Discussiones Mathematicae
Graph Theory 33 (2013) 731–745
doi:10.7151/dmgt.1700

PATH-NEIGHBORHOOD GRAPHS

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Abstract

A path-neighborhood graph is a connected graph in which every neighborhood induces a path. In the main results the 3-sun-free path-neighborhood graphs are characterized. The 3-sun is obtained from a 6-cycle by adding three chords between the three pairs of vertices at distance 2. A P_k-graph is a path-neighborhood graph in which every neighborhood is a P_k, where P_k is the path on k vertices. The P_k-graphs are characterized for $k \leq 4$.

Keywords: path-neighborhood graph, outerplanar graph, MOP, snake, 3-sun, k-fun.

2010 Mathematics Subject Classification: 05C75, 05C38, 05C99, 05C10.

1. Introduction

There is a long tradition in graph theory to characterize classes of graphs by forbidden subgraphs. There are two types of such characterizations: by forbidden subgraphs, and by forbidden *induced* subgraphs. The area of forbidden subgraphs has its origins in Kuratowski's characterization of planar graphs as being the graphs without subdivisions of K_5 or $K_{3,3}$ as subgraphs [20]. Nowadays forbidden minors is a major theme in graph theory. The area of forbidden induced subgraphs has its origins in the characterization of interval graphs by

Lekkerkerker and Boland [23]. Since then many other classes of graphs have been characterized by forbidden induced subgraphs, see e.g. [5, 15].

The opposite type of problem would be the problem of characterizing classes of graphs having certain prescribed (induced) subgraphs. A classical problem of this type is the *Reconstruction Problem*: given a graph G, is it possible to reconstruct G if all its (non-isomorphic) vertex-deleted subgraphs are given. Its origins lay with Ulam [29]. Another classical problem of this type was proposed by A.A. Zykov in 1963, [32]. It reads as follows. A graph H is said to be *realizable* by a graph G if every neighborhood in G induces a subgraph isomorphic to H. The problem was also referred to as the Trahtenbrot-Zykov problem, see [18]. Most of the work done on this problem is of the type: is a given graph H realizable or not; for a few references see e.g. [18, 6, 9, 7, 16, 8, 24]. Yet another instance is the following problem: characterize the class of graphs that have all their spanning trees in a given family of trees, all of the same order. Some first results of this type have been obtained in [19].

A broader perspective on the Trahtenbrot-Zykov problem is the following: Given a class of graphs $\mathcal{G}$, characterize the graphs in which the neighborhood of each vertex is isomorphic to some graph in $\mathcal{G}$. Parsons and Pisanski proposed this problem in 1987, see [27]. A number of interesting results of this type have been obtained in the literature. We give a few examples. The Trahtenbrot-Zykov problem is the special case where $\mathcal{G}$ consists of a single graph. First results of this type were obtained by Hall [16]. In this paper he presented amongst other things also a characterization of the connected graphs in which each neighborhood is a path on four vertices, see Theorem 17 below. Borowiecki *et al.* [4] determined bounds for the number of edges in *locally k-tree graphs*, i.e. graphs in which any neighborhood is a k-tree. The reference list of this paper contains some more examples with other types of neighborhoods. Fronček [12, 13] derived a bound for the number of edges in a graph that is locally a path, i.e. each neighborhood is a path. Diwan and Usharani [10] studied the colorability of such graphs that are planar. Parsons and Pisanski [27] studied topological characterizations of graphs in which any neighborhood is a path or a cycle. An interesting result from the viewpoint of this paper is their characterization of connected graphs in which the neighborhood of each vertex is a path of fixed length k (for $k = 4$ and 5). The case $k = 4$ is also a corollary of our main result below, so for details see below. Another interesting result from the viewpoint of this paper is that of Zelinka [31]:

Theorem A. *Let G be a finite planar 3-connected graph. Then the following two assertions are equivalent:*

(i) *G is locally snake-like.*

(ii) *Each vertex in G is adjacent to exactly one face of degree greater than 3 and each triangle in G is the boundary of a face.*

A *snake* in Zelinka's terminology is just a path. And a locally snake-like graph is a graph in which the neighborhood of each vertex induces a path. Below we call such graphs *path-neighborhood graphs.* Note that following [3] we use snake in this paper to indicate a maximal outerplanar graph with at least four vertices in which each face shares an edge with the outerface, so it is a maximal outerplanar graph with exactly two vertices of degree 2.

In this paper we continue the line of study of Zelinka, Parsons and Pisanski, and Fronček, although our results were obtained independently of these papers. We study the path-neighborhood graphs: connected graphs in which every neighborhood induces a path. Note that the paths need not be of the same length. The problem considered is to characterize the path-neighborhood graphs. Zelinka provides a special result of this type: a characterization of 3-connected, planar path-neighborhood graphs. In general it seems to be a difficult problem. As a first step we characterize the path-neighborhood graphs that are 3-sun-free. The 3-sun is depicted in Figure 2. A prime example of such a graph is the snake, a special type of maximal outerplanar graph, and we will see that maximal outerplanar graphs play an important role in the sequel. But contrary to the Zelinka result, our theorem does not restrict to planar graphs. As a corollary of our theorems we get a new proof of the characterization of the graphs in which all neighborhoods induce a path P_k of fixed length $k-1$, for $k \leq 4$.

2. First Results and Examples

A *path-neighborhood graph* is a *connected* graph in which the neighborhood of each vertex induces a path. Since the empty path does not exist, a path has vertices. Hence the one-vertex graph K_1 is not a path-neighborhood graph. Two simple examples are the 3-sun and the k-fan, see Figure 1. The 3-*sun* consists of a 6-cycle with three chords that form a triangle. It is sometimes also called a *trampoline.* The *k-fan* F_k is the graph consisting of a path R of length k and an additional vertex x adjacent to all $k+1$ vertices of R. We call x the *center* and R the *path* of the fan. Clearly, in a path-neighborhood graph a vertex x of degree $k+1$ and its neighbors induce a k-fan with center x and path R consisting of the neighbors.

First we state some simple and obvious facts.

Fact 1. *The* 3*-sun is a path-neighborhood graph. If it occurs in a path-neighborhood graph, then it must be induced.*

Fact 2. *The k-fan is a path-neighborhood graph. If it occurs in a path-neighborhood graph, then it must be induced.*

Fact 3. *A path-neighborhood graph does not contain a K_4.*

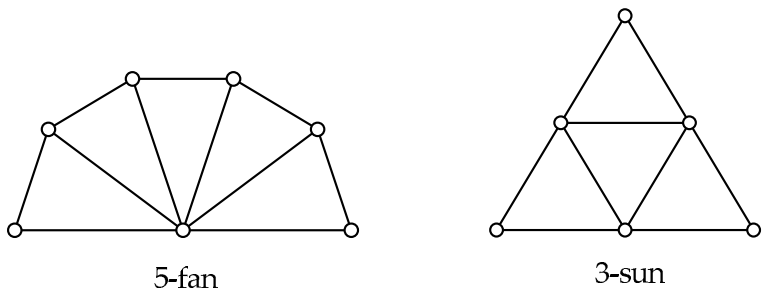

Figure 1. 5-fan and 3-sun.

Fact 4. *A path-neighborhood graph does not contain a* $K_{1,1,3}$.

Fact 5. *If a path-neighborhood graph contains a vertex of degree* 1*, then it is* K_2.

Fact 6. *If a path-neighborhood graph has maximum degree* 2*, then it is* K_3.

Proposition 7. *Let* G *be a path-neighborhood graph of maximum degree at most* 3*. Then* G *is* K_2 *or* K_3 *or the* 2*-fan.*

Proof. If G does not contain a vertex of degree 3, then, by Facts 5 and 6, G is either K_2 or K_3. So let x be a vertex of degree 3 with neighboring path $u \to y \to v$. Then y is also of degree 3 with neighboring path $u \to x \to v$. If u would have another neighbor besides x and y, then the edge xy must be in a path containing this other neighbor. This would mean that either x or y must have a fourth neighbor, impossible. So both u and v have degree 2, and G is the 2-fan on these four vertices. ■

The 3-sun and the fans are instances of a wider class of path-neighborhood graphs. An outerplanar graph is a planar graph that has a plane embedding such that all vertices lie on the outer cycle. A maximal outerplanar graph is an outerplanar graph such that the number of edges is maximum. Another way to view a maximal outerplanar graph is that it is the triangulation of a plane cycle. These graphs appeared for the first time in the literature in the classical book of Harary [17]. The following lemma appeared in a different form in [25]. It seems now to be part of folklore. Its proof is a simple exercise.

Lemma 8. *Let* G *be a maximal outerplanar graph with its plane embedding, and let* v *be any vertex. Then the neighborhood of* v *consists of a path* $v_1 \to v_2 \to \cdots \to v_k$ *and the edges* vv_1 *and* vv_k *are on the outer face, whereas the edges* $vv_2, vv_3, \ldots, vv_{k-1}$ *are interior edges.*

So maximal outerplanar graphs are path-neighborhood graphs.

A 3-sun-free maximal outerplanar graph with at least four vertices is called a *snake*, see e.g. [3]. Note that in a snake the triangulation of the plane cycle is such that each triangle has an edge on the outer cycle. Moreover, in the statement of Lemma 8, the subpath between v_2 and v_{k-1} lies entirely on the outer cycle. A snake contains exactly two vertices of degree two. The smallest snake is the 2-fan. The triangle K_3 could be considered a snake as well, but then it is degenerated in the sense that it contains three vertices of degree two. For this reason we exclude K_3.

For our purposes it is convenient to view snakes in a different way. It may seem rather complicated, but it is quite helpful in the proofs or our main theorems. We partition the outer cycle into two paths $P = u_1 \to u_2 \to \cdots \to u_p$ and $Q = v_1 \to \cdots \to v_q$. We consider two possibilities. First u_1 and u_p are the vertices of degree two. Note that we have two choices for P. In Figure 2A the two paths are indicated by thicker lines. Second u_1 and v_q are the vertices of degree 2, see Figure 2B. Again we have two choices for P. When considered in this way we could also start with the two paths P and Q, and then make a strip by a triangulation 'between' P and Q. In the first case u_1 is joined only to v_1 on Q and u_p is joined only to v_q on Q. In the second case u_1 is joined only to v_1 on Q and v_q only to u_p on P. Thus the vertices of degree two are respectively u_1, u_p, and u_1, v_q. Moreover, each vertex is joined to consecutive vertices on the other path. We call this a *triangulated strip* on P and Q. The first one is denoted by $P \rhd Q$. The second one is denoted by $P \trianglerighteq Q$. In the proofs below, paths P and Q play a different role. This is the reason for this notation and terminology. The symbols P, Q, u_i, and v_j will always be used in the above sense.

Note that any two consecutive vertices on one path have a unique common neighbor on the other path. In $P \rhd Q$ vertices u_1 and u_p are *not* a common neighbor of consecutive vertices on Q. In $P \trianglerighteq Q$ vertices u_1 and v_q are not a common neighbor of two consecutive vertices on the other path.

Lemma 9. *Let G be a path-neighborhood graph, and let S be a triangulated strip in G. If $S = P \rhd Q$, and an internal vertex of Q has a neighbor outside S, then G contains a 3-sun. If $S = P \trianglerighteq Q$, and an internal vertex of Q distinct from v_{q-1} has a neighbor outside S, then G contains a 3-sun.*

Proof. Let $S = P \rhd Q$, and write $Q = v_1 \to v_2 \to \cdots \to v_{q-1} \to v_q$, and let v_i be an internal vertex of Q with a neighbor x outside S. Note that $1 < i < q$. The neighbors of v_i in S form a path with ends v_{i-1} and v_{i+1}. So there is a path in $N(v_i)$ connecting x either to v_{i-1} or to v_{i+1}, say v_{i-1}. Let y be the last vertex on this path before v_{i-1}. Then y is a common neighbor of v_i and v_{i-1}. Let u_j be the common neighbor of v_i and v_{i-1} on P. Note that $1 < j < p$. Now either u_j and v_{i-2} are adjacent or v_{i-1} and u_{j-1} are adjacent. Also either u_j and v_{i+1}

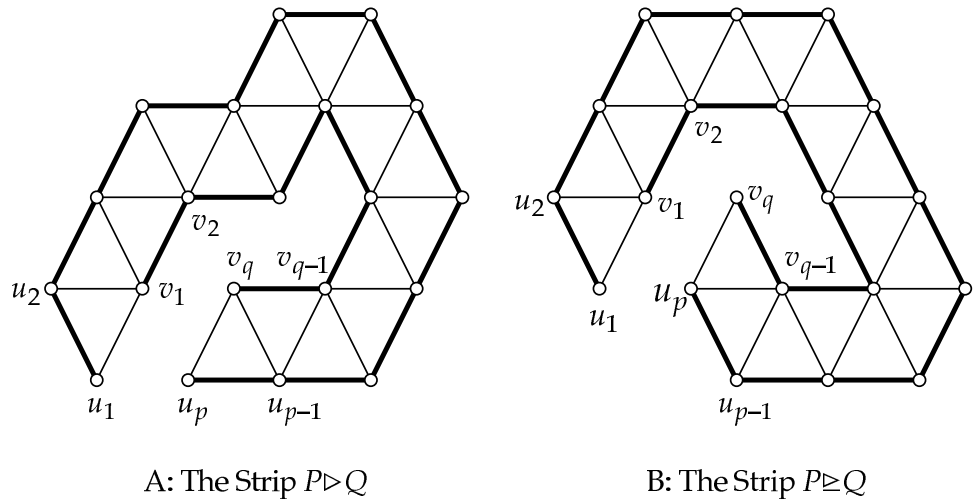

Figure 2. Snake as triangulated strip.

are adjacent or v_i and u_{j+1} are adjacent. In all cases we get a 3-sun. The case $S = P \trianglerighteq Q$ follows similarly. ■

From the triangulated strip we can make two other types of 3-sun-free path-neighborhood graphs. In all the cases below, if we connect the ends of P (or the ends of Q) by an edge, then P (or Q) should be of length at least three, so that the path is not turned into a triangle by connecting its ends. Otherwise the path may have length two. First we join u_1 and u_p, as well as v_1 and v_q. Then we join either u_1 and v_q or v_1 and u_p. So we connect the two ends of the strip such that a nice 'band' results. Such a graph could be considered as consisting of two cycles $u_1 \to u_2 \to \cdots \to u_p \to u_1$ and $v_1 \to v_2 \to \cdots \to v_q \to v_1$ with a "triangulation" in between. We call this graph the *triangulated band*, for an example see Figure 5A. Obviously, the triangulated band is planar. For the second type we proceed as follows: loosely speaking we twist the strip before we connect the two ends by which a 'Möbius band' arises. There are two ways to make the connection. First we join u_1 to u_p and v_q and we join u_p to v_1. Second we join v_1 to v_q and u_p and we join u_1 to v_q. We call these graphs a *triangulated Möbius band*. For an example see Figure 5B. The triangulated Möbius band is nonplanar, since it contains $K_{3,3}$ as a minor.

Note that, if either the triangulated band or the triangulated Möbius band occurs in a path-neighborhood graph G, then it must be induced, otherwise a cycle in some neighborhood would arise. Moreover, if there is a vertex x outside this band adjacent to some vertex w on the band, then, as in Lemma 9, it follows that G contains a 3-sun. For later reference we state this as a Lemma.

Lemma 10. *If a triangulated band or triangulated Möbius band occurs in a path-neighborhood graph then it is induced, and any vertex outside the band adjacent to the band produces a* 3*-sun.*

3. PATH-NEIGHBORHOOD GRAPHS WITHOUT A 3-SUN

In this section we characterize the 3-sun-free path-neighborhood graphs. In view of Facts 5 and 6 we only need to deal with the case in which there are vertices of degree at least three. It turns out that the 3-sun-free path-neighborhood graphs are precisely the 3-sun-free examples given in the previous section, so the trivial ones K_2 and K_3 and the snakes and the bands.

Theorem 11. *Let G be a* 3*-sun-free path-neighborhood graph with a vertex of degree* 2*. Then G is K_3 or a snake.*

Proof. Assume that G is not K_3. Let u_1 be a vertex of degree 2. Then both its neighbors have degree at least 3. If both have degree 3, then, as in Proposition 7, it follows that G is the 2-fan, and we are done.

So let u_2 be a neighbor of u_1 of degree at least 4. Let Φ_2 be the fan induced by u_2 and its neighboring path $R_2 = u_1 \to v_1 \to v_2 \to \cdots \to v_{k_2} \to u_3$. If v_1 would have degree at least 4 as well, then this would introduce a common neighbor x of v_1 and v_2 distinct from u_2, by which a 3-sun would arise. So, by Lemma 9, the only vertices in Φ_2 that still may have neighbors outside Φ_2 are u_3 and v_{k_2}. If u_3 has degree 2, then, as above, it follows that v_{k_2} has no neighbors outside Φ_2. So $G = \Phi_2$, and we are done.

So assume that u_3 has degree at least 3. Note that u_3 is adjacent to edge $u_2v_{k_2}$, where u_2 has no neighbors outside Φ_2. So the neighboring path of u_3 must be of the form $R_3 = u_2 \to v_{k_2} \to \cdots \to v_{k_3} \to v_4$, which is the path in the fan Φ_3 of u_3, where all vertices are new except u_2 and v_{k_2}. If R_3 is of length 2, then we set $k_2 = k_3$, and R_3 is just $u_2 \to v_{k_2} = v_{k_3} \to u_4$. Note that $\Phi_2 \cup \Phi_3$ is a triangulated strip. As above it follows that in $\Phi_2 \cup \Phi_3$ the only vertices that may have neighbors outside $\Phi_2 \cup \Phi_3$ are u_4 and v_{k_3}. Again, if u_4 is of degree 2, then $G = \Phi_2 \cup \Phi_3$, which is a snake, and we are done.

If u_4 has degree at least 3, then its neighboring path R_4 is of the form $u_3 \to v_{k_3} \to \cdots \to v_{k_4} \to u_5$, where all vertices except u_3 and v_{k_3} are new. As before, only u_5 and v_{k_4} may have neighbors outside the subgraph found so far. This subgraph is again a triangulated strip.

Thus we continue. Since this process has to stop, we will find a path $P = u_1 \to u_2 \to \cdots \to u_p$ and a path $Q = v_1 \to v_2 \to v_{k_{p-1}}$ with the appropriate edges in between to make it a triangulated strip $P \rhd Q$. The process stopped because u_p has degree 2. So we conclude that G is this triangulated strip, and so is a snake. ∎

Theorem 12. *Let G be a 3-sun-free path-neighborhood graph with minimum degree at least 3. Then G is a triangulated band or a triangulated Möbius band.*

Proof. The idea of the proof is to find a triangulated strip on paths P and Q such that only the ends of the paths may have adjacencies that are not yet given in the strip. By maximizing such a strip we deduce that these adjacencies cannot be outside the strip, so that the ends must be adjacent to each other in such a way that the required bands arise.

First we note that, by Proposition 7, G must contain a vertex u_2 of degree at least 4. Let Φ_2 be the fan of u_2 with path $R_2 = u_1 \to v_1 \to v_2 \to \cdots \to v_{k_2} \to v_3$. Note that $k_2 \geq 2$. Then Φ_2 is induced in G and u_2 and v_i with $1 < i < k_2$ have no neighbors outside Φ_2.

Consider u_3, and let Φ_3 be its fan. Its neighboring path R_3 contains the edge $u_2v_{k_2}$, and u_2 has no neighbors outside Φ_2, so R_3 must be of the form $u_2 \to v_{k_2} \to \cdots \to v_{k_3} \to u_4$, where all vertices ar new except u_2 and v_{k_2}. In the case that R_3 is of length 2, we set $k_2 = k_3$. Now $\Phi_2 \cup \Phi_3$ is a triangulated strip $P' \rhd Q'$ with $P' = u_1 \to u_2 \to u_3 \to u_4$ and $Q' = R_2 \to R_3$ is the concatenation of R_2 and R_3. By Lemma 9, the internal vertices of Q' have no neighbors outside the strip. By construction the internal vertices of P' have no neighbors outside the strip. Hence u_4 is adjacent to u_3 and v_{k_3} and possibly to u_1 and/or v_1 but not to any other vertex of the strip. If u_4 is not adjacent to u_1 or v_1, then we consider the fan Φ_4 on u_4 and its neighboring path $R_4 = u_3 \to v_{k_3} \to \cdots \to v_{k_4} \to u_5$, where all vertices are new except u_3 and v_{k_3}. We extend path P' with u_5 and path Q' with the other new vertices, and we get a longer triangulated strip $P' \rhd Q'$. We continue this extension process until we end up with a strip $S = P \rhd Q$ with $P = u_1 \to u_2 \to \cdots \to u_p$, $Q = v_1 \to v_2 \to \cdots \to v_{k_{p-1}}$, and u_p is adjacent to u_1 and/or v_1. We set $k_{p-1} = q$, so that the last vertex of Q is v_q.

Now consider u_1. If u_1 is not adjacent to u_p or v_q, then, as above, we can extend the strip S on the other side until this end vertex of P is adjacent to u_p and/or v_q. By renumbering the vertices we get the strip $S = P \rhd Q$ with $P = u_1 \to u_2 \to \cdots \to u_p$ and $Q = v_1 \to v_2 \to \cdots \to v_q$ such that u_1 is adjacent to u_p and/or v_q and u_p is adjacent to u_1 and/or v_1.

We consider two cases.

Case 1. *u_1 and u_p are not adjacent.* Then, necessarily, u_1 and v_q are adjacent as well as u_p and v_1. We depict this in Figure 1A. The black vertices have no other adjacencies in the figure. Now if v_1 and v_q are adjacent, then we get a triangulated Möbius band. By Lemma 10, G is this band.

So suppose that v_1 and v_q are not adjacent. Note that, by Lemma 9, u_2 and v_2 do not have neighbors outside strip S. Consider u_1. It is adjacent to edge u_2v_1 and to v_q. So its neighboring path must be of the form $R = u_2 \to v_1 \to x \to \cdots \to v_q$ plus possibly some extension beyond v_q. Consider v_1. It

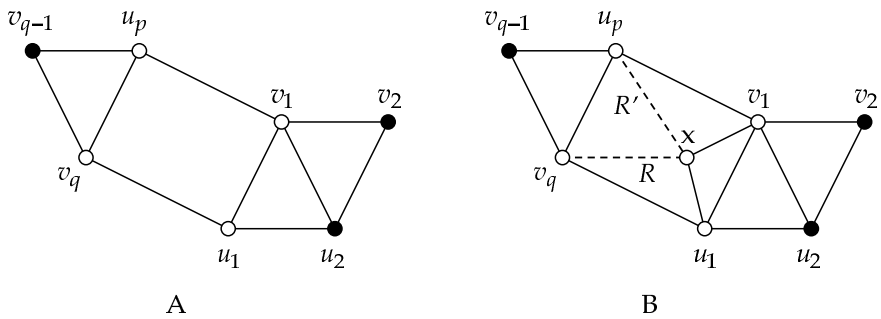

Figure 3. Case 1 in proof of Theorem 12.

is adjacent to v_2, u_2, u_1 and u_p. So its neighboring path must be of the form $R' = v_2 \to u_2 \to u_1 \to y \to \cdots \to u_p$ with possibly some extension beyond u_p. To avoid a $K_{1,1,3}$ on u_1, v_1, u_2, x, y we must have $x = y$, see Figure 1B. Let z be the neighbor of x on R different from u_1, so it is the next vertex in the direction of v_q, and let z' be the neighbor of x on R' in the direction of u_p. If $z \neq z'$, then u_1, u_2, v_1, z', x, z form a 3-sun, which is forbidden. If $z = z'$, then u_1, v_1, z form a triangle in the neighborhood of x, which is also forbidden. This implies that v_1 and v_q have to be adjacent.

Case 2. *u_1 and u_p are adjacent.* For the sake of simplicity, let us assume that both paths of the strip are from left to right with increasing indices. In Figure 1 the strips are bent to fit on the page. So u_1 and v_1 are the leftmost vertices of the paths, and u_p and v_q are the rightmost vertices.

Subcase 2.1. *u_p and v_1 are adjacent.* Now u_p is adjacent to the edges $u_{p-1}v_q$ and u_1v_1. Since u_{p-1} has no neighbors outside the strip, there must be a path R in $N(u_p)$ between v_q and either u_1 or v_1.

First assume that $R = v_q \to \cdots \to v' \to u_1$. If $v' = v_q$, then we get a triangulated Möbius band from the strip $P \rhd Q$. If $v' \neq v_q$, then we extend Q with the subpath of R between v_q and the last vertex v' before u_1, see Figure 4A, where R is indicated by dashed edges and vertices. We renumber the vertices of Q so that its last vertex is again v_q. Now we have a strip $P \trianglerighteq Q$, and the edges u_1u_p, v_1u_p, and u_1v_q connect the ends of the strip, so that again a triangulated Möbius band arises. By Lemma 10, G is this Möbius band.

Second assume that $R = v_q \to \cdots \to v' \to v_1$. If $v' = v_q$, then we get a triangulated band from the strip $P \rhd Q$. If $v' \neq v_q$, then we extend Q with the subpath of R between v_q and the last vertex v' before v_1, see Figure 4B, where R is indicated by dashed edges and vertices. We renumber the vertices of Q so that its last vertex is again v_q. Now we have a strip $P \trianglerighteq Q$, and the edges u_1u_p, v_1u_p, and v_1v_q connect the ends of the strip, so that again a triangulated band

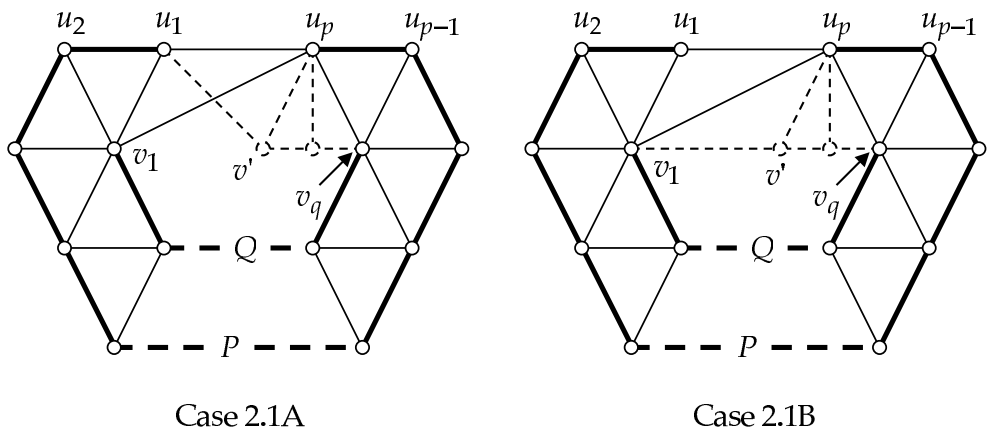

Figure 4. Case 2.1 in proof of Theorem 12.

arises. By Lemma 10, G is this triangulated band.

Subcase 2.2. *u_p and v_1 are not adjacent.* Note that u_1 is adjacent to the edge u_2v_1 and to u_p. Since u_2 has no neighbors outside the strip there must be a path R in $N(u_1)$ between u_p and v_1, say $R = u_p \to v' \to \cdots \to v'' \to v_1$. Now we extend Q to the left with the subpath of R between v' and v'', and again renumber the vertices of Q in the appropriate way. So we get a strip with paths P and Q such that u_p is adjacent to u_1 and v_1, and we are in Subcase 2.1.

This last subcase completes the proof. ■

4. Path-neighborhood Graphs with Degree Restrictions

We denote by P_k the path with k vertices. A *$P_{\leq k}$-neighborhood graph* is a path-neighborhood graph of maximum degree k. A *P_k-neighborhood graph* is a regular path-neighborhood graph of degree k, so a connected graph in which each neighborhood induces a P_k. In terms of the Trahtenbrot-Zykov Problem a P_k-neighborhood graph realizes the path P_k. The following facts can be easily deduced for $P_{\leq k}$-neighborhood graphs and P_k-neighborhood graphs, see also [16, 27].

Fact 13. *K_2 is the only P_1-neighborhood graph* (*and the only $P_{\leq 1}$-neighborhood graph*).

Fact 14. *If a path-neighborhood graph contains a pendant vertex, then it is K_2, hence the P_1-neighborhood graph.*

Fact 15. *K_3 is the only P_2-neighborhood graph.*

The following proposition was already mentioned in [32]. It is an immediate consequence of Proposition 7.

Proposition 16. *There is no P_3-neighborhood graph.*

The P_4-neighborhood graphs were already characterized by Hall as early as 1985 in [16] as the graphs consisting of a cycle of length at least 7 with additional edges connecting vertices at distance two. These graphs are also known as circulant graphs, see [26]. A new proof for Hall's characterization was given in [27] (as Theorem 5.1). Here we present these graphs in a different way so that we can deduce the characterization in a simple way from our main result Theorem 12.

The triangulated band in which all vertices have degree 4 is called a *bracelet*, see Figure 5A. It is also known as an *antiprism*. It consists of two cycles of the same length $n \geq 4$ with a triangulation in between that takes the form of a zigzag. Therefore we denote it by Z_n. The bracelet Z_4 is also known as the *brick*, cf. [18]. The triangulated Möbius band in which all vertices have degree 4 is called the *twisted bracelet*, see Figure 5B. It consists of a cycle of length n and a path on $n-1$ vertices with a triangulation in between. Loosely speaking, it starts as a zigzag, but at the end it is closed "Möbius-wise".

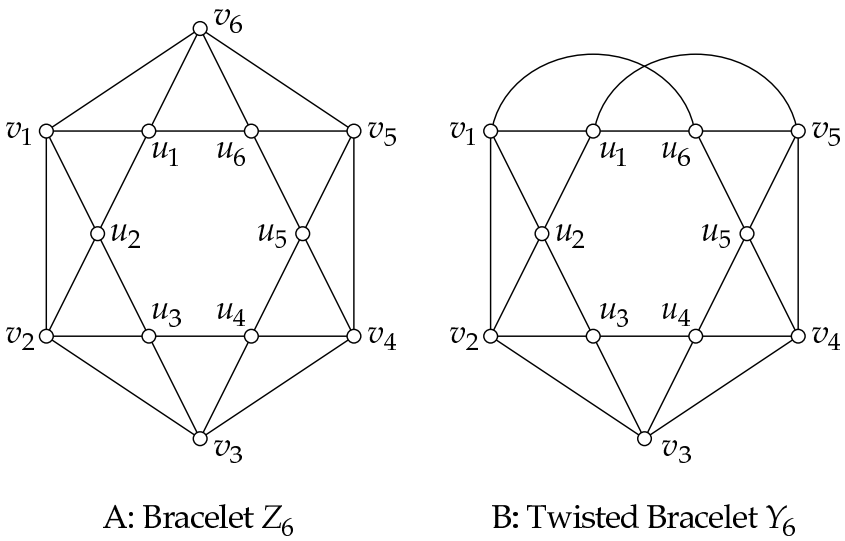

Figure 5. Bracelets.

Theorem 17. *Let G be a P_4-neighborhood graph. Then G is Z_n or Y_n, for some $n \geq 4$.*

Proof. First we prove that G is 3-sun-free. Suppose that G contains a 3-sun H. Then H is induced in G (Fact 1). Let u be a vertex of degree two in H and let

x and y be its neighbors in H. Then x and y have already a P_4 as neighborhood in H. So they do not have neighbors outside H. But for u to have a P_4 as neighborhood either x or y must be an internal vertex in this P_4, so must have a neighbor outside H. This impossibility implies that G is 3-sun-free. Now the theorem follows from Theorem 12. ■

The following theorem is an easy consequence of above results.

Theorem 18. *The $P_{\leq 4}$-neighborhood graphs are K_2, K_3, the 3-sun, the snakes of maximum degree 4, and the bracelets and twisted bracelets.*

Grünbaum constructed a very nice example of a P_5-neighborhood graph, see [18]. In [27] Parsons and Pisanski gave a sketch of a characterization of the P_5-neighborhood graphs.

5. Concluding Remarks

Parsons and Pisanski [27] proposed the problem of characterizing the graphs, in which the neighborhoods of the vertices are from a given class of graphs. Here we consider the case where all neighborhoods are paths. As a first result we have characterized the 3-sun-free path-neighborhood graphs. The snakes, 3-sun-free maximal outerplanar graphs, played a major role in the story. As a corollary, we have a new proof of the characterization of the graphs that realize P_4 in the sense of the Trahtenbrot-Zykov Problem (cf. [16, 27]). The characterization of path-neighborhood graphs in general seems to be a very difficult problem. For instance, take a maximal outerplanar graph with many pendant snake-like parts (that are 3-sun-free except for the part where they are connected with the rest of the graph), similar to paths pending at a tree. Now we can pairwise connect such pendant snakes at their ends as in making the bands above, thus creating fairly complicated path-neighborhood graphs, which are still close to maximal outerplanar graphs. Question: can we construct path-neighborhood graphs that are not even close to being maximal outerplanar, whatever that means?

In [21] the connection between path-neighborhood graphs, maximal outerplanar graphs and two other classes of graphs is pursued. These two other classes are the chordal graphs and the triangle graphs. The *triangle graph* $T(G)$ of a graph G is the graph with the triangles of G as vertices, and two such vertices are joined in $T(G)$ if, as triangles in G, they share an edge. Triangle graphs were introduced by [28], see also [11, 1, 22].

An instance of this connection is given by the following theorem in [21].

Theorem 19. *A connected graph G is a path-neighborhood graph with a tree as its triangle graph if and only if G is a maximal outerplanar graph.*

Clearly, much more could be said about the triangle graph of a path-neighborhood graph.

A variation of the above type of problems was studied by Zelinka and Fronček: the edge-neighborhood of en edge uv is the set of vertices distinct from u and v that are adjacent to u or to v or to both. Now similar questions can be raised with respect to the edge-neighborhood, see e.g. [30, 14].

Acknowledgement

We thank the referee for pointing out relevant references, which helped to improve this paper.

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Received 28 November 2011
Revised 29 August 2012
Accepted 10 September 2012