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ON THE CROSSING NUMBERS OF CARTESIAN PRODUCTS OF STARS AND GRAPHS OF ORDER SIX

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Abstract

The crossing number $\mathrm{cr}(G)$ of a graph G is the minimal number of crossings over all drawings of G in the plane. According to their special structure, the class of Cartesian products of two graphs is one of few graph classes for which some exact values of crossing numbers were obtained. The crossing numbers of Cartesian products of paths, cycles or stars with all graphs of order at most four are known. Moreover, except of six graphs, the crossing numbers of Cartesian products $G \square K_{1,n}$ for all other connected graphs G on five vertices are known. In this paper we are dealing with the Cartesian products of stars with graphs on six vertices. We give the exact values of crossing numbers for some of these graphs and we summarise all known results concerning crossing numbers of these graphs. Moreover, we give the crossing number of $G_1 \square T$ for the special graph G_1 on six vertices and for any tree T with no vertex of degree two as well as the crossing number of $K_{1,n} \square T$ for any tree T with maximum degree five.

Keywords: graph, drawing, crossing number, Cartesian product, join product, star.

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1. Introduction

Let G be a graph, whose vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. A drawing of G is a representation of G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. For simplicity, we assume that in a drawing (a) no edge passes through any vertex other than its end-points, (b) no two edges touch each other (i.e., if two edges have a common interior point, then at this point they properly cross each other), and (c) no three edges cross at the same point. The *crossing number* $\mathrm{cr}(G)$ is the smallest number of edge crossings in any drawing of G. It is easy to see that a drawing with minimum number of crossings (an *optimal* drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross.

The investigation on the crossing number of graphs is a classical and very difficult problem. According to their special structure, the class of Cartesian products of two graphs is one of few graph classes for which some exact values of crossing numbers were obtained. The *Cartesian product* $G \Box H$ of the graphs G and H has vertex set $V(G \Box H) = V(G) \times V(H)$ and any two vertices (u, u') and (v, v') are adjacent in $G \Box H$ if and only if either $u = v$ and u' is adjacent with v' in H, or $u' = v'$ and u is adjacent with v in G. Let C_n be the cycle of length n, P_n be the path of n vertices, and S_n be the star isomorphic to $K_{1,n}$. Harary et al. [7] conjectured that the crossing number of the Cartesian product $C_m \Box C_n$ of two cycles is $(m-2)n$, for all m, n satisfying $3 \leq m \leq n$. It was proved by Glebsky and Salazar [6] that for any fixed m, the conjecture holds for all $n \geq m(m+1)$. The conjecture has also been verified for $m \leq 7$. Beineke and Ringeisen in [2] started to study the crossing numbers of Cartesian products of cycles with all graphs of order at most four. In [11], [13], and [14], the crossing numbers of Cartesian products of cycles, paths and stars with all graphs of order four are given. In the paper, we are dealing with crossing numbers of Cartesian products of stars and small graphs. Some results concerning the crossing numbers of $G \Box S_n$ for graphs G on five vertices appear in [15] and [16]. The crossing numbers of Cartesian products of stars with graphs of order five are collected in [18]. The aim of the paper is to establish the crossing numbers of Cartesian products of stars with several graphs of order six. We will use some results concerning the crossing numbers of bipartite and multipartite complete graphs as well as the crossing numbers of join products of special graphs.

The join product of two graphs G and H, denoted by $G + H$, is obtained from vertex-disjoint copies of G and H by adding all edges between $V(G)$ and $V(H)$. For $|V(G)| = m$ and $|V(H)| = n$, the edge set of $G + H$ is the union of disjoint edge sets of the graphs G, H, and the complete bipartite graph $K_{m,n}$.

Kulli and Muddebihal [19] gave the characterization of all pairs of graphs which join is planar graph. Using Kleitman's result [12], the crossing numbers for join of two paths, join of two cycles, and for join of path and cycle were studied in [17].

Let D be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. For a subgraph H_i of the graph G, let $D(H_i)$ be the subdrawing of D induced by H_i. For edge-disjoint subgraphs H_i and H_j of G, we denote by $\mathrm{cr}_D(H_i, H_j)$ the number of crossings of edges in H_i and edges in H_j, and by $\mathrm{cr}_D(H_i)$ the number of crossings among edges of H_i in D. It is easy to see that for any three edge-disjoint subgraphs H_i, H_j, and H_k of the graph G the following equations hold:

$$\mathrm{cr}_D(H_i \cup H_j) = \mathrm{cr}_D(H_i) + \mathrm{cr}_D(H_j) + \mathrm{cr}_D(H_i, H_j)\,,$$

$$\mathrm{cr}_D(H_i \cup H_j, H_k) = \mathrm{cr}_D(H_i, H_k) + \mathrm{cr}_D(H_j, H_k)\,. \tag{1}$$

In the paper, some proofs are based on Kleitman's result on crossing numbers of complete bipartite graphs. More precisely, he proved that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad m \le 6\,. \tag{2}$$

Let D_n denote the discrete graph on n vertices and let G_1 be the special graph on six vertices which can be seen in Figure 1. In Section 2, we give the crossing number of the join product of G_1 with the graph D_n. Using this result and properties of the Zip product operation, for any tree T with no vertex of degree two we give the crossing number of $G_1 \Box T$ in Section 3. In Section 4, all known results concerning the crossing numbers of Cartesian products of stars with graphs on six vertices are collected. Moreover, we establish the crossing numbers of $G_i \Box S_n$ for several other graphs G_i of order six. In the proofs of the paper, we will often use the term "region" also in non-planar drawings. In this case, crossings are considered to be vertices of the "map".

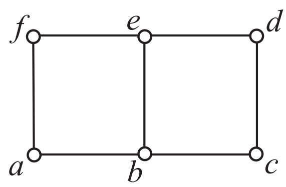

Figure 1. The graph G_1 on six vertices.

2. The Crossing Number of $G_1 + D_n$

The graph G_1 in Figure 1 consists of one 6-cycle $abcdef$, denoted by $C_6(G_1)$ in the paper, and of one edge be which, together with the edges of the 6-cycle, form two 4-cycles. The graph $G_1 + D_n$ consists of one copy of the graph G_1 and n vertices $t_1, t_2, \ldots, t_n$, where every vertex t_i, $i = 1, 2, \ldots, n$, is adjacent to every vertex of G_1, see Figure 2. For $i = 1, 2, \ldots, n$, let T^i denote the subgraph induced by six edges incident with the vertex t_i and let $F^i = G_1 \cup T^i$. To simplify the notation, let $G_1(n)$ denote the graph $G_1 + D_n$ in this paper. In Figure 2, one can easily see that

$$G_1 + D_n = G_1(n) = G_1 \cup K_{6,n} = G_1 \cup \left(\bigcup_{i=1}^{n} T^i\right). \tag{3}$$

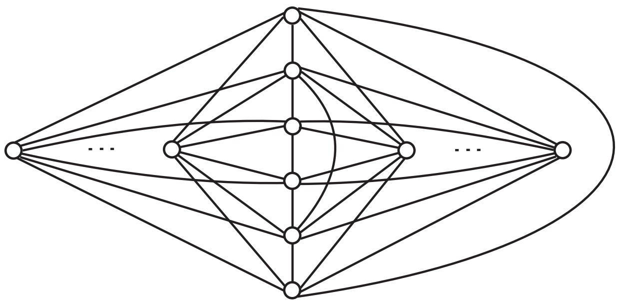

Figure 2. The drawing of the graph $G_1 + D_n$.

Lemma 1. $\mathrm{cr}(G_1 + D_2) = 2$.

Proof. The graph $G_1 + D_2$ consists of the subgraph G_1 and two subgraphs T^1 and T^2. As $G_1 + D_2$ contains $K_{3,3}$ as a subgraph, $\mathrm{cr}(G_1 + D_2) \geq 1$. If there is a drawing of $G_1 + D_2$ with only one crossing, then at least one of the subgraphs T^1 and T^2 does not cross G_1. Without loss of generality, assume that T^1 does not cross G_1. Then, in the view of the subdrawing of G_1, all vertices of G_1 are placed on the boundary of one, say outside, region. The subgraph T^1 is placed in this region and, as T^1 and G_1 do not cross each other, the edge be of G_1 does not cross T^1, too. Hence, if the edges of $C_6(G_1)$ do not cross each other, the subdrawing of $G_1 \cup T^1$ divides the plane such that at most four vertices of G_1 are on the boundary of every region as shown in Figure 3(a). If the edges of $C_6(G_1)$ cross each other, then they cross only once and no region of the subdrawing of $C_6(G_1) \cup T^1$ has more than four vertices of G_1 on its boundary, see Figure 3(b). This forces, that the subgraph T^2 crosses $G_1 \cup T^1$ at least twice in both cases. Thus, $\mathrm{cr}(G_1 + D_2) \geq 2$. On the other hand, in Figure 2 it is easy to see that $\mathrm{cr}(G_1 + D_2) \leq 2$. This completes the proof. ■

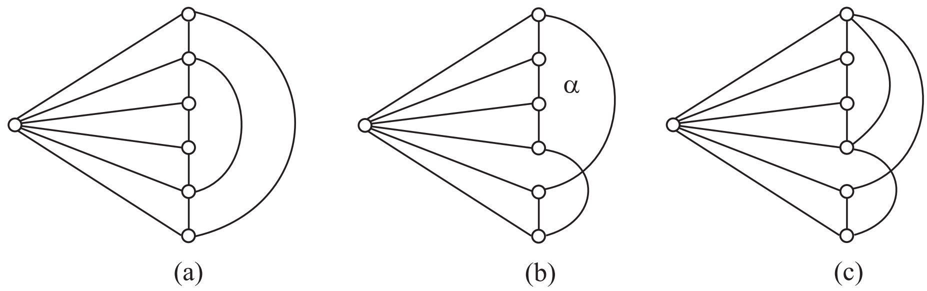

Figure 3. The forced subdrawings of $F^i = G_1 \cup T^i$.

Theorem 2. $\mathrm{cr}(G_1 + D_n) = 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor$ *for* $n \geq 1$.

Proof. The drawing in Figure 2 shows that $\mathrm{cr}(G_1 + D_n) \leq 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor$ and that the theorem is true if the equality holds. We prove the reverse inequality by induction on n. As the graph $G_1 + D_1$ is planar, the case $n = 1$ is trivial. Lemma 1 implies that the result is true for the case $n = 2$.

Suppose now that for $n \geq 3$

$$\mathrm{cr}(G_1(n-2)) \geq 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 2\left\lfloor\frac{n-2}{2}\right\rfloor \tag{4}$$

and consider such a drawing D of $G_1(n)$ that

$$\mathrm{cr}_D(G_1(n)) < 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor . \tag{5}$$

The drawing D has the following properties:

Property 1. $\mathrm{cr}_D(T^i, T^j) \neq 0$ *for all* $i, j = 1, 2, \ldots, n$, $i \neq j$.

Assume that for some $i \neq j$, $\mathrm{cr}_D(T^i, T^j) = 0$. The subgraph $G_1 \cup T^i \cup T^j$ is isomorphic to the graph $G_1 + D_2$. If the edges of G_1 do not cross each other, then, by Lemma 1, $\mathrm{cr}_D(G_1, T^i \cup T^j) \geq 2$. If both T^i and T^j cross G_1, then $\mathrm{cr}_D(G_1, T^i \cup T^j) \geq 2$ again. The last possibility is that one of T^i and T^j, say T^i, does not cross G_1, and the edges of G_1 cross each other. This forces that the edges of $C_6(G_1)$ cross each other, otherwise the edge be crosses T^i, see Figure 3(a). Hence, aside from the number of internal crossings in G_1, the subdrawing of $C_6(G_1) \cup T^i$ divides the plane in such a way that there is no region with more than four vertices of $C_6(G_1)$ on its boundary. This implies that the edges of T^j joining t_j with the vertices of G_1 cross $C_6(G_1) \cup T^i$ at least twice. As T^j does not cross T^i, $\mathrm{cr}_D(G_1, T^i \cup T^j) \geq 2$ again. Moreover, as $\mathrm{cr}(K_{6,3}) = 6$, in D, every subgraph T^k, $k = 1, 2, \ldots, n$, $k \neq i, j$, crosses $T^i \cup T^j$ at least six times. Since $G_1(n) = G_1 + D_n = G_1(n-2) \cup (T^i \cup T^j)$ and $G_1(n-2) = K_{6,n-2} \cup G_1$, using (1) and (4) we have

$$\begin{aligned}\mathrm{cr}_D(G_1(n)) &= \mathrm{cr}_D(G_1(n-2)) + \mathrm{cr}_D(T^i \cup T^j) + \mathrm{cr}_D(K_{6,n-2}, T^i \cup T^j) \\ &+ \mathrm{cr}_D(G_1, T^i \cup T^j) \geq 6\left\lfloor \tfrac{n-2}{2} \right\rfloor \left\lfloor \tfrac{n-3}{2} \right\rfloor + 2\left\lfloor \tfrac{n-2}{2} \right\rfloor + 0 + 6(n-2) + 2 \\ &= 6\left\lfloor \tfrac{n}{2} \right\rfloor \left\lfloor \tfrac{n-1}{2} \right\rfloor + 2\left\lfloor \tfrac{n}{2} \right\rfloor.\end{aligned}$$

This contradicts (5), and therefore $\mathrm{cr}_D(T^i, T^j) \neq 0$ for all $i, j = 1, 2, \ldots, n,\ i \neq j$.

Property 2. *The edges of* G_1 *are crossed less than* $2\lfloor \frac{n}{2} \rfloor$ *in* D.

Using (1) and (3) together with $\mathrm{cr}(K_{6,n}) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ we have

$$\begin{aligned}\mathrm{cr}_D(G_1(n)) &= \mathrm{cr}_D(K_{6,n}) + \mathrm{cr}_D(G_1) + \mathrm{cr}_D(K_{6,n}, G_1) \\ &\geq 6\left\lfloor \tfrac{n}{2} \right\rfloor \left\lfloor \tfrac{n-1}{2} \right\rfloor + \mathrm{cr}_D(G_1) + \mathrm{cr}_D\left(K_{6,n}, G_1\right).\end{aligned}$$

This, together with the assumption (5), implies that

$$\mathrm{cr}_D(G_1) + \mathrm{cr}_D(K_{6,n}, G_1) < 2\left\lfloor \frac{n}{2} \right\rfloor \tag{6}$$

and hence, the edges of G_1 are crossed less than $2\lfloor \frac{n}{2} \rfloor$ times in D.

The inequality (6) immediately implies the next property.

Property 3. *In* D*, there is at least one subgraph* T^i *which does not cross* G_1.

Assume, without loss of generality, that $\mathrm{cr}_D(G_1, T^n) = 0$. Then for the subgraph $F^n = G_1 \cup T^n$ of the graph $G_1(n)$ we have the next property.

Property 4. *In* D*, there is at least one subgraph* T^i*,* $i \in \{1, 2, \ldots, n-1\}$*, for which* $\mathrm{cr}_D(F^n, T^i) \leq 3$.

Otherwise, as $G_1(n) = K_{6,n-1} \cup F^n$ and $\mathrm{cr}_D(F^n) = \mathrm{cr}_D(G_1 \cup T^n) = 0$, we have

$$\begin{aligned}\mathrm{cr}_D(G_1(n)) &= \mathrm{cr}_D(K_{6,n-1}) + \mathrm{cr}_D(F^n) + \mathrm{cr}_D(K_{6,n-1}, F^n) \\ &\geq 6\left\lfloor \tfrac{n-1}{2} \right\rfloor \left\lfloor \tfrac{n-2}{2} \right\rfloor + 0 + 4(n-1) \geq 6\left\lfloor \tfrac{n}{2} \right\rfloor \left\lfloor \tfrac{n-1}{2} \right\rfloor + 2\left\lfloor \tfrac{n}{2} \right\rfloor,\end{aligned}$$

which contradicts (5).

Consider now the subdrawing D^* of D induced by F^n. As we assumed above, no edge of T^n crosses G_1. Our next analysis depends on whether or not the edges of the 6-cycle $C_6(G_1)$ cross each other in D^*. Assume first, that the edges of $C_6(G_1)$ do not cross each other. Since $\mathrm{cr}_D(G_1, T^n) = 0$, in D^*, all edges of T^n are placed in one of two regions, say outside, in the view of the subdrawing of $C_6(G_1)$ and the edge be of G_1, not belonging to $C_6(G_1)$, is placed inside the 6-cycle $C_6(G_1)$. The unique such drawing D^* is shown in Figure 3(a). If, in D, some vertex $t_i,\ i \in \{1, 2, \ldots, n-1\}$, is placed inside $C_6(G_1)$, then G_1 is crossed by at least two edges joining t_i with the vertices of G_1. Moreover, by Property 1, T^i crosses T^n and therefore, $\mathrm{cr}_D(F^n, T^i) \geq 3$. Outside $C_6(G_1)$ there are two vertices on the boundary of every region. Hence, for all other vertices t_i not placed inside $C_6(G_1)$, the edges of T^i cross the edges of F^n at least five times.

Let r be the number of vertices t_i, $i \in \{1, 2, \ldots, n-1\}$, which are placed in D inside the cycle $C_6(G_1)$. Thus, the corresponding subgraphs T^i cross the edges of $G_1 \cup T^n$ at least three times. The calculating of the necessary crossings in D gives

$$\begin{aligned} \mathrm{cr}_D(G_1(n)) &= \mathrm{cr}_D(K_{6,n-1}) + \mathrm{cr}_D(F^n) + \mathrm{cr}_D(K_{6,n-1}, F^n) \\ &\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 0 + 3r + 5(n-r-1) \\ &= 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor + (5n - 2r - 5 - 6\left\lfloor \frac{n-1}{2} \right\rfloor - 2\left\lfloor \frac{n}{2} \right\rfloor). \end{aligned}$$

This, together with the assumption (5), gives

$$2r > 5n - 5 - 6\left\lfloor \frac{n-1}{2} \right\rfloor - 2\left\lfloor \frac{n}{2} \right\rfloor \geq 2\left\lfloor \frac{n}{2} \right\rfloor.$$

This contradicts Property 2, because every of r subgraphs T^i crosses G_1 at least twice.

Let us turn to the case $\mathrm{cr}_D(C_6(G_1)) \neq 0$. As $\mathrm{cr}_D(G_1, T^n) = 0$, the subdrawing of $C_6(G_1)$ induced by D^* divides the plane in such a way that all vertices of $C_6(G_1)$ are placed on the boundary of one region, say outside. The whole subgraph T^n is placed in this region and on the boundary of every subregion formed by the edges of T^n and G_1 there are two vertices of G_1. The edges of $C_6(G_1)$ cross each other at least once and, by Property 1 and Property 4, on the boundary of at least one region inside $C_6(G_1)$ there are at least four vertices of G_1. Because of good drawing, this requirement forces that $C_6(G_1)$ has only one internal crossing and the unique subdrawing of $C_6(G_1) \cup T^n$ is shown in Figure 3(b). The edge be of G_1 not belonging to $C_6(G_1)$ must be adjacent to at least one vertex of $C_6(G_1)$ on the boundary of the region α. Assume first that both vertices of G_1 incident with the edge be are placed on the boundary of the region α. These vertices are in distance three. As two edges incident with the same vertex do not cross each other in the good drawing D^* and the edge be does not cross T^n, the unique possibility for placing the edge be is shown in Figure 3(c). It is easy to verify that, in this case, every subgraph T^i, $i = 1, 2, \ldots, n-1$, crosses the edges of F^n at least four times in D. This contradicts Property 4. Hence, the edge be is adjacent to only one vertex on the boundary of the region α in such a way that it splits the region α into two subregions. So, in D^*, there is no region with at least four vertices on its boundary. In this case, in D, every subgraph T^i, $i = 1, 2, \ldots, n-1$, crosses F^n at least four times. This contradiction with Property 4 completes the proof. ∎

3. The Crossing Number of $G_1 \square T$.

In this section, using Zip product operation introduced in [3], we establish the crossing number of the Cartesian product of the graph G_1 with any tree without vertices of degree two. For better reading, we repeat the related terms and results introduced by Bokal in [3] and [4].

For $i = 1, 2$, let H_i be a graph and let $v_i \in V(H_i)$ be its vertex of degree d. Let $N_i = N_{H_i}(v_i)$ be the set of neighbouring vertices of v_i and let $\sigma : N_1 \to N_2$ be a bijection. We call σ a *zip function* of the graphs H_1 and H_2 at vertices v_1 and v_2. The *zip product* of the graphs H_1 and H_2 according to σ is defined to be the graph $H_1 \odot_\sigma H_2$, obtained from the disjoint union of $H_1 - v_1$ and $H_2 - v_2$ after adding the edges $u\sigma(u)$, $u \in N_1$. Let $H_1 \,{}_{v_1}\!\odot_{v_2} H_2$ denote the set of all graphs obtained as a zip product $H_1 \odot_\sigma H_2$ for some zip function $\sigma : N_1 \to N_2$.

Let D_i be a drawing of a graph H_i. The drawing imposes a cyclic ordering of the edges incident with v_i, which can be extended to its neighbourhood N_i. Let the bijection $\pi_i : N_i \to \{1, \ldots, d\}$ be one of the corresponding labellings. We define $\sigma : N_1 \to N_2$, $\sigma = \pi_2^{-1}\pi_1$, to be the *zip function* of the drawings D_1 and D_2 at vertices v_1 and v_2. The *zip product* of D_1 and D_2 according to σ is the drawing $D_1 \odot_\sigma D_2$, obtained from D_1 by adding a mirrored copy of D_2 that has v_2 on the infinite face disjointly into some face of D_1 that contains the vertex v_1, removing the vertices v_1 and v_2 together with small disks around them from the drawings, and then joining the edges according to the function σ. As σ respects the ordering of the edges around v_1 and v_2, the edges between D_1 and D_2 may be drawn without introducing any new crossings. Clearly $D_1 \odot_\sigma D_2$ is a drawing of $H_1 \odot_\sigma H_2$, which implies the following lemma.

Lemma 3 [3]**.** *For $i = 1, 2$, let D_i be an optimal drawing of H_i, let $v_i \in V(H_i)$ be a vertex of degree d, and let σ be a zip function of D_1 and D_2 according to v_1 and v_2. Then* $\mathrm{cr}(H_1 \odot_\sigma H_2) \leq \mathrm{cr}(H_1) + \mathrm{cr}(H_2)$.

Let $G^{(i)} = G + D_i$ be the *suspension* of order i of a graph G. The vertices of D_i are called *apices* of $G^{(i)}$. For a multiset $L \subseteq V(H_2)$, we denote with $H_1 \Box_L H_2$ the *capped Cartesian product* of graphs H_1 and H_2, that is, the graph obtained by adding a distinct vertex v' to $H_1 \Box H_2$ for each copy of a vertex $v \in L$ and joining v' to all the vertices of $H_1 \Box \{v\}$. We call each v' a *cap* of v. Let $\chi_L(v)$ denote the multiplicity of v in L and $\ell(v) := \deg_{H_2}(v) + \chi_L(v)$. An edge $uv \in E(H_2)$ is *unbalanced* if $\ell(u) \neq \ell(v)$. Let $\beta(H_2)$ be the number of unbalanced edges of H_2.

A drawing D of $G^{(i)}$ is *apex-homogeneous* if there exists a permutation ρ of the vertices of G such that the vertex rotation around every apex in D is ρ or ρ^{-1}. Two drawings $D^{(i)}$ of $G^{(i)}$ and $D^{(j)}$ of $G^{(j)}$ are *pairwise apex-homogeneous*, if they are apex-homogeneous with respect to the same permutation ρ. A graph G has *all apex-homogeneous drawings* if there exist drawings $D^{(i)}$ of $G^{(i)}$, $i \geq 1$, such that every two of them are pairwise apex-homogeneous. The next result given by Bokal enables us to establish the crossing numbers of the Cartesian products of our graph G_1 with all trees which do not contain vertices of degree two.

Theorem 4 [4]**.** *Let G be a graph of order n, let T be a tree, and let $L \subseteq V(T)$ be a multiset with either $\ell(v) \geq 3$ or, if G has a dominating vertex, $\ell(v) \geq 2$ for* *every* $v \in V(T)$*. Define*

$$B = \sum\nolimits_{v \in V(T)} \mathrm{cr}(G^{(\ell(v))}).$$

Then, $B \leq \mathrm{cr}(G \square_L T) \leq B + \frac{\beta(T)}{2}\binom{n}{2}$*. Also,* $\mathrm{cr}(G \square_L T) = B$ *whenever* G *has all apex-homogeneous drawings.*

Consider now a graph H_{G_1} obtained by joining all vertices of G_1 to six vertices of a connected graph H such that every vertex of G_1 be adjacent to exactly one vertex of H. Let $H^*_{G_1}$ be the graph obtained from H_{G_1} by contracting the edges of G_1.

Lemma 5. $\mathrm{cr}(H^*_{G_1}) \leq \mathrm{cr}(H_{G_1})$.

Proof. Assume an optimal drawing of H_{G_1}. We remark that the edges of G_1 can cross each other in this drawing, as well as they can be crossed by some other edges of H_{G_1}. Let us denote by x_1 and x_2 the number of crossings on the edges ab and bc of G_1, respectively. Similarly, let x_3 and x_4 denote the number of crossings on the edges ed and ef, see one of the possible cases in Figure 4(a).

Let us contract the graph G_1 into the vertex b and connect this vertex with six vertices of H in such a way that the segments along the edges ab and bc in G_1 are used twice and the segments along the edges af, be, and cd only once. The segments along the edges ef and ed are not used as shown in Figure 4(b). In the worth case, at most $x_1 + x_2$ new crossings can appear only on the new edges in the segments along the edges ab and bc. But $x_3 + x_4$ crossings on the edges ed and ef do not appear in the new drawing. Hence, if $x_1 + x_2 \leq x_3 + x_4$, the resulting drawing of $H^*_{G_1}$ does not have more crossings than the original drawing of H_{G_1}. If the edges ed and ef cross some other edges of G_1 in the original drawing, these crossings do not appear in the drawing of $H^*_{G_1}$ and the number of crossings is less than in the drawing of H_{G_1}. Moreover, possible crossings among the edges of G_1 not appeared on the edges ed and ef cannot be crossings in a good drawing of $H^*_{G_1}$. Thus, if $x_1 + x_2 \leq x_3 + x_4$ our drawing after contracting the graph G_1 into vertex b does not have more crossings than the original drawing of H_{G_1}. Due to symmetry of the graph G_1, the same holds if $x_3 + x_4 \leq x_1 + x_2$.

Assume that the statement of Lemma 5 is not true. Then there is a good drawing of the graph H_{G_1} in which $x_1 + x_2 > x_3 + x_4$ and $x_3 + x_4 > x_1 + x_2$. This contradiction completes the proof. ■

In the rest of the section, we give the crossing numbers of the Cartesian products of the graph G_1 with trees T not containing vertices of degree two. For the special case $T = S_n$, the crossing number of the graph $G_1 \square S_n$ is obtained.

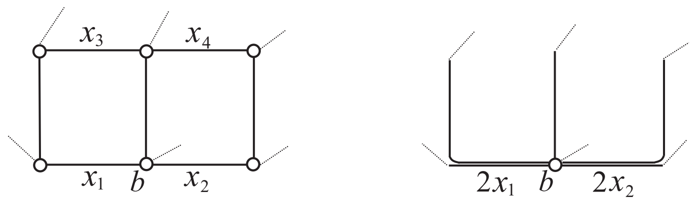

Figure 4. The contracting of G_1.

Theorem 6. *Let T be any tree of order n with no vertices of degree two and let d_i be the number of vertices of degree i in T. Then*

$$\mathrm{cr}(T\Box G_1) = \sum\nolimits_{i=1}^{\Delta(T)} (d_i \cdot cr(G_1 + D_i))\,.$$

Proof. Let T' be the tree obtained from T by removing all vertices of degree one in T. For a vertex v of T', let r_v be the number of T-leaves adjacent to v in T, and let L be the set of vertices in T', each with multiplicity r_v. As T has no vertices of degree two, $\ell(v) = d_{T'}(v) + r(v) = d_T(v) \geq 3$ for all $v \in V(T')$. The drawing in Figure 2 shows that G_1 has all apex-homogeneous drawings. Thus, by Theorem 4, $\mathrm{cr}(G_1\Box_L T') = \sum_{i=1}^{\Delta(T)}(d_i \cdot cr(G_1 + D_i))$.

The graph $G_1\Box_L T'$ is obtained from $G_1\Box T$ by contracting all the G_1 edges corresponding to $G_1\Box\{u\}$, where u is a leaf of T. Then, the iterative applications of Lemma 5 show that $\mathrm{cr}(G_1\Box_L T') \leq \mathrm{cr}(G_1\Box T)$. On the other hand, the graph $G_1\Box T$ is obtained from $G_1\Box_L T'$ by zipping a copy of $G' = G_1 + \{v\}$ at each cap of $G_1\Box_L T'$. As $G' = G_1 + \{v\}$ is planar, Lemma 3 implies that $\mathrm{cr}(G_1\Box_L T') \geq \mathrm{cr}(G_1\Box T)$, and the proof is done. ∎

By Theorem 2, $\mathrm{cr}(G_1 + D_n) = 6\lfloor \frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor$. As $\mathrm{cr}(G_1 + D_1) = 0$, for the special tree S_n we have the next result.

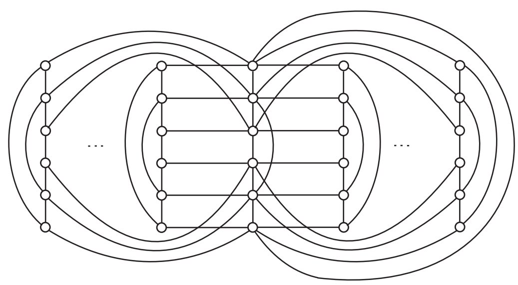

Figure 5. The drawing of $G_1\Box S_n$ with $6\lfloor \frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor$ crossings.

Corollary 7. $\mathrm{cr}(G_1\square S_n) = 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor$ *for* $n \geq 1$.

The drawing of the Cartesian product $G_1\square S_n$ with $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor$ crossings is shown in Figure 5. In the next section, we will use parts of similar drawings for some special graphs G_i of order six.

4. THE CROSSING NUMBER OF $G_i\square S_n$ FOR SOME GRAPHS OF ORDER SIX

There are 112 connected graphs of order six. In the rest of the paper, we collect old and new results concerning the crossing numbers of Cartesian products of eighteen graphs on six vertices with stars. In Section 3, the crossing number of $G_1\square S_n$ is established. In Figure 6, the other 16 graphs G_i, $i = 2, 3, \ldots, 17$, on six vertices are presented. The last graph, not presented in Figure 6, is the complete tripartite graph $K_{2,2,2}$.

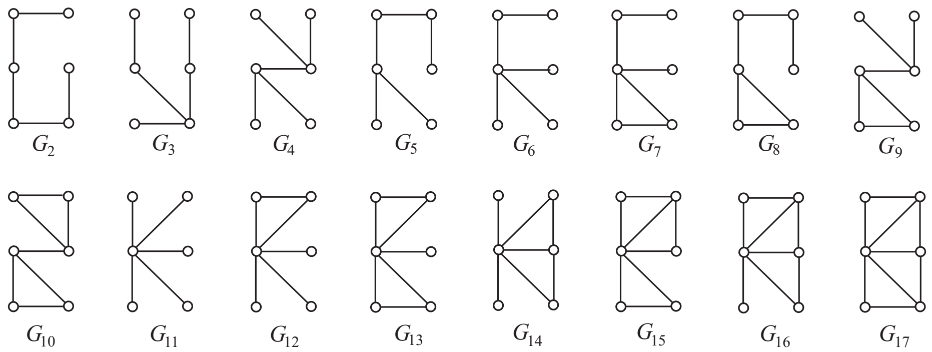

Figure 6. Sixteen graphs G_j on six vertices.

As corollary of Theorem 4, in [4] Bokal estimated the following results:

Corollary 8 [4]. *Let T be a tree and $n \geq 1$. Then, for $d_v = \deg_T(v)$,*

$$\mathrm{cr}(S_n\square T) = \sum\nolimits_{v\in V(T), d_v\geq 2} \mathrm{cr}(K_{1,d_v,n}).$$

Corollary 9 [4]. *Let $n \geq 1$ be any integer and T a subcubic tree with n_2 vertices of degree two and n_3 vertices of degree three. Then,*

$$\mathrm{cr}(S_n\square T) = \left\lfloor\frac{n}{2}\right\rfloor\Big((n_2 + 2n_3)\left\lfloor\frac{n-1}{2}\right\rfloor + n_3\Big).$$

In [10], Huang and Zhao proved that $\mathrm{cr}(K_{1,4,n}) = 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor$. Mei and Huang proved in [21] that the crossing number of the complete tripartite graph $K_{1,5,n}$ is $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor$. This, together with Asano's [1] result $\mathrm{cr}(K_{1,3,n}) = 2\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{n}{2}\rfloor$ and the known fact that $\mathrm{cr}(K_{1,2,n}) = \mathrm{cr}(K_{3,n}) = \lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ enables us to state:

Corollary 10. *Let $n \geq 1$ be any integer and T a tree with maximum degree five. Let T have n_2 vertices of degree two, n_3 vertices of degree three, n_4 vertices of degree four, and n_5 vertices of degree five. Then,*

$$\mathrm{cr}(S_n \Box T) = \left\lfloor \frac{n}{2} \right\rfloor \Big((n_2 + 2n_3 + 4n_4 + 6n_5) \left\lfloor \frac{n-1}{2} \right\rfloor + n_3 + 2n_4 + 4n_5 \Big).$$

In [3], Bokal proved the conjecture given by Jendroľ and Ščerbová [11] that $\mathrm{cr}(P_m \Box S_n) = (m-2)\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ for the path P_m of length $m-1$. Hence, $\mathrm{cr}(G_2 \Box S_n) = \mathrm{cr}(P_6 \Box S_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. For the graphs G_3, G_4, and G_5 in Figure 6, Corollary 9 implies that $\mathrm{cr}(G_3 \Box S_n) = \mathrm{cr}(G_5 \Box S_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ and $\mathrm{cr}(G_4 \Box S_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$. Moreover, by Corollary 10, $\mathrm{cr}((G_6 \Box S_n) = 5\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$. This results enables us to give the crossing numbers of Cartesian products of stars with the graphs G_7, G_8, G_9, and G_{10}. The graph G_7 contains G_6 as a subgraph, so $\mathrm{cr}(G_7 \Box S_n) \geq 5\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$. On the other hand, in Figure 7(a) there is a left-side drawing of the graph $G_7 \Box S_n$ with $\lceil \frac{n}{2} \rceil$ non-central copies of G_7 and the right $\lfloor \frac{n}{2} \rfloor$ non-central copies of G_7 are omitted (compare with the drawing of the graph $G_1 \Box S_n$ in Figure 5). The drawing in Figure 7(a) implies that there is a drawing of the graph $G_7 \Box S_n$ with $5\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ crossings and therefore, $\mathrm{cr}(G_7 \Box S_n) = 5\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$.

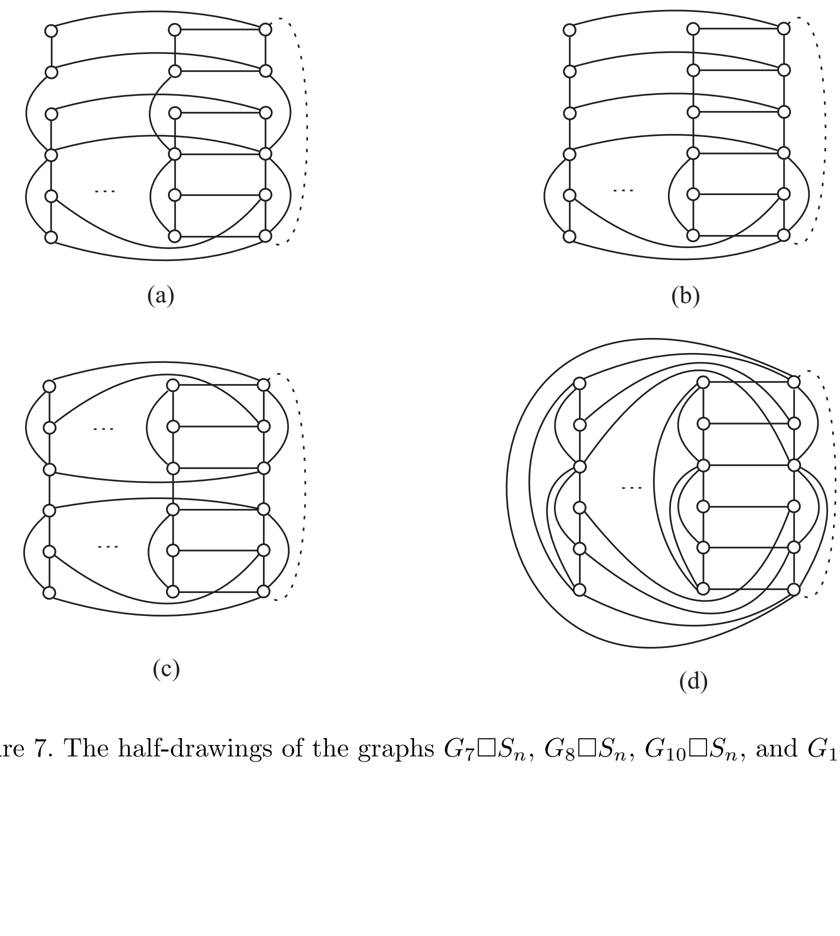

Figure 7. The half-drawings of the graphs $G_7 \Box S_n$, $G_8 \Box S_n$, $G_{10} \Box S_n$, and $G_{17} \Box S_n$.

The graph G_8 contains G_5 as a subgraph, so $\mathrm{cr}(G_8\square S_n) \geq 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{n}{2}\rfloor$. Figure 7(b) shows that there is a drawing of the graph $G_8\square S_n$ with $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{n}{2}\rfloor$ crossings. This states that $\mathrm{cr}(G_8\square S_n) = 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{n}{2}\rfloor$. Both graphs G_9 and G_{10} contain G_4. This implies that $\mathrm{cr}(G_9\square S_n) \geq 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor$ and $\mathrm{cr}(G_{10}\square S_n) \geq 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor$. In the half-drawing of the graph $G_{10}\square S_n$ in Figure 7(c) we can see that the crossing number of the graph $G_{10}\square S_n$ is at most $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor$. Using the fact that the graph $G_9\square S_n$ is a subgraph of $G_{10}\square S_n$, we have that $\mathrm{cr}(G_9\square S_n) = \mathrm{cr}(G_{10}\square S_n) = 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor$.

In [21], Mei and Huang proved that $\mathrm{cr}(K_{1,5,n}) = 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor$. Since the graph $S_5\square S_n = G_{11}\square S_n$ is a subdivision of the graph $K_{1,5,n}$, $\mathrm{cr}(G_{11}\square S_n) = 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor$. All graphs G_i, $i = 12, 13, \ldots, 17$, in Figure 6 contain the graph G_{11} as a subgraph and therefore, $\mathrm{cr}(G_i\square S_n) \geq 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor$ for $i = 12, 13, \ldots, 17$. The left-side drawing of the graph $G_{17}\square S_n$ in Figure 7(d) implies that there is a drawing of the graph $G_{17}\square S_n$ with at most $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor$ crossings. Thus, we have the same lower and upper bound for the crossing number of all six considered graphs, which implies that $\mathrm{cr}(G_i\square S_n) = 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor$ for all $i = 12, 13, \ldots, 17$.

In [9], Ho proved that the crossing number of the complete 4-partite graph $K_{2,2,2,n}$ is $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 3n$. Using this result, in [5] it is shown that the crossing number of the Cartesian product $K_{2,2,2}\square S_n$ is $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 6n$.

5. Comments

Except the graph $K_5\square S_n$, the known results concerning crossing numbers of Cartesian products of stars with graphs of order five are collected in [18]. The crossing number of $K_5\square S_n$ was presented in [20]. For six remaining graphs G_i on five vertices, the problem of estimating $\mathrm{cr}(G_i\square S_n)$ is still open, even though some incorrect result were published. For example, in [8] the incorrect proof states that for the tree T on five vertices with one vertex of degree two and one vertex of degree three, the crossing number of $T\square S_n$ is $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{n}{2}\rfloor$. This contradicts Corollary 9.

We suppose that the application of Zip product operation can be used to estimate the unknown values of the crossing number for Cartesian products of some graphs on five vertices with trees, and also with stars. The same we expect for lager graphs, namely for some of 94 remaining graphs of order six.

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