR THE STRONG EDGE CHROMATIC NUMBER OF SPARSE PLANAR GRAPHS

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Abstract

We prove that every planar graph with maximum degree Δ is strong edge $(2\Delta - 1)$-colorable if its girth is at least $40\lfloor \frac{\Delta}{2} \rfloor + 1$. The bound $2\Delta - 1$ is reached at any graph that has two adjacent vertices of degree Δ.

Keywords: planar graph, edge coloring, 2-distance coloring, strong edge-coloring.

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1. Introduction

By a graph we mean a non-oriented graph without loops and multiple edges. By $V(G)$, $E(G)$, $\Delta(G)$, and $g(G)$ we denote the sets of vertices and edges, maximum degree, and girth (i.e., the smallest length of a cycle) of a graph G, respectively. (We will drop the argument when the graph is clear from context.)

By the celebrated Vizing's Edge Coloring Theorem [29], each simple graph (not necessarily planar) has $\chi_e \le \Delta + 1$, where χ_e is its edge chromatic number. Using strong properties of graphs critical w.r.t. edge coloring, Vizing [30] proved that each planar graph with $\Delta \ge 8$ has $\chi_e = \Delta$. Sanders and Zhao [28] and, independently, Zhang [33] proved that $\chi_e = 7$ if $\Delta = 7$.

A lot of research is devoted to the vertex 2-distance coloring of planar graphs.

Definition. A coloring $\varphi : V(G) \to \{1, 2, \ldots, k\}$ of G is 2-*distance* if any two vertices at distance at most two from each other get different colors. The minimum number of colors in 2-distance colorings of G is its 2-*distance chromatic number*, denoted by $\chi_2(G)$.

The problem of 2-distance coloring of vertices arises in applications; in particular, it is one of the main models in the mobile phoning. In graph theory there is an old (1977) conjecture of Wegner [31] that $\chi_2 \le \lfloor \frac{3}{2}\Delta \rfloor + 1$ for any planar graph with $\Delta \ge 8$ (see also Jensen and Toft's monograph [24]).

The following upper bounds have been established: $\lfloor \frac{9\Delta}{5} \rfloor + 2$ for $\Delta \ge 749$ by Agnarsson and Halldorsson [1] and $\lceil \frac{9\Delta}{5} \rceil + 1$ for $\Delta \ge 47$ by Borodin, Broersma, Glebov, and van den Heuvel [3, 4]. Molloy and Salavatipour [25, 26] proved $\lceil \frac{5\Delta}{3} \rceil + 78$ for all Δ and $\lceil \frac{5\Delta}{3} \rceil + 25$ for $\Delta \ge 241$. Havet et. al. [19] gave a proof sketch of $\frac{3}{2}\Delta(1 + o(1))$; a full text can be found in [20].

In [5, 10] we give sufficient conditions (in terms of g and Δ) for the 2-distance chromatic number of a planar graph to equal the trivial lower bound $\Delta + 1$. In particular, we determine the least g such that $\chi_2 = \Delta + 1$ if Δ is large enough (depending on g) to be equal to seven. Constructions of planar graphs with $g = 6$ and $\chi_2 = \Delta + 2$ are given in [5, 15].

Dvořák, Kràl, Nejedlỳ, and Škrekovski [15] proved that every planar graph with $\Delta \ge 8821$ and $g \ge 6$ has $\chi_2 \le \Delta + 2$, and Borodin and Ivanova [6, 7] weakened the restriction on Δ to 18.

Borodin, Ivanova, and Neustroeva [11, 12] proved that $\chi_2 = \Delta + 1$ whenever $\Delta \ge 31$ for planar graphs of girth six with the additional assumption that each edge is incident with a vertex of degree two.

Ivanova [22] improved the results in [5, 10] for $\Delta \ge 5$ as follows.

Theorem 1. *If G is a planar graph, then $\chi_2(G) = \Delta + 1$ in each of the cases: $\Delta \ge 16$, $g = 7$; $\Delta \ge 10$, $8 \le g \le 9$; $\Delta \ge 6$, $10 \le g \le 11$; $\Delta = 5$, $g \ge 12$.*

A lot of attention is paid to coloring graphs with $\Delta = 3$ (called *subcubic*). For such planar graphs Dvořák, Škrekovski, and Tancer [16] proved that $\chi_2 = 4$ if $g \geq 24$ (i.e., they independently obtained a result in [10]) and $\chi_2 \leq 5$ if $g \geq 14$. The second of these results was also obtained by Montassier and Raspaud [27], which was improved by Ivanova and Solov'eva [23] and Havet [18] to $g \geq 13$ and by Borodin and Ivanova [9] to $g \geq 12$. Borodin and Ivanova [8] proved $\chi_2 = 4$ if $g \geq 22$, and Cranston and Kim [14] proved $\chi_2 \leq 6$ for $g \geq 9$.

In 1985, Erdős and Nešetřil introduced the edge analogue of 2-distance coloring into consideration.

Definition. An edge coloring $\varphi : E(G) \to \{1, 2, \ldots, k\}$ of G is *strong* if any two edges get different colors if they are adjacent (i.e., have a common end vertex) or have a common adjacent edge. The minimum number of colors in strong edge-colorings of G is its *strong edge chromatic number*, denoted by $\chi_2^e(G)$.

They conjectured that $\chi_2^e \leq \frac{5}{4}\Delta^2$ for Δ even and $\chi_2^e \leq \frac{1}{4}(5\Delta^2 - 2\Delta + 1)$ for Δ odd; they gave a construction showing that this number is necessary. Andersen proved this conjecture for the case $\Delta = 3$ [2]. For $\Delta = 4$, the conjectured bound is 20. Horák [21] proved $\chi_2^e \leq 23$, which bound was strengthened by Cranston [13] to 22. For other related results, we refer the reader to a brief survey by West [32] and a paper by Faudree *et al.* [17].

Not so much is known about the strong edge chromatic number of planar graphs. It is easy to see that for $\Delta = 2$ there are graphs with $\chi_2^e = 4$ and arbitrarily large girth. Indeed, to strong edge color the cycle C_{3k} it suffices three colors, while for C_{3k+1} and C_{3k+2} we need at least four colors, and, moreover, C_5 has $\chi_2^e(C_5) = 5$.

Clearly, each graph with two adjacent Δ-vertices has $\chi_2^e \geq 2\Delta - 1$. The purpose of our paper is to establish a precise upper bound, which is $2\Delta - 1$, for the strong edge chromatic number of sufficiently sparse planar graphs.

Theorem 2. *Each planar graph G with maximum degree $\Delta \geq 3$ and $g(G) \geq 40\lfloor \frac{\Delta}{2} \rfloor + 1$ has $\chi_2^e(G) \leq 2\Delta - 1$.*

Problem 3. Give precise upper bound for $\chi_2^e(G)$ of a planar graph G in terms of $g(G)$ and $\Delta(G)$.

Problem 4. Is every planar graph with large enough girth (depending on Δ) strong edge $(2\Delta - 1)$-choosable for each $\Delta \geq 3$?

2. Proof of Theorem 2

The main work in the proof is to show that a minimal counterexample cannot contain a long path of Δ-vertices, each with $\Delta - 2$ pendant edges. We first prove this when $\Delta = 3$, and later handle the general case $\Delta \geq 4$. To complete the proof by contradiction, we use a short argument based on Euler's formula to show that every planar graph with girth at least $40\lfloor \frac{\Delta}{2} \rfloor + 1$ must contain such a long path of Δ-vertices.

Now we proceed to the formal proof. Among all counterexamples to Theorem 2, we choose a counterexample with the minimum number of 2^+-vertices (i.e., those of degree at least two). To each 2^+-vertex v, we add $\Delta - d(v)$ pendant edges. The minimum counterexample G obtained has vertices only of degree 1 and Δ. Without loss of generality, we can assume that G is connected.

Lemma 5. *G has no Δ-vertex adjacent to $\Delta - 1$ pendant vertices.*

Proof. Delete all pendant vertices at such a Δ-vertex. Since the graph obtained has fewer Δ-vertices, it can be colored, and its coloring can be extended to G because each uncolored edge has at most $2\Delta - 2$ restrictions on the choice of color. ■

Definition. A *t-caterpillar* $C[v_0, v_{t+1}]$ consists of a path $v_0 v_1 \cdots v_{t+1}$, where each v_i, $1 \leq i \leq t$, is incident with $\Delta - 2$ pendant edges $e_{i,j}$, $1 \leq j \leq \Delta - 2$ (see Figure 1). The edges incident with v_0 other than $v_0 v_1$ are denoted $e_{0,j}$, $1 \leq j \leq \Delta - 1$.

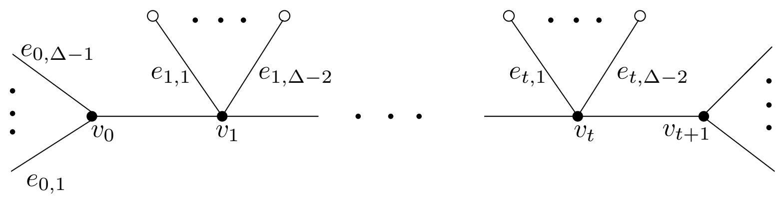

Figure 1. Caterpillar $C[v_0, v_{t+1}]$.

2.1. Subcubic graphs

Proposition 6. *If $\Delta(G) = 3$, then G has no 8-caterpillar.*

Proof. Suppose G contains $C[v_0, v_9]$ (see Figure 2). We delete $v_2, \ldots, v_7$ and all pendant vertices adjacent to $v_1, \ldots, v_8$. By the minimality of G, we have a coloring c of the graph obtained. Without loss of generality, we can assume that $c(v_0v_1) = 3$ and $\{c(e_{0,1}), c(e_{0,2})\} = \{1, 2\}$. Also, let $c(v_8v_9) = \alpha_3$, and denote the colors of the other two edges at v_9 by α_1 and α_2.

Note that the five edges at any two adjacent Δ-vertices should be colored pairwise differently. Hence, in any extension of c we should have $\{c(e_{1,1}), c(v_1v_2)\} = \{4, 5\}$, $\{c(e_{2,1}), c(v_2v_3)\} = \{1, 2\}$, and $\{c(e_{3,1}), c(v_3v_4)\} \subseteq \{3, 4, 5\}$. Similar conditions should hold at vertices v_8, v_7, v_6, and v_5 (see Figure 2).

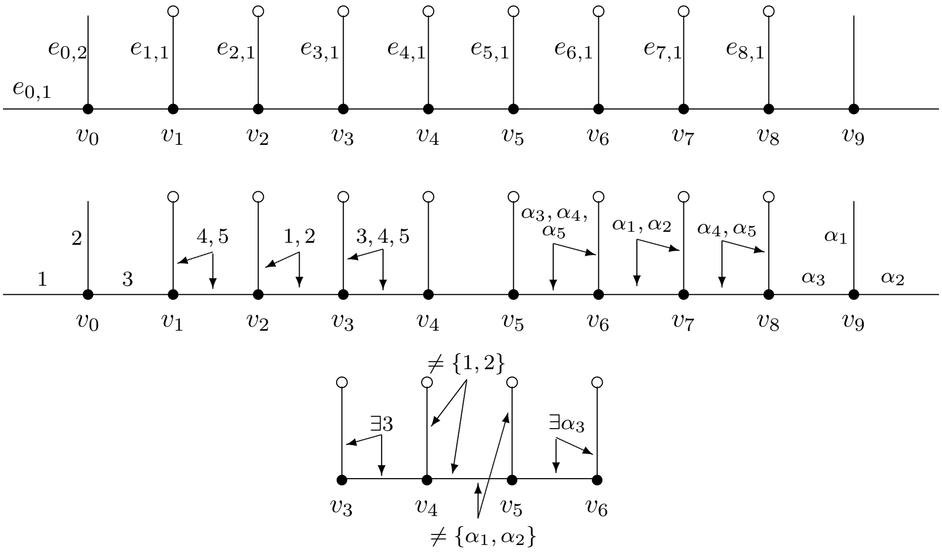

Figure 2. Reducing $C[v_0, v_9]$ for $\Delta = 3$.

It is not hard to check that the problem of extending c to the uncolored edges is reduced to coloring edges $e_{3,1}, v_3v_4, e_{4,1}, v_4v_5, e_{5,1}, v_5v_6, e_{6,1}$ so that the following conditions (V3)–(V6) are satisfied:

(V3) $3 \in \{c(e_{3,1}), c(v_3v_4)\}$;

(V4) $\{c(e_{4,1}), c(v_4v_5)\} \neq \{1, 2\}$;

(V5) $\{c(v_4v_5), c(e_{5,1})\} \neq \{\alpha_1, \alpha_2\}$;

(V6) $\alpha_3 \in \{c(v_5v_6), c(e_{6,1})\}$.

Indeed, to check (V3) it suffices to note that if $\{c(e_{3,1}), c(v_3v_4)\} = \{4, 5\}$, then we have no color for v_1v_2. Similarly, if $\{c(e_{4,1}), c(v_4v_5)\} = \{1, 2\}$, then it is impossible to color v_2v_3, which proves (V4) (see Figure 2). The same is true for (V5) and (V6).

Note also that if $e_{3,1}, v_3v_4, e_{4,1}, v_4v_5, e_{5,1}, v_5v_6, e_{6,1}$ are colored according to (V3)–(V6), then we can color the uncolored edges in this order: v_2v_3, $e_{2,1}$, v_1v_2, $e_{1,1}$, v_6v_7, $e_{7,1}$, v_7v_8, and $e_{8,1}$.

Put $\{\alpha_4, \alpha_5\} = \{1, \ldots, 5\} \setminus \{\alpha_1, \alpha_2, \alpha_3\}$. Coloring the seven "central" uncolored edges is split into three cases: by the symmetry between colors 1 and 2 on the one hand, and between 4 and 5 on the other hand, we can assume that $\alpha_3 \in \{1, 3, 5\}$. Note that all the conditions (V3)–(V6) are satisfied in the proofs obtained for each case below.

Case 1. $\alpha_3 = 1$. Put $c(e_{6,1}) = c(e_{4,1}) = 1$, $c(e_{5,1}) = 2$. If $3 \in \{\alpha_4, \alpha_5\}$ then we put $c(e_{3,1}) = c(v_5v_6) = 3$. Now put $c(v_4v_5) = 4$ if $\{\alpha_1, \alpha_2\} = \{2, 5\}$; otherwise, we put $c(v_4v_5) = 5$. Finally, we put $c(v_3v_4) \in \{4, 5\} - c(v_4v_5)$. If $3 \notin \{\alpha_4, \alpha_5\}$ then we can put $c(e_{3,1}) = c(v_5v_6) \geq 4$, $c(v_3v_4) = 3$, and $c(v_4v_5) \in \{4, 5\} - c(v_5v_6)$.

Case 2. $\alpha_3 = 3$. Put $c(e_{3,1}) = c(v_5v_6) = 3$ and $c(e_{5,1}) = 2$. If $\{\alpha_4, \alpha_5\} = \{1,2\}$ then it suffices to put $c(e_{4,1}) = c(e_{6,1}) = 1$. Now without loss of generality, we can assume that $4 \in \{\alpha_4, \alpha_5\}$; then we put $c(v_3v_4) = c(e_{61}) = 4$. Finally, we put $c(v_4v_5) = 5$ if $5 \notin \{\alpha_1, \alpha_2\}$ and $c(v_4v_5) = 1$ otherwise.

Case 3. $\alpha_3 = 5$. We put $c(e_{3,1}) = c(v_5v_6) = 5$ and $c(v_3v_4) = 3$. If $3 \in \{\alpha_4, \alpha_5\}$, then we put $c(e_{6,1}) = 3$ and further put $c(v_4v_5) = 4$ if $4 \notin \{\alpha_1, \alpha_2\}$ and $c(e_{4,1}) = 4$ otherwise. If $3 \notin \{\alpha_4, \alpha_5\}$, i.e. $3 \in \{\alpha_1, \alpha_2\}$ and $\{1,2\} \cap \{\alpha_4, \alpha_5\} \neq \varnothing$, then we can assume by symmetry that $1 \in \{\alpha_4, \alpha_5\}$, and it suffices to put $c(e_{6,1}) = c(e_{4,1}) = 1$, $c(e_{5,1}) = 2$, and $c(v_4v_5) = 4$. ∎

We can rewrite Proposition 6 as follows:

Lemma 7. *For $\Delta = 3$, suppose that $c(v_0v_1) = 3$ and $\{c(e_{0,1}), c(e_{0,2})\} = \{1,2\}$; then for every three colors α, β, and γ we can color the caterpillar $C[v_0, v_8]$ so that $c(v_8v_9) = \gamma$ and $\{c(e_{8,1}), c(v_7v_8)\} = \{\alpha, \beta\}$.*

Informally speaking, we can 5-color the caterpillar $C[v_0, v_8]$ for arbitrary color assigned to edge v_8v_9 and any two other colors assigned to the pair of edges $\{e_{8,1}, v_7v_8\}$. However, we do not claim that we can choose the color of $e_{8,1}$ as we wish.

2.2. Case $\Delta \geq 4$

Proposition 8. *If $\Delta(G) \geq 4$, then G has no $8\lfloor\frac{\Delta}{2}\rfloor$-caterpillar.*

Proof. Suppose G contains $C[v_0, v_{L+1}]$, where $L = 8\lfloor\frac{\Delta}{2}\rfloor$. We delete $v_2, \ldots, v_{L-1}$ and all pendant vertices adjacent to $v_1, \ldots, v_L$. By the minimality of G, we have a coloring c of the graph obtained.

Without loss of generality, we can assume that $c(v_0v_1) = \Delta$ and the other $\Delta - 1$ edges at v_0 are colored with $1, 2, \ldots, \Delta - 1$. Also, suppose that $c(v_Lv_{L+1}) = \rho'_\Delta$ and the other $\Delta - 1$ edges at v_{L+1} are colored with $\rho'_1, \rho'_2, \ldots, \rho'_{\Delta-1}$ (see Figure 3).

Let $\{\rho_1^{(L)}, \ldots, \rho_{\Delta-1}^{(L)}\} = \{1, \ldots, 2\Delta - 1\} \setminus \{\rho'_1, \ldots, \rho'_\Delta\}$. Then we have

$\{c(v_{L-1}v_L), c(e_{L,1}), \ldots, c(e_{L,\Delta-2})\} = \{\rho_1^{(L)}, \ldots, \rho_{\Delta-1}^{(L)}\}$, and we put $\rho_\Delta^{(L)} = c(v_Lv_{L+1})$.

This obvious equivalence makes it possible to split coloring our $C[v_0, v_{L+1}]$ into manageable pieces of length eight by suitably precoloring neighborhoods of vertices v_{8k}, where $1 \leq k \leq L - 1$, as described in lemmas below.

Definition. Colors $1, \ldots, \Delta - 1$ are *minor*, while colors $\Delta + 1, \ldots, 2\Delta - 1$ are *major*.

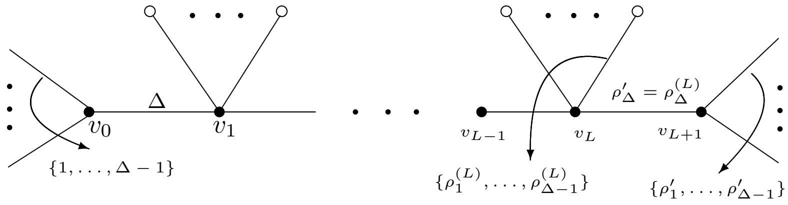

Figure 3. Shift of the precoloring from v_{L+1} to v_L in Proposition 8.

Lemma 9. *Let $\Delta \geq 4$. Suppose we have a partial coloring c of $C[v_0, v_8]$ such that $c(v_0v_1) = \Delta$ and $\{c(e_{0,1}), \ldots, c(e_{0,\Delta-1})\} = \{1, \ldots, \Delta-1\}$; then for any color set $R = \{\rho_1, \ldots, \rho_\Delta\}$ such that at most two of ρ_i's are major there is an extension of c to $C[v_0, v_8]$ such that $c(v_8v_9) = \rho_\Delta$ and $\{c(v_7v_8), c(e_{8,1}), \ldots, c(e_{8,\Delta-2})\} = \{\rho_1, \ldots, \rho_{\Delta-1}\}$, except for the case when ρ_Δ is minor, $\Delta \in R$, and R contains precisely two major colors* (*see Figure* 4).

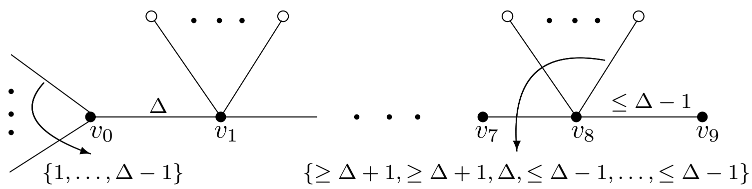

Figure 4. The exception in Lemma 9.

Proof. Since R contains at most two major colors, it follows that it contains at least $\Delta - 3$ minor colors. Moreover, R contains $\Delta - 3$ minor colors different from ρ_Δ. Indeed, this is obvious if R contains at least $\Delta - 2$ minor colors. So suppose R contains precisely $\Delta - 3$ minor colors. It follows from the assumption of Lemma 9 that the other three elements of R are Δ and two major colors. Due to the exception described in the statement of Lemma 9, we have $\rho_\Delta \geq \Delta$, as desired.

Without loss of generality, we can assume that R contains $\Delta-3$ minor colors $R_m = \{\rho_1, \ldots, \rho_{\Delta-3}\}$. We put $\{c(e_{i,1}), \ldots, c(e_{i,\Delta-3})\} = R_m$ for all $i \in \{2, 4, 6, 8\}$ (see Figure 5). Since there are $\Delta - 1$ major colors, it follows that there is a set R_s of $\Delta - 3$ major colors such that $R_s \cap R = \varnothing$. For all $i \in \{1, 3, 5, 7\}$, we put $\{c(e_{i,1}), \ldots, c(e_{i,\Delta-3})\} = R_s$.

Let m_1 and m_2 be the two minor colors avoiding R_m, and let s_1 and s_2 be the two major colors avoiding R_s. Note that $\{\rho_{\Delta-2}, \rho_{\Delta-1}, \rho_\Delta\} \subset \{m_1, m_2, \Delta, s_1, s_2\}$ by our construction. So we are in the situation of Lemma 7 (which deals with the case $\Delta = 3$) with respect to the yet uncolored edges of $C[v_0, v_8]$, where $\{m_1, m_2, \Delta, s_1, s_2\}$ plays the role of $\{1, 2, 3, 4, 5\}$. Thus c can be extended to $C[v_0, v_8]$ as desired. $\square$

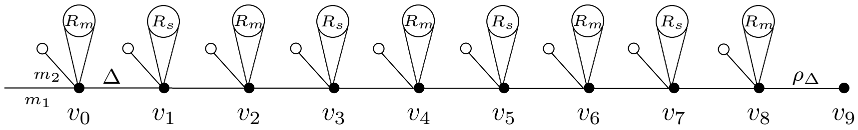

Figure 5. Proof in Lemma 9.

By Lemma 9, we can color any caterpillar $C[v_{8k}, v_{8k+8}]$ if its end vertices are precolored the same:

Corollary 10. *Let $\Delta \geq 4$, and let $k \geq 1$ be an integer. Suppose c is a partial coloring of $C[v_{8k}, v_{8k+8}]$ such that $c(v_{8k}v_{8k+1}) = c(v_{8k+8}v_{8k+9}) = \rho_\Delta$ and $\{c(v_{8k-1}v_{8k}), c(e_{8k,1}), \dots, c(e_{8k,\Delta-2})\} = \{c(v_{8k+7}v_{8k+8}), c(e_{8k+8,1}), \dots, c(e_{8k+8,\Delta-2}) = \{\rho_1, \dots, \rho_{\Delta-1}\}$; then c can be extended to $C[v_{8k}, v_{8k+8}]$.*

Clearly, the statement of Corollary 10 is equivalent to the special case of Lemma 9 when $R = \{1, \dots, \Delta\}$.

Our next lemma easily resolves the exceptional case arising in Lemma 9.

Lemma 11. *Let $\Delta \geq 4$. Suppose we have a partial coloring c of $C[v_0, v_{16}]$ such that $c(v_0v_1) = \Delta$ and $\{c(e_{0,1}), \dots, c(e_{0,\Delta-1})\} = \{1, \dots, \Delta - 1\}$; then for any color set $R = \{\rho_1, \dots, \rho_\Delta\}$ such that precisely two of ρ_i's are major and $\Delta \in R$ there is an extension of c to $C[v_0, v_{16}]$ in which $c(v_{16}v_{17})$ is some minor color from R and $\{c(v_{15}v_{16}), c(e_{16,1}), \dots, c(e_{16,\Delta-2})\} = R - c(v_{16}v_{17})$.*

Proof. Given R, we define a coloring of edges at the intermediate vertex v_8 as follows: $c(v_8v_9) = \Delta$ and $\{c(v_7v_8), c(e_{8,1}), \dots, c(e_{8,\Delta-2})\} = R - \Delta$. It follows that this coloring can be extended to $C[v_0, v_8]$ by Lemma 9 and to $C[v_8, v_{16}]$ by Corollary 10. □

Lemma 12. *Let $\Delta \geq 4$, let $k \geq 1$ be an integer, and we have a color set $R = \{\rho_1, \dots, \rho_\Delta\}$ such that at least three of ρ_i's are major. Suppose c is a partial coloring of $C[v_{8k}, v_{8k+8}]$ in which $c(v_{8k+8}v_{8k+9}) = \rho_\Delta$ and $\{c(v_{8k+7}v_{8k+8}), c(e_{8k+8,1}), \dots, c(e_{8k+8,\Delta-2})\} = \{\rho_1, \dots, \rho_{\Delta-1}\}$; then there exists a color set $\Lambda = \{\lambda_1, \dots, \lambda_\Delta\}$ such that*

1. *Λ contains two fewer major colors than R and contains two more minor colors than R, and*
2. *there is an extension of c to $C[v_{8k}, v_{8k+8}]$ such that $c(v_{8k}v_{8k+1}) = \lambda_\Delta = \rho_\Delta$ and $\{\lambda_1, \dots, \lambda_{\Delta-1}\} = \{c(v_{8k+7}v_{8k+8}), c(e_{8k,1}), \dots, c(e_{8k,\Delta-2})\}$.*

Proof. We first put $\lambda_\Delta = \rho_\Delta$. It follows from the assumption on R that $R - \rho_\Delta$ contains at least two major colors, say ρ_1 and ρ_2. Since there are $\Delta - 1$ minor colors, it follows that $R - \rho_\Delta$ avoids at least two minor colors, say m_1 and m_2. We put $\Lambda = \{m_1, m_2, \rho_3, \dots, \rho_\Delta\}$.

Now Lemma 9 can be applied to $C[v_{8k}, v_{8k+8}]$ (it does not matter that k is not necessarily zero). Indeed, colors $\{m_1, m_2, \rho_3, \ldots, \rho_{\Delta-1}\}$ can be regarded as "minor", and ρ_Δ is an analog of color Δ. We see that R contains precisely two "major" colors from the viewpoint of Lemma 9 (i.e., they do not appear in Λ), namely, ρ_1 and ρ_2, which are also major in the ordinary sense. Furthermore, R is not an exception for Λ, since $\lambda_\Delta = \rho_\Delta$. $\square$

Finishing the proof of Proposition 8.
We are now able to construct color sets $R^{(8k)} = \{\rho_1^{(8k)}, \ldots, \rho_\Delta^{(8k)}\}$, to be precolorings at vertices v_{8k} of our $C[v_0, v_L]$, for k from $\lfloor \frac{\Delta}{2} \rfloor - 1$ to 1 by induction. (Recall that still $L = 8\lfloor \frac{\Delta}{2} \rfloor$.)

Induction base. We are given a set $R^{(L)} = \{\rho_1^{(L)}, \ldots, \rho_\Delta^{(L)}\}$ defined at the beginning of Section 2.2, and we must color the edges incident with v_L as follows: $c(v_L v_{L+1}) = \rho_\Delta^{(L)}$ and $\{c(v_{L-1}v_L), c(e_{L,1}), \ldots, c(e_{L,\Delta-2})\} = \{\rho_1^{(L)}, \ldots, \rho_{\Delta-1}^{(L)}\}$.

Induction step ($k+1 \to k$). We are given a set $R^{(8k+8)}$, and we must color the edges incident with v_{8k+8} as follows: $c(v_{8k+8}v_{8k+9}) = \rho_\Delta^{(8k+8)}$ and $\{c(v_{8k+7}v_{8k+8}), c(e_{8k+8,1}), \ldots, c(e_{8k+8,\Delta-2})\} = \{\rho_1^{(8k+8)}, \ldots, \rho_{\Delta-1}^{(8k+8)}\}$.

We now construct a set $R^{(8k)}$ to color the edges incident with v_{8k} so that $c(v_{8k}v_{8k+1}) = \rho_\Delta^{(8k)}$, $\{c(v_{8k-1}v_{8k}), c(e_{8k,1}), \ldots, c(e_{8k,\Delta-2})\} = \{\rho_1^{(8k)}, \ldots, \rho_{\Delta-1}^{(8k)}\}$, and $C[v_{8k}, v_{8k+8}]$ can be colored.

Case 1. $R^{(8k+8)}$ contains at least three major colors. We apply Lemma 12. The resulting set $R^{(8k)}$ has two fewer major colors than $R^{(8k+8)}$.

Case 2. $R^{(8k+8)}$ contains at most two major colors. Here, we put $R^{(8k+8)} = R^{(8k)}$.

So, we have constructed sets $R^{(8k)}$ for all $k \geq 1$. By construction, we can color the caterpillar $C[v_8, v_L]$ in portions of length eight as described in Lemmas 9, 11, 12, and Corollary 10. We are done if we can color caterpillar $C[v_0, v_8]$, so suppose we cannot.

Note that then our $R^{(8)}$ contains at most two major colors, for if it contains at least three of them, then $R^{(16)}$ contains at least five (as mentioned in Case 1 above), and so on, and finally, $R^{(L)}$ contains at least $2\lfloor \frac{\Delta}{2} \rfloor + 1 \geq \Delta$ major colors, which is impossible.

So, $R^{(8)}$ is an exceptional set described in Lemma 9, which means that $\rho_\Delta^{(8)}$ is minor, $R^{(8)}$ contains Δ and precisely two major colors. Let us prove that $R^{(16)} = R^{(8)}$, i.e., $R^{(8)}$ was obtained from $R^{(16)}$ as in Case 2 above. Indeed, otherwise the argument in the previous paragraph shows that $R^{(L)}$ contains at least $2\lfloor \frac{\Delta}{2} \rfloor \geq \Delta - 1$ major colors, which is possible only if Δ is odd. However, we should also have $\rho_\Delta^{(L)} = \cdots = \rho_\Delta^{(8)} \leq \Delta - 1$ (by (2) in Lemma 12) and $\Delta \in R^{(8k)}$ whenever $1 \leq k \leq \lfloor \frac{\Delta}{2} \rfloor$ (by (1) in Lemma 12). This implies that $R^{(L)}$ contains

Δ, one minor color ($\rho_\Delta^{(L)} = \rho_\Delta^{(8)}$), and all $\Delta - 1$ major colors, which is impossible since $|R^{(L)}| = \Delta$.

Now we delete this invalid set $R^{(8)}$ and get in the situation of Lemma 11. Thus, we can color the caterpillar $C[v_0, v_{16}]$ in addition to the already colored $C[v_{16}, v_L]$. This completes the proof of Proposition 8. ■

Completing the proof of Theorem 2.

So, G may have only $\leq (8\lfloor\frac{\Delta}{2}\rfloor - 1)$-caterpillars. We delete all pendant vertices to obtain graph G'. By Lemma 5, G' has no pendant vertices. Now contract all k-threads, when $k \geq 1$, of G' (i.e., paths consisting of k vertices of degree 2) to edges.

Euler's formula $|V| - |E| + |F| = 2$ for the pseudograph G^* obtained can be rewritten as $(4|E| - 6|V|) + (2|E| - 6|F|) = -12$, where F is a set of faces of G^*. Hence,

$$\sum_{v \in V}(2d(v) - 6) + \sum_{f \in F}(r(f) - 6) < 0,$$

where $d(v)$ is the degree of vertex v, and $r(f)$ is the size of face f. Since the minimum degree of G^* is at least 3, it follows that there is a face f of size at most 5 in G^*. Restore all 2-vertices of contracted k-threads; then each edge of f becomes a path of at most $8\lfloor\frac{\Delta}{2}\rfloor$ edges, which implies that $r(f) \leq 40\lfloor\frac{\Delta}{2}\rfloor$ in G, contrary to the assumption on $g(G)$. Theorem 2 is proved.

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