*Discussiones Mathematicae*
*Graph Theory 33 (2013) 717–730*
*doi:10.7151/dmgt.1703*

# THE DISTANCE ROMAN DOMINATION NUMBERS OF GRAPHS

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**Abstract**

Let $k$ be a positive integer, and let $G$ be a simple graph with vertex set $V(G)$. A *$k$-distance Roman dominating function* on $G$ is a labeling $f : V(G) \to \{0, 1, 2\}$ such that for every vertex with label 0, there is a vertex with label 2 at distance at most $k$ from each other. The *weight* of a $k$-distance Roman dominating function $f$ is the value $\omega(f) = \sum_{v \in V} f(v)$. The *$k$-distance Roman domination number* of a graph $G$, denoted by $\gamma_R^k(D)$, equals the minimum weight of a $k$-distance Roman dominating function on G. Note that the 1-distance Roman domination number $\gamma_R^1(G)$ is the usual *Roman domination number* $\gamma_R(G)$. In this paper, we investigate properties of the $k$-distance Roman domination number. In particular, we prove that for any connected graph $G$ of order $n \geq k + 2$, $\gamma_R^k(G) \leq 4n/(2k + 3)$ and we characterize all graphs that achieve this bound. Some of our results extend these ones given by Cockayne *et al.* in 2004 and Chambers *et al.* in 2009 for the Roman domination number.

**Keywords:** $k$-distance Roman dominating function, $k$-distance Roman domination number, Roman dominating function, Roman domination number.

**2010 Mathematics Subject Classification:** 05C69.

## 1. Terminology and Introduction

In this paper, $G$ is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Denote by $K_n$ the *complete graph*, by $C_n$ the *cycle* and by $P_n$ the *path* of order $n$, respectively. Given two graphs $G_1$ and $G_2$ such that $V(G_1) \cap V(G_2) = \emptyset$, the *disjoint union* is the graph $G_1 \cup G_2$ with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Let $k$ be a positive integer. For two vertices $x$ and $y$, let $d(x, y)$ denote the distance between $x$ and $y$ in $G$. The *girth* $g(G)$ of a graph $G$ is the length of its shortest cycle. For a vertex $v \in V(G)$, the *open $k$-neighborhood* $N_{k,G}(v)$ is the set $\{u \in V(G) \mid u \neq v \text{ and } d(u, v) \leq k\}$ and the *closed $k$-neighborhood* $N_{k,G}[v]$ is the set $N_{k,G}(v) \cup \{v\}$. The *open $k$-neighborhood* $N_{k,G}(S)$ of a set $S \subseteq V$ is the set $\bigcup_{v \in S} N_{k,G}(v)$, and the *closed-$k$-neighborhood* $N_{k,G}[S]$ of $S$ is the set $N_{k,G}(S) \cup S$. The *$k$-degree* of a vertex $v$ is defined as $\deg_{k,G}(v) = |N_{k,G}(v)|$. The minimum and maximum $k$-degree of a graph $G$ are denoted by $\delta_k(G)$ and $\Delta_k(G)$, respectively. If $\delta_k(G) = \Delta_k(G)$, then the graph $G$ is called *distance-$k$-regular*. The *$k$-th power* $G^k$ of a graph $G$ is the graph with vertex set $V(G)$ where two different vertices $u$ and $v$ are adjacent if and only if the distance $d(u, v)$ is at most $k$ in $G$. Now we observe that $N_{k,G}(v) = N_{1,G^k}(v) = N_{G^k}(v)$, $N_{k,G}[v] = N_{1,G^k}[v] = N_{G^k}[v]$, $\deg_{k,G}(v) = \deg_{1,G^k}(v) = \deg_{G^k}(v)$, $\delta_k(G) = \delta_1(G^k) = \delta(G^k)$ and $\Delta_k(G) = \Delta_1(G^k) = \Delta(G^k)$. Consult [6, 10] for the notation and terminology which are not defined here.

Let $k \geq 1$ be an integer. A set $D \subseteq V(G)$ is a $k$-distance dominating set of $G$ if every vertex in $V(G) - D$ is within distance $k$ of at least one vertex in $D$. The $k$-distance domination number $\gamma^k(G)$ of $G$ is the minimum cardinality among all $k$-distance dominating sets of $G$.

A *$k$-distance Roman dominating function* ($k$DRDF) on a graph $G = (V, E)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that for every vertex $v$ for which $f(v) = 0$, there is a vertex $u$ for which $f(u) = 2$ and $d(u, v) \leq k$. The *weight* of a $k$DRDF $f$ is the value $\omega(f) = \sum_{v \in V} f(v)$. The *$k$-distance Roman domination number* of a graph $G$, denoted by $\gamma_R^k(G)$, equals the minimum weight of a $k$DRDF on $G$. A *$\gamma_R^k(G)$-function* is a $k$-distance Roman dominating function of $G$ with weight $\gamma_R^k(G)$. A $k$-distance Roman dominating function $f : V \longrightarrow \{0, 1, 2\}$ can be represented by the ordered partition $(V_0, V_1, V_2)$ (or $(V_0^f, V_1^f, V_2^f)$ to refer $f$) of $V$, where $V_i = \{v \in V \mid f(v) = i\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2|$. Since $V_1^f \cup V_2^f$ is a $k$-distance dominating set when $f$ is a $k$DRDF, and since placing weight 2 at the vertices of a $k$-distance dominating set yields a $k$DRDF, we have

$$\gamma^k(G) \leq \gamma_R^k(G) \leq 2\gamma^k(G). \tag{1}$$

Note that the 1-distance Roman domination number $\gamma_R^1(G)$ is the usual *Roman domination number* $\gamma_R(G)$. The definition of the Roman dominating function was

given multiplicity by Steward [9] and ReVelle and Rosing [8]. Cockayne *et al.* [3] as well as Chambers *et al.* [2] have given a lot of results on Roman domination.

Our purpose in this paper is to initiate the study of the $k$-distance Roman domination number and establish some bounds for the $k$-distance Roman domination number of a graph. Some of our results extend many well-known results.

## 2. Some Basic Results

We start with some known results on the classical Roman domination number.

**Theorem A** [4]**.** *For any graph $G$ of order $n$ and maximum degree $\Delta \geq 1$,*

$$\gamma_R(G) \geq \frac{2n}{\Delta + 1}.$$

**Theorem B** [3]**.** *For any graph $G$ of order $n$ and minimum degree $\delta$,*

$$\gamma_R(G) \leq \frac{2 + \ln((1+\delta)/2)}{\delta + 1} n.$$

**Theorem C** [2]**.** *For any tree $T$ of order $n \geq 3$, $\gamma_R(T) \leq 4n/5$.*

**Theorem D** [2]**.** *If $G$ is a graph of order $n \geq 3$, then*

$$\gamma_R(G) + \gamma_R(\overline{G}) \leq n + 3.$$

*Furthermore, equality holds only when $G$ or $\overline{G}$ is $C_5$ or $\frac{n}{2}K_2$.*

The next two observations are straightforward to verify.

**Observation 1.** *Let $f = (V_0, V_1, V_2)$ be any $\gamma_R^k$-function of a graph $G$. Then*

(a) $\Delta_k(G[V_1]) \leq 1$.

(b) *If $w \in V_1$, then $N_{k,G}(w) \cap V_2 = \emptyset$.*

(c) *If $u \in V_0$, then $|V_1 \cap N_{k,G}(u)| \leq 2$.*

(d) *$V_2$ is a $\gamma^k$-set of the induced subgraph $G[V_0 \cup V_2]$.*

(e) *Let $H = G[V_0 \cup V_2]$. Then each vertex $v \in V_2$ with $N_{k,G}(v) \cap V_2 \neq \emptyset$ has at least two private neighbors relative to $V_2$ in the graph $H$.*

**Observation 2.** *Let $k \geq 1$ be an integer, and let $G$ be a graph of order $n \geq 2$. If $\mathrm{diam}(G) \leq k$, then $\gamma_R^k(G) = \gamma_R(K_n) = 2$.*

**Observation 3.** *If $k \geq 1$ is an integer and $G$ is a graph of order $n$ with $\Delta_k(G) \geq 1$, then*

$$\gamma_R^k(G) \geq \frac{2n}{\Delta_k(G) + 1}.$$

***Proof.*** Using the facts $\gamma_R^k(G) = \gamma_R(G^k)$, $\Delta_k(G) = \Delta(G^k)$ and Theorem A, we obtain

$$\gamma_R^k(G) = \gamma_R(G^k) \geq \frac{2n}{\Delta(G^k)+1} = \frac{2n}{\Delta_k(G)+1}.$$

■

Applying Theorem B, we obtain analogously the next result.

**Observation 4.** *For any graph $G$ of order $n$,*

$$\gamma_R^k(G) \leq \frac{2+\ln((1+\delta_k(G))/2)}{\delta_k(G)+1}n.$$

**Observation 5.** *If $k \geq 1$ is an integer and $G$ is a graph of order $n$ with $\Delta_k(G) \geq 1$, then*

$$\gamma_R^k(G) \leq n - \Delta_k(G) + 1.$$

***Proof.*** Let $v$ be a vertex of $G$ such that $\deg_{k,G}(v) = \Delta_k(G)$. Then $f = (N_{k,G}(v), V(G) - N_{k,G}[v], \{v\})$ is a $k$DRDF on $G$ with weight $n - \Delta_k(G) + 1$ and therefore $\gamma_R^k(G) \leq n - \Delta_k(G) + 1$. ■

Let $k \geq 1$ be an integer, and let $H$ be a graph with $\Delta_k(H) = n(H) - 1 \geq 2$. Now let $G = rK_1 \cup sK_2 \cup H$ for two integers $r, s \geq 0$. Then $\Delta_k(G) = \Delta_k(H)$ and

$$\gamma_R^k(G) = r + 2s + 2 = n(G) - \Delta_k(G) + 1.$$

This family of graphs demonstrates that the uppper bound in Observation 5 is sharp.

**Observation 6.** *Let $k \geq 1$ be an integer, and let $G$ be a graph of order $n \geq 2$. Then $\gamma_R^k(G) = 2$ if and only if $n = 2$ or $n \geq 3$ and $\Delta_k(G) = n - 1$.*

***Proof.*** Assume first that $n = 2$ or $n \geq 3$ and $\Delta_k(G) = n - 1$. If $n = 2$, then $\gamma_R^k(G) = 2$. If $n \geq 3$ and $\Delta_k(G) = n - 1$, then Observation 5 implies that

$$2 \leq \gamma_R^k(G) \leq n - \Delta_k(G) + 1 = 2$$

and therefore $\gamma_R^k(G) = 2$.

Conversely, assume that $\gamma_R^k(G) = 2$. If $\Delta_k(G) = 0$, then it follows that $n = 2$. If $\Delta_k(G) \geq 1$, then we deduce from Observation 3 that

$$2 = \gamma_R^k(G) \geq \frac{2n}{\Delta_k(G)+1}$$

and hence $\Delta_k(G)+1 \geq n$. This leads to $\Delta_k(G) = n-1$, and the proof is complete.

■

**Observation 7.** *Let $k \geq 1$ be an integer, and let $G$ be a graph of order $n$. Then $\gamma_R^k(G) = n$ if and only if $G = rK_1 \cup sK_2$ for some integers $r, s \geq 0$.*

***Proof.*** If $G = rK_1 \cup sK_2$ for some integers $r, s \geq 0$, then obviously $\gamma_R^k(G) = n$.

Conversely, assume that $\gamma_R^k(G) = n$. If $\Delta_k(G) \geq 2$, then Observation 5 leads to the contradiction $\gamma_R^k(G) \leq n - 1$. Thus $\Delta_k(G) \leq 1$ and so $G = rK_1 \cup sK_2$ for some integers $r, s \geq 0$. ■

**Observation 8.** *Let $k \geq 1$ be an integer, and let $G$ be a graph of order $n \geq 4$. Then $\gamma_R^k(G) = 3$ if and only if $\Delta_k(G) = n - 2$.*

***Proof.*** Assume first that $\Delta_k(G) = n-2$. Observation 6 implies that $\gamma_R^k(G) \geq 3$. Since we deduce from Observation 5 that $\gamma_R^k(G) \leq n - \Delta_k(G) + 1 = 3$, we obtain $\gamma_R^k(G) = 3$.

Conversely, assume that $\gamma_R^k(G) = 3$. By Observation 6, we have $\Delta_k(G) \leq n - 2$. Let now $f = (V_0, V_1, V_2)$ be a $\gamma_R^k(G)$-function. We deduce from the assumption $n \geq 4$ that $|V_1| = |V_2| = 1$. Let $V_2 = \{v\}$ and $V_1 = \{w\}$. Since $\gamma_R^k(G) = 3$, it is obvious that $N_{k,G}[v] = V(G) - \{w\}$ and thus $\Delta_k(G) \geq n - 2$. This yields $\Delta_k(G) = n - 2$, and the proof is complete. ■

**Observation 9.** *Let $k \geq 2$ be an integer, and let $G$ be a graph of order $n \geq 3$. Then $\gamma_R^k(G) = n - 1$ if and only if $G = K_3 \cup rK_1 \cup sK_2$ or $G = P_3 \cup rK_1 \cup sK_2$ for some integers $r, s \geq 0$.*

***Proof.*** If $G = K_3 \cup rK_1 \cup sK_2$ or $G = P_3 \cup rK_1 \cup sK_2$ for some integers $r, s \geq 0$, then obviously $\gamma_R^k(G) = n - 1$.

Conversely, assume that $\gamma_R^k(G) = n - 1$. If $\Delta_k(G) \geq 3$, then Observation 5 implies the contradiction $\gamma_R^k(G) \leq n - \Delta_k(G) + 1 \leq n - 2$. Therefore $\Delta_k(G) \leq 2$. If $\Delta_k(G) \leq 1$, then we deduce from Observation 7 the contradiction $\gamma_R^k(G) = n$. Consequently, $\Delta_k(G) = 2$. If $G$ contains at least two components $H_1$ and $H_2$ with $\Delta_k(H_1) = \Delta_k(H_2) = 2$, then $\gamma_R^k(G) \leq n - 2$, a contradiction. Hence $G$ has exactly one component $H$ with $\Delta_k(H) = 2$, and the remaining components are isolated vertices or isomorphic to $K_2$. If $|V(H)| \geq 4$, then the assumption $k \geq 2$ shows that $\Delta_k(G) = \Delta_k(H) \geq 3$, a contradiction. Hence $|V(H)| = 3$ and so $G = K_3 \cup rK_1 \cup sK_2$ or $G = P_3 \cup rK_1 \cup sK_2$ for some integers $r, s \geq 0$. ■

The proof of the next result is similar to that of Observation 9 and is therefore omitted.

**Observation 10.** *Let $G$ be a graph of order $n \geq 3$. Then $\gamma_R(G) = n - 1$ if and only if $G = H \cup rK_1 \cup sK_2$ for some integers $r, s \geq 0$, where $H \in \{C_3, C_4, C_5, P_3, P_4, P_5\}$.*

**Observation 11.** *Let $k \geq 3$ be an integer, and let $G$ be a graph of order $n \geq 2$. Then $\gamma_R^k(G) = 2$ or $\gamma_R^k(\overline{G}) = 2$.*

***Proof.*** If $\mathrm{diam}(G) \leq 3$, then it follows from Observation 2 that $\gamma_R^k(G) = 2$. If $\mathrm{diam}(G) \geq 4$, then a result of Bondy and Murty [1] (page 14) implies that $\mathrm{diam}(\overline{G}) \leq 2$. Applying again Observation 2, we see that $\gamma_R^k(\overline{G}) = 2$. ■

**Observation 12.** *Let $G$ be a graph of order $n \geq 2$. If $\mathrm{diam}(G) \neq 3$, then $\gamma_R^2(G) = 2$ or $\gamma_R^2(\overline{G}) = 2$.*

***Proof.*** If $\mathrm{diam}(G) \leq 2$, then it follows from Observation 2 that $\gamma_R^2(G) = 2$. If $\mathrm{diam}(G) \geq 3$, then the assumption $\mathrm{diam}(G) \neq 3$ implies that $\mathrm{diam}(G) \geq 4$. As above, we deduce that $\mathrm{diam}(\overline{G}) \leq 2$, and Observation 2 leads to $\gamma_R^2(\overline{G}) = 2$. ■

**Observation 13.** *Let $k \geq 1$ be an integer, and let $G$ be a graph of order $n \geq 2$. Then $\gamma_R^k(G) = 2\gamma^k(G)$ if and only if $G$ has a $\gamma_R^k(G)$-function $f = (V_0, V_1, V_2)$ with $|V_1| = 0$.*

***Proof.*** Assume first that $\gamma_R^k(G) = 2\gamma^k(G)$. Let $S$ be a $k$-distance dominating set of $G$ such that $|S| = \gamma^k(G)$. Then $f = (V(G) - S, \emptyset, S) = (V_0, V_1, V_2)$ is a $k$DRDF on $G$ such that

$$\omega(f) = 2|S| = 2\gamma^k(G) = \gamma_R^k(G)$$

and therefore $f$ is a $\gamma_R^k(G)$-function with $|V_1| = 0$.

Conversely, let $f = (V_0, V_1, V_2)$ be a $\gamma_R^k(G)$-function with $|V_1| = 0$ and thus $\gamma_R^k(G) = 2|V_2|$. Then $V_2$ is also $k$-distance dominating set of $G$, and hence we deduce that $2\gamma^k(G) \leq 2|V_2| = \gamma_R^k(G)$. Applying the second inequality in (1), we obtain the identity $\gamma_R^k(G) = 2\gamma^k(G)$, and the proof is complete. ■

The special case $k = 1$ of Observation 13 can be found in the article [3].

Next we will prove a Nordhaus-Gaddum inequality.

**Theorem 14.** *Let $k \geq 2$ be an integer, and let $G$ be a graph of order $n \geq 3$. Then*

$$\gamma_R^k(G) + \gamma_R^k(\overline{G}) \leq n + 2.$$

*Furthermore, equality holds in the bound if and only if $G$ or $\overline{G}$ is isomorphic to $rK_1 \cup sK_2$ for two integers $r, s \geq 0$.*

***Proof.*** If neither $G$ nor $\overline{G}$ is isomorphic to $C_5$ or to $\frac{n}{2}K_2$, then it follows from Theorem D that

$$\gamma_R^k(G) + \gamma_R^k(\overline{G}) \leq \gamma_R(G) + \gamma_R(\overline{G}) \leq n + 2.$$

If $G = C_5$ or $\overline{G} = C_5$, then $\gamma_R^k(G) + \gamma_R^k(\overline{G}) = 4 < 7 = n + 2$, and if $G = \frac{n}{2}K_2$ or $\overline{G} = \frac{n}{2}K_2$, then $\gamma_R^k(G) + \gamma_R^k(\overline{G}) = n + 2$, and the desired Nordhaus-Gaddum bound is proved.

If $G$ or $\overline{G}$ is isomorphic to $rK_1 \cup sK_2$ for two integers $r, s \geq 0$, then obviously $\gamma_R^k(G) + \gamma_R^k(\overline{G}) = n + 2$.

Next assume that $\gamma_R^k(G) + \gamma_R^k(\overline{G}) = n + 2$. We distinguish two cases.

*Case* 1. Assume that $k \geq 3$. If $\mathrm{diam}(G) \leq 3$, then $\gamma_R^k(G) = 2$ and therefore $\gamma_R^k(\overline{G}) = n$. According to Observation 7, we observe that $\overline{G} = rK_1 \cup sK_2$ for two integers $r, s \geq 0$. If $\mathrm{diam}(G) \geq 4$, then $\mathrm{diam}(\overline{G}) \leq 2$. It follows that $\gamma_R^k(\overline{G}) = 2$ and thus $\gamma_R^k(G) = n$. Applying again Observation 7, we see that $G = rK_1 \cup sK_2$ for two integers $r, s \geq 0$.

*Case* 2. Assume that $k = 2$. If $\mathrm{diam}(G) \leq 2$, then $\gamma_R^2(G) = 2$ and therefore $\gamma_R^2(\overline{G}) = n$. According to Observation 7, we observe that $\overline{G} = rK_1 \cup sK_2$ for two integers $r, s \geq 0$. If $\mathrm{diam}(G) \geq 4$, then $\mathrm{diam}(\overline{G}) \leq 2$. It follows that $\gamma_R^2(\overline{G}) = 2$ and thus $\gamma_R^2(G) = n$, and so $G = rK_1 \cup sK_2$ for two integers $r, s \geq 0$. If $\mathrm{diam}(\overline{G}) \leq 2$ or $\mathrm{diam}(\overline{G}) \geq 4$, then we obtain analogously that $G$ or $\overline{G}$ is isomorphic to $rK_1 \cup sK_2$ for two integers $r, s \geq 0$.

There remains the case that $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 3$. Let $x$ and $y$ be two vertices of $G$ such that $d(x, y) = \mathrm{diam}(G) = 3$. Obviously, $f = (V(G) - \{x, y\}, \emptyset, \{x, y\})$ is a 2DRDF on $\overline{G}$, since there is no vertex in $G$ adjacent to both $x$ and $y$. Therefore $\gamma_R^2(\overline{G}) \leq 4$. Analogously, we obtain $\gamma_R^2(G) \leq 4$ and hence

$$\gamma_R^2(G) + \gamma_R^2(\overline{G}) \leq 8 < n + 2$$

when $n \geq 7$.

Finally, assume that $4 \leq n \leq 6$. If $4 \leq n \leq 5$, then $\Delta_2(G) = \Delta_2(\overline{G}) = n - 1$ and hence $\gamma_R^2(G) = \gamma_R^2(\overline{G}) = 2$ and consequently

$$\gamma_R^2(G) + \gamma_R^2(\overline{G}) = 4 < n + 2.$$

If $n = 6$, then $\Delta_2(G) = \Delta_2(\overline{G}) \geq 4$ and hence $\gamma_R^2(G) \leq 3$ and $\gamma_R^2(\overline{G}) \leq 3$. It follows that

$$\gamma_R^2(G) + \gamma_R^2(\overline{G}) \leq 6 < n + 2,$$

and the proof is complete. ■

## 3. Bounds on the $k$-distance Roman Domination Number

**Theorem 15.** *If $k \geq 1$ is an integer and $G$ a connected graph of order $n$ with $n - \Delta(G) - k \geq 0$, then*

$$\gamma_R^k(G) \leq n - \Delta(G) - k + 2.$$

***Proof.*** Let $v$ be a vertex of $G$ such that $\deg_G(v) = \Delta(G)$. If $d(u, v) \leq k$ for each $u \in V(G)$, then obviously $\gamma_R^k(G) = 2$ and we are done. If $d(w, v) > k$ for some

$w \in V(G)$, then choose a vertex $u$ in $G$ such that $d(u,v) = k+1$. Let $P$ be a shortest $(u,v)$-path. Then clearly $d(v,z) \leq k$ for each $z \in (V(P) - \{u\}) \cup N_G(v)$ and hence $f = ((V(P) - \{u,v\}) \cup N_G(v), V(G) - ((V(P) - \{u\}) \cup N_G(v)), \{v\})$ is a $k$DRDF on $G$ with weight $n - \Delta(G) - k + 2$ and therefore $\gamma_R^k(G) \leq n - \Delta(G) - k + 2$. ■

The special case $k = 1$ of Theorem 15 can be found in [2].

**Theorem 16.** *If $k \geq 1$ is an integer and $G$ a graph of order $n$ with $\Delta = \Delta(G) \geq 3$, then*

$$\gamma_R^k(G) \geq \frac{2n(\Delta - 2)}{\Delta(\Delta - 1)^k - 2}.$$

***Proof.*** Each vertex $v \in V(G)$ dominates at most $\Delta$ vertices at distance 1 from $v$, at most $\Delta(\Delta - 1)$ vertices at distance 2 from $v$, at most $\Delta(\Delta - 1)^2$ vertices of at distance 3 from $v$, and so on. Thus

$$\Delta_k(G) \leq \Delta + \Delta(\Delta - 1) + \Delta(\Delta - 1)^2 + \cdots + \Delta(\Delta - 1)^{k-1} = \Delta \cdot \frac{(\Delta - 1)^k - 1}{\Delta - 2}.$$

Applying Observation 3, we obtain the desired lower bound as follows

$$\gamma_R^k(G) \geq \frac{2n}{\Delta_k(G) + 1} \geq \frac{2n}{\Delta \cdot \frac{(\Delta-1)^k - 1}{\Delta - 2} + 1} = \frac{2n(\Delta - 2)}{\Delta(\Delta - 1)^k - 2}.$$

■

In the case that $\Delta(G) = 2$, the proof of Theorem 16 leads to the next lower bound, and Proposition 19 below shows that this bound is sharp.

**Theorem 17.** *If $k \geq 1$ is an integer and $G$ a graph of order $n$ with $\Delta(G) = 2$, then*

$$\gamma_R^k(G) \geq \frac{2n}{2k + 1}.$$

**Theorem 18.** *If $k \geq 1$ is an integer and $G$ a connected graph of order $n \geq 2$, then*

$$\gamma_R^k(G) \geq \left\lceil \frac{\mathrm{diam}(G) + 2}{k + 1} \right\rceil.$$

***Proof.*** The statement is obviously true for $K_2$. Let $G$ be a connected graph of order $n \geq 3$ and $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R^k$(G)-function. Suppose that $P = v_1 v_2 \cdots v_{\mathrm{diam}(G)+1}$ is a diametral path in $G$. This diametral path includes at most two edges from the induced subgraph $G[N[v]]$ for each $v \in V_1^f$ and at most $2k$ edges from the induced subgraph $G[N_{k,G}[v]]$ for each $v \in V_2^f$. Let $E' = \{v_i v_{i+1} \mid 1 \leq i \leq \mathrm{diam}(G)\} \cap (\bigcup_{v \in V_1^f} E(G[N[v]]) \cup \bigcup_{v \in V_2^f} E(G[N_{k,G}[v]]))$.

Then the diametral path contains at most $|V_2^f| - 1$ edges not in $E'$, joining the neighborhoods at distance $k$ of the vertices of $V_2^f$. Since $G$ is a connected graph of order at least 3, $V_2^f \neq \emptyset$. Hence,

$$\begin{aligned} \operatorname{diam}(G) &\leq 2k|V_2^f| + 2|V_1^f| + (|V_2^f| - 1) \\ &= (2k+2)|V_2^f| + (k+1)|V_1^f| - |V_2^f| - 1 - |V_1^f| \\ &\leq (k+1)\gamma_R^k(G) - 2, \end{aligned}$$

and the result follows. ■

The next proposition is straightforward to verify.

**Proposition 19.** *For $n \geq 3$,*

$$\gamma_R^k(C_n) = \begin{cases} \frac{2n}{2k+1} & n \equiv 0 \pmod{2k+1}, \\ 2\lfloor \frac{n}{2k+1} \rfloor + 1 & n \equiv 1 \pmod{2k+1}, \\ 2\lfloor \frac{n}{2k+1} \rfloor + 2 & \text{otherwise.} \end{cases}$$

**Theorem 20** [7]**.** *For a graph $G$ of order $n$ with $g(G) \geq 3$, $\gamma_R(G) \geq \left\lceil \frac{2g(G)}{3} \right\rceil$.*

**Theorem 21.** *If $k \geq 1$ is an integer and $G$ a connected graph of order $n \geq 2$ and $\infty > g(G) \geq 2k+1$, then*

$$\gamma_R^k(G) \geq \left\lceil \frac{2g(G)}{2k+1} \right\rceil.$$

***Proof.*** By Theorem 20 we may assume that $k \geq 2$. First note that if $G$ is an $n$-cycle then the result follows from Proposition 19. Now, let $C$ be a cycle of length $g(G)$ in $G$. Since $k \geq 2$, $g(G) \geq 5$. Then a vertex not in $V(C)$, can dominate at most one vertex of $C$ for otherwise we obtain a cycle of length less than $g(G)$ which is a contradiction. On the other hand, each vertex in $V(C)$ dominates at most $2k+1$ vertex of $V(C)$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R^k(G)$-function. Then obviously, $\gamma_R^k(G) = |V_1| + 2|V_2| \geq |V_1| + 2\frac{g(G)}{2k+1} \geq \frac{2g(G)}{2k+1}$. This leads to the desired bound, and the proof is complete. ■

The special case $k = 1$ of Theorems 18 and 21 can be found in [7].

## 4. CONNECTED GRAPHS

Let $k \geq 1$ be an integer. For $n$-vertex graphs, always $\gamma_R^k(G) \leq n$, with equality when $G = \overline{K_n}$. In this section we prove that $\gamma_R^k(G) \leq 4n/(2k+3)$ when $G$ is a connected $n$-vertex graph. Since deleting an edge cannot decrease $\gamma_R^k(G)$, it suffices to prove the bound for trees.

A *leaf* of a graph $G$ is a vertex of degree 1, while a *support vertex* of $G$ is a vertex adjacent to a leaf. For a vertex $v$ in a rooted tree $T$, let $D(v)$ denote the set of descendants of $v$ and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$.

**Theorem 22.** *If $k \geq 1$ is an integer and $T$ is a tree of order $n \geq k+2$, then $\gamma_R^k(T) \leq 4n/(2k+3)$.*

***Proof.*** By Theorem C we may assume that $k \geq 2$. The proof is by induction on $n$. The base step handles trees with few vertices or diameter $2k$ and $2k+1$. If $k+2 \leq n \leq 2k+1$ or $\mathrm{diam}(T) \leq 2k$, then $T$ has a $k$-distance dominating vertex, and $\gamma_R^k(T) = 2 < 4n/(2k+3)$. If $\mathrm{diam}(T) = 2k+1$, then $T$ has a $k$-distance dominating set of size 2, which yields $\gamma_R^k(T) \leq 4$. This is sufficiently small for trees with at least $2k+4$ vertices. Let $P = v_1v_2 \cdots v_{2k+2}$ be a longest path in $T$. For $n \in \{2k+2, 2k+3\}$ and $\mathrm{diam}(T) = 2k+1$, we may assume, without loss of generality, that $\deg(v_{2k+1}) = 2$. Then the function $f : V(G) \to \{0,1,2\}$ defined by $f(v_{k+1}) = 2, f(v_{2k+2}) = 1$ and $f(x) = 0$ otherwise, is a $k$DRDF on $G$ and hence $\gamma_R^k(T) \leq 3$, which is small enough.

Hence we may assume that $\mathrm{diam}(T) \geq 2k+2$. For a subtree $T'$ with $n'$ vertices, where $n' \geq k+2$, the induction hypothesis yields a $k$DRDF $f'$ of $T'$ with weight at most $\frac{4}{2k+3}n'$. We find a subtree $T'$ such that adding a bit more weight to $f'$ will yield a small enough $k$DRDF $f$ for $T$. Let $P = v_1v_2 \cdots v_rv_{r+1} \cdots v_{r+k+1}$ be a longest path in $T$ chosen to maximize the $\sum_{j=1}^{k} \deg_T(v_{r+j})$ and let $T$ be rooted in $v_1$. We consider three cases.

*Case* 1. $\sum_{j=1}^{k} \deg_T(v_{r+j}) > 2k$. Let $T' = T - T_{v_{r+1}}$. Since $\mathrm{diam}(T) \geq 2k+2$, we have $n' \geq k+2$. Define $f$ on $V(T)$ by letting $f(x) = f'(x)$ except for $f(v_{r+1}) = 2$ and $f(x) = 0$ for each $x \in V(T_{v_{r+1}}) - \{v_{r+1}\}$. Note that $f$ is a $k$DRDF for $T$ and that

$$w(f) = w(f') + 2 \leq \frac{4(n-k-2)}{2k+3} + 2 < \frac{4n}{2k+3}.$$

*Case* 2. $\sum_{j=1}^{k} \deg_T(v_{r+j}) = 2k$ and $\deg(v_r) = 2$. Let $T' = T - T_{v_r}$. If $n' = k+1$, then $T$ is a path on $2k+3$ vertices and has a $k$DRDF of weight 4. Otherwise, the induction hypothesis applies. Define $f$ on $V(T)$ by letting $f(x) = f'(x)$ except for $f(v_{r+1}) = 2$ and $f(v_r) = f(v_{r+2}) = \cdots = f(v_{r+k+1}) = 0$. Again $f$ is a $k$DRDF, and the computation $w(f) < \frac{4n}{2k+3}$ is the same as in Case 1.

*Case* 3. $\sum_{j=1}^{k} \deg_T(v_{r+j}) = 2k$ and $\deg(v_r) > 2$. Consider two subcases.

*Subcase* 3.1. $d(v_r, u) \leq k+1$ for each $u \in V(T)$. Let $S = \{u_1, \ldots, u_t\}$ be the set of vertices in distance $k+1$ from $v_r$. Obviously $v_1, v_{r+k+1} \in S$ and so $t \geq 2$.

On the other hand, it is clear that $n \geq t(k+1)+1$. Define $f : V(G) \to \{0,1,2\}$ by $f(v_r) = 2, f(u_1) = \cdots = f(u_t) = 1$ and $f(x) = 0$ otherwise. Clearly $f$ is a $k$DRDF on $G$ and hence

$$\gamma_R^k(T) \leq t+2 \leq \frac{4(t(k+1)+1)}{2k+3} \leq \frac{4n}{2k+3},$$

with equality if and only if $t = 2$ and $n = 2(k+1)+1$, and this if and only if $T = P_{2k+3}$.

*Subcase* 3.2. $d(v_r, u) \geq k+2$ for some $u \in V(T)$. It is clear that $d(v_1, v_r) \geq k+2$. By the choice of $P$, $d(v_r, w) \leq k+1$ for each $w \in V(T_{v_r})$. Let $T_1$ and $T_2$ be the connected components of $T - v_{r-1}v_r$. Let $f_i = (V_0^{f_i}, V_1^{f_i}, V_2^{f_i})$ be a $\gamma_R^k(T_i)$-function for $i = 1, 2$. Obviously $f = (V_0^{f_1} \cup V_0^{f_2}, V_1^{f_1} \cup V_1^{f_2}, V_2^{f_1} \cup V_2^{f_2})$ is a $k$DRDF of $T$. By induction hypothesis we obtain

$$\gamma_R^k(T) \leq \gamma_R^k(T_1) + \gamma_R^k(T_2) = \omega(f_1) + \omega(f_2) \leq \frac{4|V(T_1)|}{2k+3} + \frac{4|V(T_2)|}{2k+3} = \frac{4n}{2k+3}. \tag{2}$$

This completes the proof. ■

Let $k$ be a positive integer and let $F_{m,k}$ consist of the disjoint union of $m$ copies of $P_{2k+3}$ plus a path through the central vertices of these copies, as illustrated in Figure 1 for $k = 2$. If $v_1 v_2 \ldots v_{2k+3}$ is an induced path in a graph, then a $k$DRDF must put total weight at least 4 on $\{v_1, v_2, \ldots, v_{2k+3}\}$. In $F_m$, there are $m$ disjoint induced paths on $2k+3$ vertices, so $\gamma_R^k(T) \geq 4|V(T)|/(2k+3)$ for each $T \in F_m$. Such copies of $P_{2k+3}$ can be assembled in many ways, and this allows us to characterize the trees achieving equality in Theorem 22.

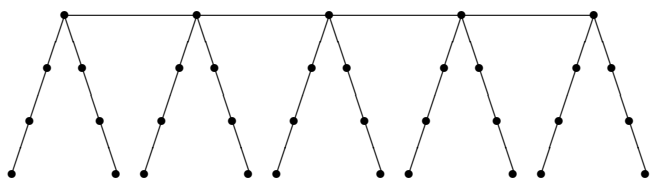

Figure 1. A member of $F_{5,2}$.

**Theorem 23.** *If $k \geq 1$ is an integer and $T$ is an $n$-vertex tree, then $\gamma_R^k(T) = 4n/(2k+3)$ if and only if $V(T)$ can be partitioned into sets inducing $P_{2k+3}$ such that the subgraph induced by the central vertices of these paths is connected.*

***Proof.*** We have observed that if an induced subgraph $H$ of $G$ is isomorphic to $P_{2k+3}$, and its noncentral vertices have no neighbors outside $H$ in $G$, then every $k$DRDF of $G$ puts weight at least 4 on $V(H)$. Thus in any tree with such a vertex partition, weight at least 4 is needed on every set in the partition.

To show that equality requires this structure, we examine the proof of Theorem 22 more closely. The proof is by induction on $n$. In the base cases and Cases 1 and 2, we produce a $k$DRDF with weight less than $4n/(2k+3)$. In Case 3 and Subcase 3.1 with diameter $2k+2$, equality requires $T = P_{2k+3}$.

Define $T_1, T_2$ as in the inductive part of Case 3. The bound holds with equality only if $\gamma_R^k(T_1) = \frac{4|V(T_1)|}{2k+3}$ and $\gamma_R^k(T_2) = \frac{4|V(T_2)|}{2k+3}$. It follows from $\gamma_R^k(T_2) = \frac{4|V(T_2)|}{2k+3}$ and the first paragraph in the proof of Theorem 22 that $\mathrm{diam}(T_2) = 2k+2$. Since $d(v_r, u) \leq k+1$ for each $u \in T_2$, from the proof of Subcase 3.1 we deduce that $T_2 = P_{2k+3}$ with central vertex $v_r$. By induction hypothesis, $V(T)$ can be partitioned into sets inducing $P_{2k+3}$ such that the subgraph induced by the central vertices of these paths is connected. Suppose $\{u_1, \ldots, u_{2k+3}\}$ is the partition set inducing $P_{2k+3} = u_1u_2 \cdots u_{2k+3}$ containing $v_{r-1}$. We claim that $v_{r-1} = u_{k+2}$. Otherwise, we may assume, without loss of generality that, $v_{r-1} \in \{u_{k+3}, \ldots, u_{2k+3}\}$. Define $f : V(G) \to \{0,1,2\}$ by $f(v_r) = f(u_{k+1}) = 2, f(u_{2k+3}) = 1$ and let $f$ assign 2 to all other central vertices and 1 to all other leaves. It is easy to see that $f$ is a $k$DRDF of $T$ with weight less than $4n/(2k+3)$ which is a contradiction. Thus $v_{r-1}$ is the central vertex of the path $P_{2k+3} = u_1u_2 \cdots u_{2k+3}$ and the proof is complete. ■

**Theorem 24.** *If $k$ is a positive integer and $G$ is a connected $n$-vertex graph with $n \geq k+2$, then*

$$\gamma_R^k(G) \leq 4n/(2k+3).$$

*Moreover, the equality holds if and only if $G$ is $C_{2k+3}$ or obtained from $\frac{n}{2k+3}P_{2k+3}$ by adding a connected subgraph on the set of centers of the components of $\frac{n}{2k+3}$ $P_{2k+3}$.*

***Proof.*** If $G$ has the specified form, then as remarked earlier every $k$DRDF puts weight at least 4 on the vertex set of each copy of $P_{2k+3}$.

Now suppose that $\gamma_R^k(G) = \frac{4n}{2k+3}$. Since adding edges cannot increase $\gamma_R^k(G)$, every spanning tree of $G$ has the form specified in Theorem 22. Given a spanning tree $T$, let $S_1, \ldots, S_k$ be the $(2k+3)$-sets in the special partition of $V(T)$. The assignment of weight 4 that guards $S_i$ can be chosen independently of any other $S_j$. If any edge of $G$ joins vertices of $S_i$ and $S_j$ that are not the centers of the paths they induce, then a $k$DRDF with weight less than $\frac{4n}{2k+3}$ can be built as in the proof of Theorem 23. ■

The special case $k = 1$ of Theorems 22, 23 and 24 can be found in [2]. As an application of Theorem 24, we prove the next result.

**Corollary 25.** *Let $f = (V_0, V_1, V_2)$ be any $\gamma_R^k(G)$-function of a connected graph $G$ of order $n \geq 3$. Then*

(1) $1 \leq |V_2| \leq \frac{2n}{2k+3}$ *and a graph* $G$ *admits a* $\gamma_R^k(G)$*-function such that* $|V_2| = \frac{2n}{2k+3}$ *if and only if* $G$ *is* $C_{2k+3}$ *or is obtained from* $\frac{n}{2k+3}P_{2k+3}$ *by adding a connected subgraph on the set of centers of the components of* $\frac{n}{2k+3}P_{2k+3}$.

(2) $0 \leq |V_1| \leq \frac{4n}{2k+3} - 2$.

(3) $n - \frac{4n}{2k+3} + 1 \leq |V_0| \leq n - 1$.

***Proof.*** (1) If $V_2 = \emptyset$, then $V_1 = V(G)$ and $V_0 = \emptyset$. The RDF $(\emptyset, V(G), \emptyset)$ is not minimum since $|V_1| + 2|V_2| > \frac{4n}{2k+3}$. Hence $|V_2| \geq 1$. On the other hand, $|V_2| \leq \frac{2n}{2k+3} - |V_1|/2 \leq \frac{2n}{2k+3}$.

Let $G$ admit a $\gamma_R^k(G)$-function $f = (V_0, V_1, V_2)$ such that $|V_2| = \frac{2n}{2k+3}$. Then by Theorem 24, $\frac{4n}{2k+3} \leq |V_1| + 2|V_2| = \gamma_R^k(G) \leq \frac{4n}{2k+3}$ and the result follows from Theorem 24 again.

Conversely, let $G$ be $C_{2k+3}$ or obtained from $\frac{n}{2k+3}P_{2k+3}$ by adding a connected subgraph on the set of centers of the components of $\frac{n}{2k+3}P_{2k+3}$. If $G = C_{2k+3}$, then assign 2 to two vertices at distance 2 and 0 to the other vertices. If $G$ is obtained from $\frac{n}{2k+3}P_{2k+3}$ by adding a connected subgraph on the set of centers of the components of $\frac{n}{2k+3}P_{2k+3}$, then assign 2 to the neighbors of centers of the components $P_{2k+3}$ and 0 to the other vertices. Obviously $f$ is a $\gamma_R^k(G)$-function with the desired property.

(2) Since $|V_2| \geq 1$, $|V_1| \leq \frac{4n}{2k+3} - 2|V_2| \leq \frac{4n}{2k+3} - 2$.

(3) The upper bound comes from $|V_0| \leq n - |V_2| \leq n - 1$. For the lower bound, adding side by side $2|V_0| + 2|V_1| + 2|V_2| = 2n$, $-|V_1| - 2|V_2| \geq \frac{-4n}{2k+3}$ and $|V_1| \leq \frac{4n}{2k+3} - 2$ gives $2|V_0| \geq 2n - \frac{8n}{2k+3} + 2$. Therefore $|V_0| \geq n - \frac{4n}{2k+3} + 1$. ■

The special case $k = 1$ of Corollary 25 can be found in [5].

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Received 20 December 2011
Revised 4 September 2012
Accepted 5 September 2012