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THE DISTANCE ROMAN DOMINATION NUMBERS OF GRAPHS

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Abstract

Let k be a positive integer, and let G be a simple graph with vertex set $V(G)$. A *k-distance Roman dominating function* on G is a labeling $f : V(G) \to \{0, 1, 2\}$ such that for every vertex with label 0, there is a vertex with label 2 at distance at most k from each other. The *weight* of a k-distance Roman dominating function f is the value $\omega(f) = \sum_{v \in V} f(v)$. The *$k$-distance Roman domination number* of a graph G, denoted by $\gamma_R^k(D)$, equals the minimum weight of a k-distance Roman dominating function on G. Note that the 1-distance Roman domination number $\gamma_R^1(G)$ is the usual *Roman domination number* $\gamma_R(G)$. In this paper, we investigate properties of the k-distance Roman domination number. In particular, we prove that for any connected graph G of order $n \geq k + 2$, $\gamma_R^k(G) \leq 4n/(2k + 3)$ and we characterize all graphs that achieve this bound. Some of our results extend these ones given by Cockayne *et al.* in 2004 and Chambers *et al.* in 2009 for the Roman domination number.

Keywords: k-distance Roman dominating function, k-distance Roman domination number, Roman dominating function, Roman domination number.

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1. Terminology and Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Denote by K_n the *complete graph*, by C_n the *cycle* and by P_n the *path* of order n, respectively. Given two graphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \emptyset$, the *disjoint union* is the graph $G_1 \cup G_2$ with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Let k be a positive integer. For two vertices x and y, let $d(x, y)$ denote the distance between x and y in G. The *girth* $g(G)$ of a graph G is the length of its shortest cycle. For a vertex $v \in V(G)$, the *open k-neighborhood* $N_{k,G}(v)$ is the set $\{u \in V(G) \mid u \neq v \text{ and } d(u, v) \leq k\}$ and the *closed k-neighborhood* $N_{k,G}[v]$ is the set $N_{k,G}(v) \cup \{v\}$. The *open k-neighborhood* $N_{k,G}(S)$ of a set $S \subseteq V$ is the set $\bigcup_{v \in S} N_{k,G}(v)$, and the *closed-$k$-neighborhood* $N_{k,G}[S]$ of S is the set $N_{k,G}(S) \cup S$. The *k-degree* of a vertex v is defined as $\deg_{k,G}(v) = |N_{k,G}(v)|$. The minimum and maximum k-degree of a graph G are denoted by $\delta_k(G)$ and $\Delta_k(G)$, respectively. If $\delta_k(G) = \Delta_k(G)$, then the graph G is called *distance-k-regular*. The *k-th power* G^k of a graph G is the graph with vertex set $V(G)$ where two different vertices u and v are adjacent if and only if the distance $d(u, v)$ is at most k in G. Now we observe that $N_{k,G}(v) = N_{1,G^k}(v) = N_{G^k}(v)$, $N_{k,G}[v] = N_{1,G^k}[v] = N_{G^k}[v]$, $\deg_{k,G}(v) = \deg_{1,G^k}(v) = \deg_{G^k}(v)$, $\delta_k(G) = \delta_1(G^k) = \delta(G^k)$ and $\Delta_k(G) = \Delta_1(G^k) = \Delta(G^k)$. Consult [6, 10] for the notation and terminology which are not defined here.

Let $k \geq 1$ be an integer. A set $D \subseteq V(G)$ is a k-distance dominating set of G if every vertex in $V(G) - D$ is within distance k of at least one vertex in D. The k-distance domination number $\gamma^k(G)$ of G is the minimum cardinality among all k-distance dominating sets of G.

A *k-distance Roman dominating function* (kDRDF) on a graph $G = (V, E)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that for every vertex v for which $f(v) = 0$, there is a vertex u for which $f(u) = 2$ and $d(u, v) \leq k$. The *weight* of a kDRDF f is the value $\omega(f) = \sum_{v \in V} f(v)$. The *$k$-distance Roman domination number* of a graph G, denoted by $\gamma_R^k(G)$, equals the minimum weight of a kDRDF on G. A *$\gamma_R^k(G)$-function* is a k-distance Roman dominating function of G with weight $\gamma_R^k(G)$. A k-distance Roman dominating function $f : V \longrightarrow \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) (or (V_0^f, V_1^f, V_2^f) to refer f) of V, where $V_i = \{v \in V \mid f(v) = i\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2|$. Since $V_1^f \cup V_2^f$ is a k-distance dominating set when f is a kDRDF, and since placing weight 2 at the vertices of a k-distance dominating set yields a kDRDF, we have

$$\gamma^k(G) \leq \gamma_R^k(G) \leq 2\gamma^k(G). \tag{1}$$

Note that the 1-distance Roman domination number $\gamma_R^1(G)$ is the usual *Roman domination number* $\gamma_R(G)$. The definition of the Roman dominating function was given multiplicity by Steward [9] and ReVelle and Rosing [8]. Cockayne *et al.* [3] as well as Chambers *et al.* [2] have given a lot of results on Roman domination.

Our purpose in this paper is to initiate the study of the k-distance Roman domination number and establish some bounds for the k-distance Roman domination number of a graph. Some of our results extend many well-known results.

2. Some Basic Results

We start with some known results on the classical Roman domination number.

Theorem A [4]**.** *For any graph G of order n and maximum degree $\Delta \geq 1$,*

$$\gamma_R(G) \geq \frac{2n}{\Delta + 1}.$$

Theorem B [3]**.** *For any graph G of order n and minimum degree δ,*

$$\gamma_R(G) \leq \frac{2 + \ln((1+\delta)/2)}{\delta + 1} n.$$

Theorem C [2]**.** *For any tree T of order $n \geq 3$, $\gamma_R(T) \leq 4n/5$.*

Theorem D [2]**.** *If G is a graph of order $n \geq 3$, then*

$$\gamma_R(G) + \gamma_R(\overline{G}) \leq n + 3.$$

Furthermore, equality holds only when G or $\overline{G}$ is C_5 or $\frac{n}{2}K_2$.

The next two observations are straightforward to verify.

Observation 1. *Let $f = (V_0, V_1, V_2)$ be any γ_R^k-function of a graph G. Then*
(a) $\Delta_k(G[V_1]) \leq 1$.
(b) *If $w \in V_1$, then $N_{k,G}(w) \cap V_2 = \emptyset$.*
(c) *If $u \in V_0$, then $|V_1 \cap N_{k,G}(u)| \leq 2$.*
(d) *V_2 is a γ^k-set of the induced subgraph $G[V_0 \cup V_2]$.*
(e) *Let $H = G[V_0 \cup V_2]$. Then each vertex $v \in V_2$ with $N_{k,G}(v) \cap V_2 \neq \emptyset$ has at least two private neighbors relative to V_2 in the graph H.*

Observation 2. *Let $k \geq 1$ be an integer, and let G be a graph of order $n \geq 2$. If $\operatorname{diam}(G) \leq k$, then $\gamma_R^k(G) = \gamma_R(K_n) = 2$.*

Observation 3. *If $k \geq 1$ is an integer and G is a graph of order n with $\Delta_k(G) \geq 1$, then*

$$\gamma_R^k(G) \geq \frac{2n}{\Delta_k(G) + 1}.$$

Proof. Using the facts $\gamma_R^k(G) = \gamma_R(G^k)$, $\Delta_k(G) = \Delta(G^k)$ and Theorem A, we obtain

$$\gamma_R^k(G) = \gamma_R(G^k) \geq \frac{2n}{\Delta(G^k)+1} = \frac{2n}{\Delta_k(G)+1}.$$

■

Applying Theorem B, we obtain analogously the next result.

Observation 4. *For any graph G of order n,*

$$\gamma_R^k(G) \leq \frac{2+\ln((1+\delta_k(G))/2)}{\delta_k(G)+1}n.$$

Observation 5. *If $k \geq 1$ is an integer and G is a graph of order n with $\Delta_k(G) \geq 1$, then*

$$\gamma_R^k(G) \leq n - \Delta_k(G) + 1.$$

Proof. Let v be a vertex of G such that $\deg_{k,G}(v) = \Delta_k(G)$. Then $f = (N_{k,G}(v), V(G) - N_{k,G}[v], \{v\})$ is a kDRDF on G with weight $n - \Delta_k(G) + 1$ and therefore $\gamma_R^k(G) \leq n - \Delta_k(G) + 1$. ■

Let $k \geq 1$ be an integer, and let H be a graph with $\Delta_k(H) = n(H) - 1 \geq 2$. Now let $G = rK_1 \cup sK_2 \cup H$ for two integers $r, s \geq 0$. Then $\Delta_k(G) = \Delta_k(H)$ and

$$\gamma_R^k(G) = r + 2s + 2 = n(G) - \Delta_k(G) + 1.$$

This family of graphs demonstrates that the uppper bound in Observation 5 is sharp.

Observation 6. *Let $k \geq 1$ be an integer, and let G be a graph of order $n \geq 2$. Then $\gamma_R^k(G) = 2$ if and only if $n = 2$ or $n \geq 3$ and $\Delta_k(G) = n - 1$.*

Proof. Assume first that $n = 2$ or $n \geq 3$ and $\Delta_k(G) = n - 1$. If $n = 2$, then $\gamma_R^k(G) = 2$. If $n \geq 3$ and $\Delta_k(G) = n - 1$, then Observation 5 implies that

$$2 \leq \gamma_R^k(G) \leq n - \Delta_k(G) + 1 = 2$$

and therefore $\gamma_R^k(G) = 2$.

Conversely, assume that $\gamma_R^k(G) = 2$. If $\Delta_k(G) = 0$, then it follows that $n = 2$. If $\Delta_k(G) \geq 1$, then we deduce from Observation 3 that

$$2 = \gamma_R^k(G) \geq \frac{2n}{\Delta_k(G)+1}$$

and hence $\Delta_k(G)+1 \geq n$. This leads to $\Delta_k(G) = n-1$, and the proof is complete.

■

Observation 7. *Let $k \geq 1$ be an integer, and let G be a graph of order n. Then $\gamma_R^k(G) = n$ if and only if $G = rK_1 \cup sK_2$ for some integers $r, s \geq 0$.*

Proof. If $G = rK_1 \cup sK_2$ for some integers $r, s \geq 0$, then obviously $\gamma_R^k(G) = n$.

Conversely, assume that $\gamma_R^k(G) = n$. If $\Delta_k(G) \geq 2$, then Observation 5 leads to the contradiction $\gamma_R^k(G) \leq n-1$. Thus $\Delta_k(G) \leq 1$ and so $G = rK_1 \cup sK_2$ for some integers $r, s \geq 0$. ■

Observation 8. *Let $k \geq 1$ be an integer, and let G be a graph of order $n \geq 4$. Then $\gamma_R^k(G) = 3$ if and only if $\Delta_k(G) = n-2$.*

Proof. Assume first that $\Delta_k(G) = n-2$. Observation 6 implies that $\gamma_R^k(G) \geq 3$. Since we deduce from Observation 5 that $\gamma_R^k(G) \leq n - \Delta_k(G) + 1 = 3$, we obtain $\gamma_R^k(G) = 3$.

Conversely, assume that $\gamma_R^k(G) = 3$. By Observation 6, we have $\Delta_k(G) \leq n-2$. Let now $f = (V_0, V_1, V_2)$ be a $\gamma_R^k(G)$-function. We deduce from the assumption $n \geq 4$ that $|V_1| = |V_2| = 1$. Let $V_2 = \{v\}$ and $V_1 = \{w\}$. Since $\gamma_R^k(G) = 3$, it is obvious that $N_{k,G}[v] = V(G) - \{w\}$ and thus $\Delta_k(G) \geq n-2$. This yields $\Delta_k(G) = n-2$, and the proof is complete. ■

Observation 9. *Let $k \geq 2$ be an integer, and let G be a graph of order $n \geq 3$. Then $\gamma_R^k(G) = n-1$ if and only if $G = K_3 \cup rK_1 \cup sK_2$ or $G = P_3 \cup rK_1 \cup sK_2$ for some integers $r, s \geq 0$.*

Proof. If $G = K_3 \cup rK_1 \cup sK_2$ or $G = P_3 \cup rK_1 \cup sK_2$ for some integers $r, s \geq 0$, then obviously $\gamma_R^k(G) = n-1$.

Conversely, assume that $\gamma_R^k(G) = n-1$. If $\Delta_k(G) \geq 3$, then Observation 5 implies the contradiction $\gamma_R^k(G) \leq n - \Delta_k(G) + 1 \leq n-2$. Therefore $\Delta_k(G) \leq 2$. If $\Delta_k(G) \leq 1$, then we deduce from Observation 7 the contradiction $\gamma_R^k(G) = n$. Consequently, $\Delta_k(G) = 2$. If G contains at least two components H_1 and H_2 with $\Delta_k(H_1) = \Delta_k(H_2) = 2$, then $\gamma_R^k(G) \leq n-2$, a contradiction. Hence G has exactly one component H with $\Delta_k(H) = 2$, and the remaining components are isolated vertices or isomorphic to K_2. If $|V(H)| \geq 4$, then the assumption $k \geq 2$ shows that $\Delta_k(G) = \Delta_k(H) \geq 3$, a contradiction. Hence $|V(H)| = 3$ and so $G = K_3 \cup rK_1 \cup sK_2$ or $G = P_3 \cup rK_1 \cup sK_2$ for some integers $r, s \geq 0$. ■

The proof of the next result is similar to that of Observation 9 and is therefore omitted.

Observation 10. *Let G be a graph of order $n \geq 3$. Then $\gamma_R(G) = n-1$ if and only if $G = H \cup rK_1 \cup sK_2$ for some integers $r, s \geq 0$, where $H \in \{C_3, C_4, C_5, P_3, P_4, P_5\}$.*

Observation 11. *Let $k \geq 3$ be an integer, and let G be a graph of order $n \geq 2$. Then $\gamma_R^k(G) = 2$ or $\gamma_R^k(\overline{G}) = 2$.*

Proof. If $\operatorname{diam}(G) \le 3$, then it follows from Observation 2 that $\gamma_R^k(G) = 2$. If $\operatorname{diam}(G) \ge 4$, then a result of Bondy and Murty [1] (page 14) implies that $\operatorname{diam}(\overline{G}) \le 2$. Applying again Observation 2, we see that $\gamma_R^k(\overline{G}) = 2$. ∎

Observation 12. *Let G be a graph of order $n \ge 2$. If $\operatorname{diam}(G) \ne 3$, then $\gamma_R^2(G) = 2$ or $\gamma_R^2(\overline{G}) = 2$.*

Proof. If $\operatorname{diam}(G) \le 2$, then it follows from Observation 2 that $\gamma_R^2(G) = 2$. If $\operatorname{diam}(G) \ge 3$, then the assumption $\operatorname{diam}(G) \ne 3$ implies that $\operatorname{diam}(G) \ge 4$. As above, we deduce that $\operatorname{diam}(\overline{G}) \le 2$, and Observation 2 leads to $\gamma_R^2(\overline{G}) = 2$. ∎

Observation 13. *Let $k \ge 1$ be an integer, and let G be a graph of order $n \ge 2$. Then $\gamma_R^k(G) = 2\gamma^k(G)$ if and only if G has a $\gamma_R^k(G)$-function $f = (V_0, V_1, V_2)$ with $|V_1| = 0$.*

Proof. Assume first that $\gamma_R^k(G) = 2\gamma^k(G)$. Let S be a k-distance dominating set of G such that $|S| = \gamma^k(G)$. Then $f = (V(G) - S, \emptyset, S) = (V_0, V_1, V_2)$ is a kDRDF on G such that

$$\omega(f) = 2|S| = 2\gamma^k(G) = \gamma_R^k(G)$$

and therefore f is a $\gamma_R^k(G)$-function with $|V_1| = 0$.

Conversely, let $f = (V_0, V_1, V_2)$ be a $\gamma_R^k(G)$-function with $|V_1| = 0$ and thus $\gamma_R^k(G) = 2|V_2|$. Then V_2 is also k-distance dominating set of G, and hence we deduce that $2\gamma^k(G) \le 2|V_2| = \gamma_R^k(G)$. Applying the second inequality in (1), we obtain the identity $\gamma_R^k(G) = 2\gamma^k(G)$, and the proof is complete. ∎

The special case $k = 1$ of Observation 13 can be found in the article [3].

Next we will prove a Nordhaus-Gaddum inequality.

Theorem 14. *Let $k \ge 2$ be an integer, and let G be a graph of order $n \ge 3$. Then*

$$\gamma_R^k(G) + \gamma_R^k(\overline{G}) \le n + 2.$$

Furthermore, equality holds in the bound if and only if G or $\overline{G}$ is isomorphic to $rK_1 \cup sK_2$ for two integers $r, s \ge 0$.

Proof. If neither G nor $\overline{G}$ is isomorphic to C_5 or to $\frac{n}{2}K_2$, then it follows from Theorem D that

$$\gamma_R^k(G) + \gamma_R^k(\overline{G}) \le \gamma_R(G) + \gamma_R(\overline{G}) \le n + 2.$$

If $G = C_5$ or $\overline{G} = C_5$, then $\gamma_R^k(G) + \gamma_R^k(\overline{G}) = 4 < 7 = n + 2$, and if $G = \frac{n}{2}K_2$ or $\overline{G} = \frac{n}{2}K_2$, then $\gamma_R^k(G) + \gamma_R^k(\overline{G}) = n + 2$, and the desired Nordhaus-Gaddum bound is proved.

If G or $\overline{G}$ is isomorphic to $rK_1 \cup sK_2$ for two integers $r, s \geq 0$, then obviously $\gamma_R^k(G) + \gamma_R^k(\overline{G}) = n + 2$.

Next assume that $\gamma_R^k(G) + \gamma_R^k(\overline{G}) = n + 2$. We distinguish two cases.

Case 1. Assume that $k \geq 3$. If $\mathrm{diam}(G) \leq 3$, then $\gamma_R^k(G) = 2$ and therefore $\gamma_R^k(\overline{G}) = n$. According to Observation 7, we observe that $\overline{G} = rK_1 \cup sK_2$ for two integers $r, s \geq 0$. If $\mathrm{diam}(G) \geq 4$, then $\mathrm{diam}(\overline{G}) \leq 2$. It follows that $\gamma_R^k(\overline{G}) = 2$ and thus $\gamma_R^k(G) = n$. Applying again Observation 7, we see that $G = rK_1 \cup sK_2$ for two integers $r, s \geq 0$.

Case 2. Assume that $k = 2$. If $\mathrm{diam}(G) \leq 2$, then $\gamma_R^2(G) = 2$ and therefore $\gamma_R^2(\overline{G}) = n$. According to Observation 7, we observe that $\overline{G} = rK_1 \cup sK_2$ for two integers $r, s \geq 0$. If $\mathrm{diam}(G) \geq 4$, then $\mathrm{diam}(\overline{G}) \leq 2$. It follows that $\gamma_R^2(\overline{G}) = 2$ and thus $\gamma_R^2(G) = n$, and so $G = rK_1 \cup sK_2$ for two integers $r, s \geq 0$. If $\mathrm{diam}(\overline{G}) \leq 2$ or $\mathrm{diam}(\overline{G}) \geq 4$, then we obtain analogously that G or $\overline{G}$ is isomorphic to $rK_1 \cup sK_2$ for two integers $r, s \geq 0$.

There remains the case that $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 3$. Let x and y be two vertices of G such that $d(x, y) = \mathrm{diam}(G) = 3$. Obviously, $f = (V(G) - \{x, y\}, \emptyset, \{x, y\})$ is a 2DRDF on $\overline{G}$, since there is no vertex in G adjacent to both x and y. Therefore $\gamma_R^2(\overline{G}) \leq 4$. Analogously, we obtain $\gamma_R^2(G) \leq 4$ and hence

$$\gamma_R^2(G) + \gamma_R^2(\overline{G}) \leq 8 < n + 2$$

when $n \geq 7$.

Finally, assume that $4 \leq n \leq 6$. If $4 \leq n \leq 5$, then $\Delta_2(G) = \Delta_2(\overline{G}) = n - 1$ and hence $\gamma_R^2(G) = \gamma_R^2(\overline{G}) = 2$ and consequently

$$\gamma_R^2(G) + \gamma_R^2(\overline{G}) = 4 < n + 2.$$

If $n = 6$, then $\Delta_2(G) = \Delta_2(\overline{G}) \geq 4$ and hence $\gamma_R^2(G) \leq 3$ and $\gamma_R^2(\overline{G}) \leq 3$. It follows that

$$\gamma_R^2(G) + \gamma_R^2(\overline{G}) \leq 6 < n + 2,$$

and the proof is complete. ■

3. Bounds on the k-distance Roman Domination Number

Theorem 15. *If $k \geq 1$ is an integer and G a connected graph of order n with $n - \Delta(G) - k \geq 0$, then*

$$\gamma_R^k(G) \leq n - \Delta(G) - k + 2.$$

Proof. Let v be a vertex of G such that $\deg_G(v) = \Delta(G)$. If $d(u, v) \leq k$ for each $u \in V(G)$, then obviously $\gamma_R^k(G) = 2$ and we are done. If $d(w, v) > k$ for some

$w \in V(G)$, then choose a vertex u in G such that $d(u,v) = k+1$. Let P be a shortest (u,v)-path. Then clearly $d(v,z) \le k$ for each $z \in (V(P) - \{u\}) \cup N_G(v)$ and hence $f = ((V(P) - \{u,v\}) \cup N_G(v), V(G) - ((V(P) - \{u\}) \cup N_G(v)), \{v\})$ is a kDRDF on G with weight $n - \Delta(G) - k + 2$ and therefore $\gamma_R^k(G) \le n - \Delta(G) - k + 2$. ■

The special case $k = 1$ of Theorem 15 can be found in [2].

Theorem 16. *If $k \ge 1$ is an integer and G a graph of order n with $\Delta = \Delta(G) \ge 3$, then*

$$\gamma_R^k(G) \ge \frac{2n(\Delta - 2)}{\Delta(\Delta - 1)^k - 2}.$$

Proof. Each vertex $v \in V(G)$ dominates at most Δ vertices at distance 1 from v, at most $\Delta(\Delta - 1)$ vertices at distance 2 from v, at most $\Delta(\Delta - 1)^2$ vertices of at distance 3 from v, and so on. Thus

$$\Delta_k(G) \le \Delta + \Delta(\Delta - 1) + \Delta(\Delta - 1)^2 + \cdots + \Delta(\Delta - 1)^{k-1} = \Delta \cdot \frac{(\Delta - 1)^k - 1}{\Delta - 2}.$$

Applying Observation 3, we obtain the desired lower bound as follows

$$\gamma_R^k(G) \ge \frac{2n}{\Delta_k(G) + 1} \ge \frac{2n}{\Delta \cdot \frac{(\Delta-1)^k - 1}{\Delta - 2} + 1} = \frac{2n(\Delta - 2)}{\Delta(\Delta - 1)^k - 2}.$$

■

In the case that $\Delta(G) = 2$, the proof of Theorem 16 leads to the next lower bound, and Proposition 19 below shows that this bound is sharp.

Theorem 17. *If $k \ge 1$ is an integer and G a graph of order n with $\Delta(G) = 2$, then*

$$\gamma_R^k(G) \ge \frac{2n}{2k + 1}.$$

Theorem 18. *If $k \ge 1$ is an integer and G a connected graph of order $n \ge 2$, then*

$$\gamma_R^k(G) \ge \left\lceil \frac{\mathrm{diam}(G) + 2}{k + 1} \right\rceil.$$

Proof. The statement is obviously true for K_2. Let G be a connected graph of order $n \ge 3$ and $f = (V_0^f, V_1^f, V_2^f)$ be a γ_R^k(G)-function. Suppose that $P = v_1 v_2 \cdots v_{\mathrm{diam}(G)+1}$ is a diametral path in G. This diametral path includes at most two edges from the induced subgraph $G[N[v]]$ for each $v \in V_1^f$ and at most $2k$ edges from the induced subgraph $G[N_{k,G}[v]]$ for each $v \in V_2^f$. Let $E' = \{v_i v_{i+1} \mid 1 \le i \le \mathrm{diam}(G)\} \cap (\bigcup_{v \in V_1^f} E(G[N[v]]) \cup \bigcup_{v \in V_2^f} E(G[N_{k,G}[v]]))$.

Then the diametral path contains at most $|V_2^f|-1$ edges not in E', joining the neighborhoods at distance k of the vertices of V_2^f. Since G is a connected graph of order at least 3, $V_2^f \neq \emptyset$. Hence,

$$\begin{aligned} \operatorname{diam}(G) &\leq 2k|V_2^f| + 2|V_1^f| + (|V_2^f|-1) \\ &= (2k+2)|V_2^f| + (k+1)|V_1^f| - |V_2^f| - 1 - |V_1^f| \\ &\leq (k+1)\gamma_R^k(G) - 2, \end{aligned}$$

and the result follows. ■

The next proposition is straightforward to verify.

Proposition 19. *For $n \geq 3$,*

$$\gamma_R^k(C_n) = \begin{cases} \frac{2n}{2k+1} & n \equiv 0 \pmod{2k+1}, \\ 2\lfloor \frac{n}{2k+1} \rfloor + 1 & n \equiv 1 \pmod{2k+1}, \\ 2\lfloor \frac{n}{2k+1} \rfloor + 2 & \text{otherwise.} \end{cases}$$

Theorem 20 [7]**.** *For a graph G of order n with $g(G) \geq 3$, $\gamma_R(G) \geq \left\lceil \frac{2g(G)}{3} \right\rceil$.*

Theorem 21. *If $k \geq 1$ is an integer and G a connected graph of order $n \geq 2$ and $\infty > g(G) \geq 2k+1$, then*

$$\gamma_R^k(G) \geq \left\lceil \frac{2g(G)}{2k+1} \right\rceil.$$

Proof. By Theorem 20 we may assume that $k \geq 2$. First note that if G is an n-cycle then the result follows from Proposition 19. Now, let C be a cycle of length $g(G)$ in G. Since $k \geq 2$, $g(G) \geq 5$. Then a vertex not in $V(C)$, can dominate at most one vertex of C for otherwise we obtain a cycle of length less than $g(G)$ which is a contradiction. On the other hand, each vertex in $V(C)$ dominates at most $2k+1$ vertex of $V(C)$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R^k(G)$-function. Then obviously, $\gamma_R^k(G) = |V_1| + 2|V_2| \geq |V_1| + 2\frac{g(G)}{2k+1} \geq \frac{2g(G)}{2k+1}$. This leads to the desired bound, and the proof is complete. ■

The special case $k = 1$ of Theorems 18 and 21 can be found in [7].

4. Connected Graphs

Let $k \geq 1$ be an integer. For n-vertex graphs, always $\gamma_R^k(G) \leq n$, with equality when $G = \overline{K_n}$. In this section we prove that $\gamma_R^k(G) \leq 4n/(2k+3)$ when G is a connected n-vertex graph. Since deleting an edge cannot decrease $\gamma_R^k(G)$, it suffices to prove the bound for trees.

A *leaf* of a graph G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. For a vertex v in a rooted tree T, let $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

Theorem 22. *If $k \geq 1$ is an integer and T is a tree of order $n \geq k+2$, then $\gamma_R^k(T) \leq 4n/(2k+3)$.*

Proof. By Theorem C we may assume that $k \geq 2$. The proof is by induction on n. The base step handles trees with few vertices or diameter $2k$ and $2k+1$. If $k+2 \leq n \leq 2k+1$ or $\operatorname{diam}(T) \leq 2k$, then T has a k-distance dominating vertex, and $\gamma_R^k(T) = 2 < 4n/(2k+3)$. If $\operatorname{diam}(T) = 2k+1$, then T has a k-distance dominating set of size 2, which yields $\gamma_R^k(T) \leq 4$. This is sufficiently small for trees with at least $2k+4$ vertices. Let $P = v_1v_2\cdots v_{2k+2}$ be a longest path in T. For $n \in \{2k+2, 2k+3\}$ and $\operatorname{diam}(T) = 2k+1$, we may assume, without loss of generality, that $\deg(v_{2k+1}) = 2$. Then the function $f : V(G) \to \{0,1,2\}$ defined by $f(v_{k+1}) = 2, f(v_{2k+2}) = 1$ and $f(x) = 0$ otherwise, is a kDRDF on G and hence $\gamma_R^k(T) \leq 3$, which is small enough.

Hence we may assume that $\operatorname{diam}(T) \geq 2k+2$. For a subtree T' with n' vertices, where $n' \geq k+2$, the induction hypothesis yields a kDRDF f' of T' with weight at most $\frac{4}{2k+3}n'$. We find a subtree T' such that adding a bit more weight to f' will yield a small enough kDRDF f for T. Let $P = v_1v_2\cdots v_rv_{r+1}\cdots v_{r+k+1}$ be a longest path in T chosen to maximize the $\sum_{j=1}^{k}\deg_T(v_{r+j})$ and let T be rooted in v_1. We consider three cases.

Case 1. $\sum_{j=1}^{k}\deg_T(v_{r+j}) > 2k$. Let $T' = T - T_{v_{r+1}}$. Since $\operatorname{diam}(T) \geq 2k+2$, we have $n' \geq k+2$. Define f on $V(T)$ by letting $f(x) = f'(x)$ except for $f(v_{r+1}) = 2$ and $f(x) = 0$ for each $x \in V(T_{v_{r+1}}) - \{v_{r+1}\}$. Note that f is a kDRDF for T and that

$$w(f) = w(f') + 2 \leq \frac{4(n-k-2)}{2k+3} + 2 < \frac{4n}{2k+3}.$$

Case 2. $\sum_{j=1}^{k}\deg_T(v_{r+j}) = 2k$ and $\deg(v_r) = 2$. Let $T' = T - T_{v_r}$. If $n' = k+1$, then T is a path on $2k+3$ vertices and has a kDRDF of weight 4. Otherwise, the induction hypothesis applies. Define f on $V(T)$ by letting $f(x) = f'(x)$ except for $f(v_{r+1}) = 2$ and $f(v_r) = f(v_{r+2}) = \cdots = f(v_{r+k+1}) = 0$. Again f is a kDRDF, and the computation $w(f) < \frac{4n}{2k+3}$ is the same as in Case 1.

Case 3. $\sum_{j=1}^{k}\deg_T(v_{r+j}) = 2k$ and $\deg(v_r) > 2$. Consider two subcases.

Subcase 3.1. $d(v_r, u) \leq k+1$ for each $u \in V(T)$. Let $S = \{u_1, \ldots, u_t\}$ be the set of vertices in distance $k+1$ from v_r. Obviously $v_1, v_{r+k+1} \in S$ and so $t \geq 2$.

On the other hand, it is clear that $n \geq t(k+1)+1$. Define $f : V(G) \to \{0,1,2\}$ by $f(v_r) = 2, f(u_1) = \cdots = f(u_t) = 1$ and $f(x) = 0$ otherwise. Clearly f is a kDRDF on G and hence

$$\gamma_R^k(T) \leq t+2 \leq \frac{4(t(k+1)+1)}{2k+3} \leq \frac{4n}{2k+3},$$

with equality if and only if $t = 2$ and $n = 2(k+1)+1$, and this if and only if $T = P_{2k+3}$.

Subcase 3.2. $d(v_r, u) \geq k+2$ for some $u \in V(T)$. It is clear that $d(v_1, v_r) \geq k+2$. By the choice of P, $d(v_r, w) \leq k+1$ for each $w \in V(T_{v_r})$. Let T_1 and T_2 be the connected components of $T - v_{r-1}v_r$. Let $f_i = (V_0^{f_i}, V_1^{f_i}, V_2^{f_i})$ be a $\gamma_R^k(T_i)$-function for $i = 1,2$. Obviously $f = (V_0^{f_1} \cup V_0^{f_2}, V_1^{f_1} \cup V_1^{f_2}, V_2^{f_1} \cup V_2^{f_2})$ is a kDRDF of T. By induction hypothesis we obtain

$$(2) \quad \gamma_R^k(T) \leq \gamma_R^k(T_1) + \gamma_R^k(T_2) = \omega(f_1) + \omega(f_2) \leq \frac{4|V(T_1)|}{2k+3} + \frac{4|V(T_2)|}{2k+3} = \frac{4n}{2k+3}.$$

This completes the proof. ■

Let k be a positive integer and let $F_{m,k}$ consist of the disjoint union of m copies of P_{2k+3} plus a path through the central vertices of these copies, as illustrated in Figure 1 for $k = 2$. If $v_1v_2 \ldots v_{2k+3}$ is an induced path in a graph, then a kDRDF must put total weight at least 4 on $\{v_1, v_2, \ldots, v_{2k+3}\}$. In F_m, there are m disjoint induced paths on $2k+3$ vertices, so $\gamma_R^k(T) \geq 4|V(T)|/(2k+3)$ for each $T \in F_m$. Such copies of P_{2k+3} can be assembled in many ways, and this allows us to characterize the trees achieving equality in Theorem 22.

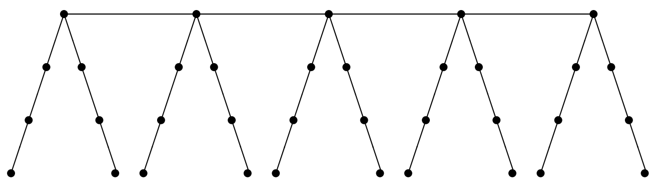

Figure 1. A member of $F_{5,2}$.

Theorem 23. *If $k \geq 1$ is an integer and T is an n-vertex tree, then $\gamma_R^k(T) = 4n/(2k+3)$ if and only if $V(T)$ can be partitioned into sets inducing P_{2k+3} such that the subgraph induced by the central vertices of these paths is connected.*

Proof. We have observed that if an induced subgraph H of G is isomorphic to P_{2k+3}, and its noncentral vertices have no neighbors outside H in G, then every kDRDF of G puts weight at least 4 on $V(H)$. Thus in any tree with such a vertex partition, weight at least 4 is needed on every set in the partition.

To show that equality requires this structure, we examine the proof of Theorem 22 more closely. The proof is by induction on n. In the base cases and Cases 1 and 2, we produce a kDRDF with weight less than $4n/(2k+3)$. In Case 3 and Subcase 3.1 with diameter $2k+2$, equality requires $T = P_{2k+3}$.

Define T_1, T_2 as in the inductive part of Case 3. The bound holds with equality only if $\gamma_R^k(T_1) = \frac{4|V(T_1)|}{2k+3}$ and $\gamma_R^k(T_2) = \frac{4|V(T_2)|}{2k+3}$. It follows from $\gamma_R^k(T_2) = \frac{4|V(T_2)|}{2k+3}$ and the first paragraph in the proof of Theorem 22 that $\mathrm{diam}(T_2) = 2k+2$. Since $d(v_r, u) \leq k+1$ for each $u \in T_2$, from the proof of Subcase 3.1 we deduce that $T_2 = P_{2k+3}$ with central vertex v_r. By induction hypothesis, $V(T)$ can be partitioned into sets inducing P_{2k+3} such that the subgraph induced by the central vertices of these paths is connected. Suppose $\{u_1, \ldots, u_{2k+3}\}$ is the partition set inducing $P_{2k+3} = u_1u_2\cdots u_{2k+3}$ containing v_{r-1}. We claim that $v_{r-1} = u_{k+2}$. Otherwise, we may assume, without loss of generality that, $v_{r-1} \in \{u_{k+3}, \ldots, u_{2k+3}\}$. Define $f : V(G) \to \{0,1,2\}$ by $f(v_r) = f(u_{k+1}) = 2, f(u_{2k+3}) = 1$ and let f assign 2 to all other central vertices and 1 to all other leaves. It is easy to see that f is a kDRDF of T with weight less than $4n/(2k+3)$ which is a contradiction. Thus v_{r-1} is the central vertex of the path $P_{2k+3} = u_1u_2\cdots u_{2k+3}$ and the proof is complete. ■

Theorem 24. *If k is a positive integer and G is a connected n-vertex graph with $n \geq k+2$, then*

$$\gamma_R^k(G) \leq 4n/(2k+3).$$

Moreover, the equality holds if and only if G is C_{2k+3} or obtained from $\frac{n}{2k+3}P_{2k+3}$ by adding a connected subgraph on the set of centers of the components of $\frac{n}{2k+3}$ P_{2k+3}.

Proof. If G has the specified form, then as remarked earlier every kDRDF puts weight at least 4 on the vertex set of each copy of P_{2k+3}.

Now suppose that $\gamma_R^k(G) = \frac{4n}{2k+3}$. Since adding edges cannot increase $\gamma_R^k(G)$, every spanning tree of G has the form specified in Theorem 22. Given a spanning tree T, let $S_1, \ldots, S_k$ be the $(2k+3)$-sets in the special partition of $V(T)$. The assignment of weight 4 that guards S_i can be chosen independently of any other S_j. If any edge of G joins vertices of S_i and S_j that are not the centers of the paths they induce, then a kDRDF with weight less than $\frac{4n}{2k+3}$ can be built as in the proof of Theorem 23. ■

The special case $k = 1$ of Theorems 22, 23 and 24 can be found in [2]. As an application of Theorem 24, we prove the next result.

Corollary 25. *Let $f = (V_0, V_1, V_2)$ be any $\gamma_R^k(G)$-function of a connected graph G of order $n \geq 3$. Then*

(1) $1 \le |V_2| \le \frac{2n}{2k+3}$ *and a graph* G *admits a* $\gamma_R^k(G)$*-function such that* $|V_2| = \frac{2n}{2k+3}$ *if and only if* G *is* C_{2k+3} *or is obtained from* $\frac{n}{2k+3}P_{2k+3}$ *by adding a connected subgraph on the set of centers of the components of* $\frac{n}{2k+3}P_{2k+3}$.

(2) $0 \le |V_1| \le \frac{4n}{2k+3} - 2$.

(3) $n - \frac{4n}{2k+3} + 1 \le |V_0| \le n - 1$.

Proof. (1) If $V_2 = \emptyset$, then $V_1 = V(G)$ and $V_0 = \emptyset$. The RDF $(\emptyset, V(G), \emptyset)$ is not minimum since $|V_1| + 2|V_2| > \frac{4n}{2k+3}$. Hence $|V_2| \ge 1$. On the other hand, $|V_2| \le \frac{2n}{2k+3} - |V_1|/2 \le \frac{2n}{2k+3}$.

Let G admit a $\gamma_R^k(G)$-function $f = (V_0, V_1, V_2)$ such that $|V_2| = \frac{2n}{2k+3}$. Then by Theorem 24, $\frac{4n}{2k+3} \le |V_1| + 2|V_2| = \gamma_R^k(G) \le \frac{4n}{2k+3}$ and the result follows from Theorem 24 again.

Conversely, let G be C_{2k+3} or obtained from $\frac{n}{2k+3}P_{2k+3}$ by adding a connected subgraph on the set of centers of the components of $\frac{n}{2k+3}P_{2k+3}$. If $G = C_{2k+3}$, then assign 2 to two vertices at distance 2 and 0 to the other vertices. If G is obtained from $\frac{n}{2k+3}P_{2k+3}$ by adding a connected subgraph on the set of centers of the components of $\frac{n}{2k+3}P_{2k+3}$, then assign 2 to the neighbors of centers of the components P_{2k+3} and 0 to the other vertices. Obviously f is a $\gamma_R^k(G)$-function with the desired property.

(2) Since $|V_2| \ge 1$, $|V_1| \le \frac{4n}{2k+3} - 2|V_2| \le \frac{4n}{2k+3} - 2$.

(3) The upper bound comes from $|V_0| \le n - |V_2| \le n - 1$. For the lower bound, adding side by side $2|V_0| + 2|V_1| + 2|V_2| = 2n$, $-|V_1| - 2|V_2| \ge \frac{-4n}{2k+3}$ and $|V_1| \le \frac{4n}{2k+3} - 2$ gives $2|V_0| \ge 2n - \frac{8n}{2k+3} + 2$. Therefore $|V_0| \ge n - \frac{4n}{2k+3} + 1$. ■

The special case $k = 1$ of Corollary 25 can be found in [5].

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