

## A NOTE ON THE PERMANENTAL ROOTS OF BIPARTITE GRAPHS<sup>1</sup>

HEPING ZHANG, SHUNYI LIU

AND

WEI LI

*School of Mathematics and Statistics*  
*Lanzhou University*  
*Lanzhou, Gansu 730000, P.R. China*

**e-mail:** zhanghp@lzu.edu.cn  
liushuny@gmail.com  
li\_w07@lzu.edu.cn

### Abstract

It is well-known that any graph has all real eigenvalues and a graph is bipartite if and only if its spectrum is symmetric with respect to the origin. We are interested in finding whether the permanental roots of a bipartite graph  $G$  have symmetric property as the spectrum of  $G$ . In this note, we show that the permanental roots of bipartite graphs are symmetric with respect to the real and imaginary axes. Furthermore, we prove that any graph has no negative real permanental root, and any graph containing at least one edge has complex permanental roots.

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### 1. INTRODUCTION

In the literature there are many polynomials associated to graphs. For example, characteristic polynomial [10], chromatic polynomial [3], matching polynomial [11] and permanental polynomial [15]. No doubt, the most extensively examined such polynomial is the characteristic polynomial.

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It is well-known that the characteristic roots of any graph are all real, and a graph is bipartite if and only if its spectrum is symmetric with respect to the origin (see, for example, [10], p. 87). An interesting question is whether the permanent roots of a bipartite graph  $G$  have symmetric property as the spectrum of  $G$ .

We answer this question affirmatively and show that the permanent roots of a bipartite graph are symmetric with respect to the real and imaginary axes.

The *permanent* of an  $n \times n$  matrix  $M$  with entries  $m_{ij}$  ( $i, j = 1, 2, \dots, n$ ) is defined by

$$(1) \quad \text{per}(M) = \sum_{\sigma} \prod_{i=1}^n m_{i\sigma(i)},$$

where the sum is taken over all permutations  $\sigma$  of  $\{1, 2, \dots, n\}$ . In strong contrast to determinants, computing permanents, even of matrices in which all entries are 0 or 1, is #P-complete [17].

Let  $G$  be a graph and  $A(G)$  the adjacency matrix of  $G$ . Recall that the *characteristic polynomial* of  $G$  is defined by

$$(2) \quad \phi(G, x) = \det(xI - A(G)).$$

In analogy to Equation (2), one defines the *permanent polynomial* of  $G$ ,  $\pi(G, x)$ , as the permanent of the characteristic matrix of  $A(G)$ , i.e.

$$(3) \quad \pi(G, x) = \text{per}(xI - A(G)).$$

A root of  $\pi(G, x)$  is called a *permanent root* of  $G$ . The *per-spectrum*  $pS(G)$  [4] of  $G$  is the multiset of permanent roots of  $G$ .

It seems that the permanent polynomial was first considered by Turner [16]. He in fact considered a graph polynomial which generalizes both the permanent and characteristic polynomials. The permanent polynomial was first systematically studied by Merris *et al.* [15], and the study of analogous objects in chemical literature was started by Kasum *et al.* [13]. The literature on permanent polynomial is far less than that on characteristic polynomial (see, for example, [2, 4, 5, 6, 7, 8, 9, 12, 13, 15, 18, 19]). This may be due to the difficulty of actually computing the permanent  $\text{per}(xI - A(G))$ .

In [4], Borowiecki proved that  $G$  has  $pS(G) = \{i\lambda_1, i\lambda_2, \dots, i\lambda_n\}$  if and only if  $G$  is bipartite without cycles of length  $4k$  ( $k = 1, 2, \dots$ ), where  $i^2 = -1$  and  $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$  is the spectrum (i.e. the multiset of characteristic roots) of  $G$ . In [5], Borowiecki and Jóźwiak posed the following problem.

**Problem 1.** Characterize those graphs which have pure imaginary per-spectrum.

Yan and Zhang [18] gave a partial solution to this problem. They obtained that if  $G$  is a bipartite graph containing no subgraphs which are even subdivisions of  $K_{2,3}$ , then the permanent roots of  $G$  are pure imaginary.

Cash [7] developed a computer program for the calculation of the permanental polynomial of molecular graphs, and applied it to a variety of benzenoid hydrocarbons [7] and fullerenes [8]. The mathematical properties of the coefficients and roots of the permanental polynomials of these chemical graphs were investigated.

In this note, we present a preliminary treatment of the permanental roots of graphs. The structure of this paper is the following. In the next section, we prove that any graph has no negative real permanental root, and any graph containing at least one edge has complex permanental roots. In Section 3, we show that a graph is bipartite if and only if its per-spectrum is symmetric with respect to the real and imaginary axes.

## 2. SOME PROPERTIES OF THE PERMANENTAL ROOTS OF GRAPHS

By the definition of permanent, we immediately obtain the following result.

**Theorem 2.** *Let  $G$  be a disconnected graph with components  $G_1, G_2, \dots, G_\omega$  ( $\omega \geq 2$ ). Then  $\pi(G, x) = \prod_{i=1}^\omega \pi(G_i, x)$ .*

Clearly, by Theorem 2, the per-spectrum of a disconnected graph  $G$  is the union of the per-spectrum of each connected component  $G_i$  of  $G$ .

A graph  $G$  is said to be a *Sachs graph* if each component of  $G$  is a single edge or a cycle. Merris *et al.* [15] obtained a modified Sachs theorem on the permanental polynomial of a graph.

**Theorem 3.** *Let  $G$  be a graph on  $n$  vertices with  $\pi(G, x) = \sum_{k=0}^n b_k x^{n-k}$ . Then*

$$(4) \quad b_k = (-1)^k \sum_H 2^{c(H)}, \quad 1 \leq k \leq n,$$

where the sum is taken over all Sachs subgraphs  $H$  of  $G$  on  $k$  vertices, and  $c(H)$  is the number of cycles in  $H$ .

By the definition of permanental polynomial, we have  $b_0 = 1$ . It follows from Theorem 3 that  $b_i \leq 0$  if  $i$  is odd and  $b_i \geq 0$  otherwise. In particular,  $b_1$  always equals 0,  $b_2$  is the number of edges of  $G$ , and  $-b_3$  is twice the number of triangles in  $G$ . For a bipartite graph  $G$ ,  $b_i = 0$  for all odd  $i$  since there exists no Sachs subgraph of an odd number of vertices in  $G$ . In fact, Borowiecki and Józwiak [5] obtained the following result.

**Theorem 4.** *Let  $G$  be a graph on  $n$  vertices with  $\pi(G, x) = \sum_{k=0}^n b_k x^{n-k}$ . Then  $G$  is bipartite if and only if  $b_i = 0$  for all odd  $i$ .*

An interval is called a *root-free interval* for a permanental polynomial  $\pi(G, x)$  if  $G$  has no permanental root in this interval. Likewise, an interval is called a

*root-free interval* for a family  $S$  of graphs if every graph in  $S$  has no permanental root in this interval. We shall show that  $(-\infty, 0)$  is a root-free interval for the family of all graphs.

**Proposition 5.** *For every graph  $G$ ,  $\pi(G, x)$  has no negative real root.*

**Proof.** Let  $\pi(G, x) = \sum_{k=0}^n b_k x^{n-k}$ . If  $n$  is odd, then  $b_k \lambda^{n-k} \leq 0$  and  $\lambda^n < 0$  for all real  $\lambda < 0$ . If  $n$  is even, then  $b_k \lambda^{n-k} \geq 0$  and  $\lambda^n > 0$  for all real  $\lambda < 0$ . Therefore, for all real  $\lambda < 0$ ,  $\pi(G, \lambda) < 0$  if  $n$  is odd and  $\pi(G, \lambda) > 0$  otherwise. ■

Note that 0 may be a permanental root of some graphs (for instance, 0 is a permanental root of a tree with an odd number of vertices). Thus  $(-\infty, 0)$  is a maximal root-free interval for the permanental polynomials of all graphs.

The following result shows that the multiplicity of 0 as a permanental root of  $G$  can be determined by the maximum number of vertices of a Sachs subgraph of  $G$ .

**Lemma 6.** *Let  $G$  be a graph on  $n$  vertices. Then the multiplicity of 0 as a root of  $\pi(G, x)$  is equal to  $n - p$ , where  $p$  is the maximum number of vertices of a Sachs subgraph of  $G$ .*

Lemma 6 can be easily obtained from Theorem 3. As immediate consequences, we obtain the following two corollaries.

**Corollary 7.** *A graph  $G$  has a zero permanental root if and only if  $G$  has no spanning Sachs subgraph.*

**Corollary 8.** *If  $G$  is bipartite, then the multiplicity of 0 as a root of  $\pi(G, x)$  is equal to the deficiency of  $G$  (i.e. the number of vertices left uncovered by any maximum matching of  $G$  [14]).*

It is well-known that all the characteristic roots of a graph are real. However, there exists no graph containing at least one edge whose permanental roots are all real. Before proving this statement, we need the following useful and classical result.

**Lemma 9** (Descartes' rule of signs [1]). *Let  $P(x)$  be a polynomial in one variable  $x$  with real coefficients. If  $P(x)$  is arranged in ascending or descending powers, then the number of real positive roots of the polynomial is no more than the number of sign variations in consecutive coefficients, and differs from this upper bound by an even integer. Multiple roots of the same value are counted separately.*

**Proposition 10.** *Let  $G$  be a graph containing at least one edge. Then  $\pi(G, x)$  has complex roots.*

**Proof.** Let  $\pi(G, x) = \sum_{k=0}^p b_k x^{n-k}$ , where  $p$  is the maximum number of vertices of a Sachs subgraph of  $G$ . By Lemma 6,  $G$  has exactly  $n - p$  zero permanental roots. Since  $b_1 = 0$ , there are at most  $p - 2$  sign changes in  $\pi(G, x)$ . Hence, by Lemma 9,  $\pi(G, x)$  has at most  $p - 2$  real positive roots. By Proposition 5,  $\pi(G, x)$  has no real negative root. It follows that  $\pi(G, x)$  has at most  $n - 2$  real roots. Thus  $G$  has at least two complex permanental roots. ■

### 3. THE PERMANENTAL ROOTS OF BIPARTITE GRAPHS

Firstly, we show that any bipartite graph has no non-zero real permanental root.

**Proposition 11.** *If  $G$  is bipartite, then  $G$  has no real permanental root except (possible) 0.*

**Proof.** Let  $G$  be a bipartite graph. By Theorems 3 and 4, all the coefficients of  $\pi(G, x)$  are nonnegative. It is easy to verify that  $\pi(G, \lambda) > 0$  for all real  $\lambda > 0$ , which implies that  $G$  has no positive real permanental root. By Proposition 5,  $G$  has no negative real permanental root. This completes the proof. ■

Proposition 11 implies that all non-zero permanental roots of a bipartite graph are complex. By the complex conjugate root theorem, if  $z \in \mathbb{C}$  is a complex root of  $\pi(G, x)$  then so is  $\bar{z}$  (the complex conjugate of  $z$ ). Moreover, we will show that all non-zero permanental roots of a bipartite graph occur in purely imaginary pairs,  $(ib, -ib)$ ,  $b \in \mathbb{R}$ , and quadruplets,  $\pm a \pm ib$ ,  $a, b \in \mathbb{R}$ .

**Lemma 12.** *Let  $P(x) = x^n + b_1 x^{n-2} + b_2 x^{n-4} + \dots + b_p x^{n-2p}$ , where  $p \leq \lfloor \frac{n}{2} \rfloor$ ,  $b_i \in \mathbb{R}$ , and  $b_p \neq 0$ . Then the roots of  $P(x)$  are symmetric with respect to the real and imaginary axes, i.e. all non-zero roots occur in real pairs  $(a, -a)$ ,  $a \in \mathbb{R}$ , purely imaginary pairs,  $(ib, -ib)$ ,  $b \in \mathbb{R}$ , and quadruplets,  $\pm a \pm ib$ ,  $a, b \in \mathbb{R}$ .*

**Proof.** Let  $f(t) = t^p + b_1 t^{p-1} + b_2 t^{p-2} + \dots + b_{p-1} t + b_p$ . Then  $P(x) = x^{n-2p}(x^{2p} + b_1 x^{2p-2} + b_2 x^{2p-4} + \dots + b_p) = x^{n-2p} f(x^2)$ . Clearly, the roots of  $P(x)$  consist of 0 ( $n - 2p$  times) and the roots of  $f(x^2)$ . Since  $b_p \neq 0$ , 0 is not a root of  $f(t)$ . Suppose that the roots of  $f(t)$  are  $z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_s, \bar{z}_s, a_1, a_2, \dots, a_l, c_1, c_2, \dots, c_m$ , where  $z_j$  ( $j = 1, \dots, s$ ) are complex numbers,  $a_j$  ( $j = 1, \dots, l$ ) are positive numbers,  $c_j$  ( $j = 1, \dots, m$ ) are negative numbers, and  $2s + l + m = p$ . Let  $z_j = r_j e^{i\theta_j}$  ( $\theta_j \neq k\pi, k = 0, \pm 1, \pm 2, \dots$ ), where  $i^2 = -1$ . Then  $\bar{z}_j = r_j e^{-i\theta_j}$ .

The roots of the equation  $x^2 = z_j$  are  $x_0 = \sqrt{r_j} e^{i\frac{\theta_j}{2}} = \sqrt{r_j}(\cos \frac{\theta_j}{2} + i \sin \frac{\theta_j}{2})$  and  $x_1 = \sqrt{r_j} e^{i\frac{\theta_j+2\pi}{2}} = \sqrt{r_j}(-\cos \frac{\theta_j}{2} - i \sin \frac{\theta_j}{2})$ . The roots of the equation  $x^2 = \bar{z}_j$  are  $x'_0 = \sqrt{r_j} e^{i\frac{-\theta_j}{2}} = \sqrt{r_j}(\cos \frac{\theta_j}{2} - i \sin \frac{\theta_j}{2})$  and  $x'_1 = \sqrt{r_j} e^{i\frac{-\theta_j+2\pi}{2}} = \sqrt{r_j}(-\cos \frac{\theta_j}{2} + i \sin \frac{\theta_j}{2})$ . The roots of the equation  $x^2 = a_j$  are  $x_{1,2} = \pm \sqrt{a_j}$  and the roots of the

equation  $x^2 = c_j$  are  $x_{1,2} = \pm i\sqrt{-c_j}$ . Therefore, the roots of  $P(x)$  are  $\{0, \dots, 0\} \cup \{\pm\sqrt{r_j}(\cos \frac{\theta_j}{2} \pm i \sin \frac{\theta_j}{2}) | j = 1, \dots, s\} \cup \{\pm\sqrt{a_j} | j = 1, \dots, l\} \cup \{\pm i\sqrt{-c_j} | j = 1, \dots, m\}$ . This completes the proof.  $\blacksquare$

By Theorem 4, the permanental polynomial of a bipartite graph has the same form as that of Lemma 12. Therefore, we immediately obtain the following result.

**Proposition 13.** *The per-spectrum of a bipartite graph is symmetric with respect to the real and imaginary axes.*

Now we are in position to present the main result of this paper, which gives a characterization of bipartite graphs  $G$  in terms of the per-spectrum of  $G$ .

**Theorem 14.**  *$G$  is bipartite if and only if the per-spectrum of  $G$  is symmetric with respect to the real and imaginary axes.*

**Proof.** The necessity of this theorem holds from Proposition 13. We proceed to prove the sufficiency of this theorem. Suppose that  $G$  is a graph on  $n$  vertices and the per-spectrum of  $G$  is symmetric with respect to the real and imaginary axes. Let  $\pi(G, x) = \sum_{k=0}^p b_k x^{n-k}$ , where  $p$  is the maximum number of vertices of a Sachs subgraph of  $G$ . Let  $f(t) = t^p + b_2 t^{p-2} + b_3 t^{p-3} + \dots + b_{p-1} t + b_p$ . Then  $\pi(G, x) = x^{n-p} f(x)$ . Clearly, the roots of  $\pi(G, x)$  consist of 0 ( $n-p$  times) and the roots of  $f(x)$ . By Proposition 5,  $G$  has no positive real permanental root. In fact, if  $G$  has a positive real permanental root, then  $G$  has a negative real permanental root, a contradiction. Suppose that the permanental roots of  $G$  are  $\{0, \dots, 0\} \cup \{\pm a_j \pm ic_j | a_j, c_j \in \mathbb{R}, a_j > 0, c_j > 0, j = 1, \dots, s\} \cup \{\pm id_j | d_j \in \mathbb{R}, d_j > 0, j = 1, \dots, l\}$ , where  $4s + 2l = p$ . Let  $z_j = a_j + ic_j$ . Then  $f(x) = \prod_{j=1}^s ((x - z_j)(x + z_j)(x - \bar{z}_j)(x + \bar{z}_j)) \prod_{j=1}^l ((x - id_j)(x + id_j)) = \prod_{j=1}^s (x^4 - 2(a_j^2 - c_j^2)x^2 + (a_j^2 + c_j^2)^2) \prod_{j=1}^l (x^2 + d_j^2)$ .

Since the coefficients of the odd-power terms of  $f(x)$  are all equal to 0, we can assume that  $f(x) = x^p + b'_2 x^{p-2} + b'_4 x^{p-4} + \dots + b'_{p-2} x^2 + b'_p$ . Therefore,  $\pi(G, x) = x^{n-p}(x^p + b'_2 x^{p-2} + b'_4 x^{p-4} + \dots + b'_{p-2} x^2 + b'_p) = x^n + b'_2 x^{n-2} + b'_4 x^{n-4} + \dots + b'_{p-2} x^{n-p+2} + b'_p x^{n-p}$ . By Theorem 4,  $G$  is a bipartite graph. This completes the proof.  $\blacksquare$

**Remark 15.** It is worth pointing out that by the above argument we can show that  $G$  is a bipartite graph if and only if the spectrum of  $G$  is symmetric with respect to the origin, whereas in most cases the eigenvector method is used to prove this classical result.

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