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HYPERGRAPHS WITH PENDANT PATHS ARE NOT CHROMATICALLY UNIQUE

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Abstract

In this note it is shown that every hypergraph containing a pendant path of length at least 2 is not chromatically unique. The same conclusion holds for h-uniform r-quasi linear 3-cycle if $r \geq 2$.

Keywords: sunflower hypergraph, chromatic polynomial, chromatic uniqueness, pendant path.

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1. Notation and Preliminary Results

A simple hypergraph $H = (V, \mathcal{E})$, with order $n = |V|$ and size $m = |\mathcal{E}|$, consists of a vertex-set $V(H) = V$ and an edge-set $E(H) = \mathcal{E}$, where $E \subseteq V$ and $|E| \geq 2$ for each edge E in $\mathcal{E}$. H is h-uniform, or is an h-hypergraph, if $|E| = h$ for each E in $\mathcal{E}$ and H is linear if no two edges intersect in more than one vertex [1]. Let $r \geq 1$ and $h \geq 2r$. H is said to be r-quasi linear (or shortly quasi linear) [13] if any two edges intersect in 0 or r vertices. Examples of quasi linear hypergraphs are t-stars [5, 8], also called sunflower hypergraphs [7, 11, 12]. We say that a hypergraph S is a t-star with kernel K, where $K \subseteq V(S)$ and $t \geq 1$, if S has exactly t edges and $e \cap e' = K$ for all distinct edges e and e' of S. A system of t pairwise disjoint edges (matching) is a t-star with empty kernel. In [12] a sunflower hypergraph was denoted by $SH(n, p, h)$; it is an h-hypergraph having a kernel of cardinality $h - p$, n vertices and k edges, where $n = h + (k - 1)p$ and $1 \leq p \leq h - 1$.

A hypergraph for which no edge is a subset of any other is called Sperner. Two vertices $u, v \in V(H)$ belong to the same component if there are vertices $x_0 = u, x_1, \ldots, x_k = v$ and edges $E_1, \ldots, E_k$ of H such that $x_{i-1},\ x_i \in E_i$ for each i $(1 \leq i \leq k)$ [1]. H is said to be connected if it has only one component. An h-uniform hypertree is a connected linear h-hypergraph without cycles [1]. We shall define two classes of quasi linear uniform hypergraphs called quasi linear elementary path and quasi linear elementary cycle and denoted by $P_m^{h,r}$ and $C_m^{h,r}$, respectively, as follows: $P_m^{h,r}$ consists of m edges $E_1, \ldots, E_m$ such that $|E_1| = \cdots = |E_m| = h, |E_i \cap E_{i+1}| = r$ for any $1 \leq i \leq m-1$ and every edge has in common with other edges only the common vertices with its neighboring edges (r for E_1 and E_m, and $2r$ for the remaining edges). $C_m^{h,r}$ may be defined in a similar way; in this case $|E_m \cap E_1| = r$.

If $\lambda \in \mathbb{N}$, a λ-coloring of a hypergraph H is a function $f : V(H) \to \{1, \ldots, \lambda\}$ such that for each edge E of H there exist x, y in E for which $f(x) \neq f(y)$. The number of λ-colorings of H is given by a polynomial $P(H, \lambda)$ of degree $|V(H)|$ in λ, called the chromatic polynomial of H. $P(H, \lambda)$ can be obtained applying inclusion-exclusion principle, in the same way as for graphs, getting the following formula [10]:

$$P(H, \lambda) = \sum\nolimits_{W \subseteq E(H)} (-1)^{|W|} \lambda^{c(W)}, \tag{1}$$

where $c(W)$ denotes the number of components of the spanning subhypergraph induced by edges from W. All h-uniform hypertrees have the same chromatic polynomial.

Lemma 1 [6]**.** *If T_k^h is any h-uniform hypertree with k edges, then*

$$P(T_k^h, \lambda) = \lambda(\lambda^{h-1} - 1)^k. \tag{2}$$

Two hypergraphs H and G are said to be chromatically equivalent or χ-equivalent, written $H \sim G$, if $P(H, \lambda) = P(G, \lambda)$. Let us restrict ourselves to the class of Sperner hypergraphs. A simple hypergraph H is said to be chromatically unique if H is isomorphic to H' for every simple hypergraph H' such that $H' \sim H$; that is, the structure of H is uniquely determined up to isomorphism by its chromatic polynomial. The notion of χ-unique graphs was first introduced and studied by Chao and Whitehead [4] (see also [9]). It is clear that all h-hypergraphs are Sperner. The notion of χ-uniqueness in the class of h-hypergraphs may be defined as follows: An h-hypergraph H is said to be h-chromatically unique if H is isomorphic to H' for every h-hypergraph H' such that $H' \sim H$.

Non-trivial chromatically unique hypergraphs are extremely rare. One example of a non-trivial chromatically unique hypergraph was proposed by Borowiecki and Łazuka; it is $SH(n, 1, h)$.

Theorem 2 [3]**.** *$SH(n, 1, h)$ is chromatically unique.*

The proof of this result was completed in [11]. Note that for $p = h-1$, $SH(n, h-1, h)$ is an h-uniform hypertree. The chromaticity of $SH(n, p, h)$ may be stated as follows.

Theorem 3 [12]**.** *Let $n = h + (k-1)p$, where $h \geq 3$, $k \geq 1$ and $1 \leq p \leq h-1$. Then SH(n,p,h) is h-chromatically unique for every $1 \leq p \leq h-2$; for $p = h-1$ $SH(n, p, h)$ is h-chromatically unique for $k = 1$ or $k = 2$ but it has not this property for $k \geq 3$. Moreover, $SH(n, p, h)$ is not chromatically unique for every $p, k \geq 2$.*

$SH(n, p, h)$ is quasi linear with $r = h - p$ and it is a path for $k = 2$. The chromaticity of non-uniform hypertrees was studied by Walter [14].

2. Main Results

Consider the hypergraph H represented in Figure 1, where H_1 is a subhypergraph of H, U and W are two edges such that: $U \cap V(H_1) = A \neq \emptyset$; $U \cap W = B \neq \emptyset$; $W \cap V(H_1) = \emptyset$. Such a path consisting of edges U and W will be called a pendant path of length 2. Denote $|A| = s \geq 1$; $|U \backslash (A \cup B)| = p \geq 1$; $|B| = t \geq 1$; $|W \backslash U| = q \geq 1$.

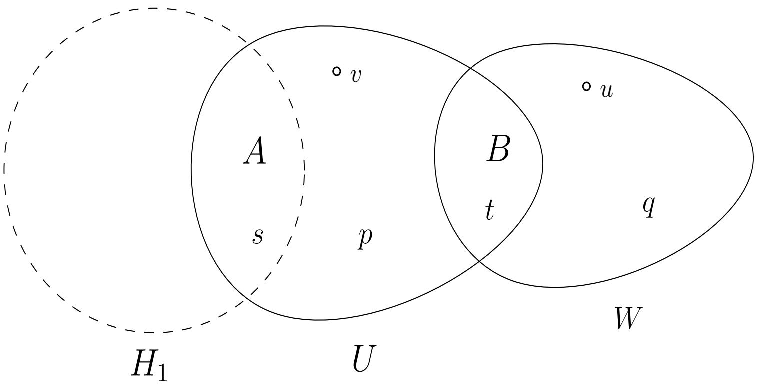

Figure 1. Hypergraph H.

Theorem 4. *Every hypergraph containing a pendant path of length at least 2 is not chromatically unique.*

Proof. For the hypergraph H from Figure 1 defined as above we shall define another Sperner hypergraph F such that $P(H, \lambda) = P(F, \lambda)$. For this consider two distinct vertices $u \in W \backslash U$ and $v \in U \backslash (A \cup B)$ and three edges: $U' = U$,

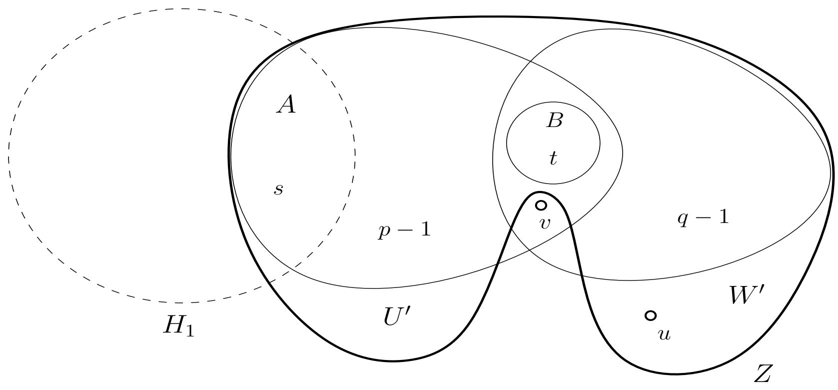

Figure 2. Structure of hypergraph F.

$W' = W \cup \{v\} \setminus \{u\}$ and $Z = U \cup W \setminus \{v\}$. We have $|U' \cap W'| = |U \cap W| + 1 = t + 1$. F is defined as follows: $V(F) = V(H)$ and $E(F) = E(H_1) \cup \{U', W', Z\}$ (see Figure 2).

Let $\varphi(H_1, \lambda)$ and $\xi(H_1, \lambda)$ denote the number of λ-colorings of H_1 such that A is monochromatic and A is not monochromatic, respectively; the corresponding numbers of λ-colorings of H are denoted by $\varphi(H, \lambda)$ and $\xi(H, \lambda)$, respectively.

If A is monochromatic and B is monochromatic, having the same color as A, then the number of λ-colorings of H is $\varphi(H_1, \lambda)(\lambda^p - 1)(\lambda^q - 1)$; if A is monochromatic and B is monochromatic having a color different from the color of A this number equals $\varphi(H_1, \lambda)(\lambda - 1)\lambda^p(\lambda^q - 1)$ and if A is monochromatic and B is not monochromatic we get $\varphi(H_1, \lambda)\lambda^{p+q}(\lambda^t - \lambda)$, which implies that

$$\varphi(H, \lambda) = \varphi(H_1, \lambda)((\lambda^p - 1)(\lambda^q - 1) + (\lambda - 1)\lambda^p(\lambda^q - 1) + \lambda^{p+q}(\lambda^t - \lambda)).$$

In a similar manner if A is not monochromatic and B is monochromatic, then the number of λ-colorings of H equals $\xi(H_1, \lambda)\lambda^{p+1}(\lambda^q - 1)$; if A and B are not monochromatic we get $\xi(H_1, \lambda)\lambda^{p+q}(\lambda^t - \lambda)$, thus yielding

$$\xi(H, \lambda) = \xi(H_1, \lambda)(\lambda^{p+1}(\lambda^q - 1) + \lambda^{p+q}(\lambda^t - \lambda))$$

and the chromatic polynomial of H is $P(H, \lambda) = \varphi(H, \lambda) + \xi(H, \lambda)$.

We shall prove that F has the same chromatic polynomial, by showing that $\varphi(H, \lambda) = \varphi(F, \lambda)$ and $\xi(H, \lambda) = \xi(F, \lambda)$.

If A is not monochromatic, by considering the cases B monochromatic and B not monochromatic we easily deduce that

$$\xi(F, \lambda) = \xi(H_1, \lambda)(\lambda^{p+1}(\lambda^q - 1) + \lambda^{p+q}(\lambda^t - \lambda)) = \xi(H, \lambda).$$

If A is monochromatic and B is not monochromatic then the number of λ-colorings of F is equal to $\varphi(H_1, \lambda)\lambda^{p+q}(\lambda^t - \lambda)$.

If A is monochromatic and B is monochromatic, we shall consider the subcases: a) the colors of A and B coincide; b) the colors of A and B are different.

a) Suppose that the common color of A and B is λ_0. If f is a coloring having required properties, we obtain four subcases:

$f(u) = f(v) = \lambda_0$, when the number of λ-colorings of F is equal to $\varphi(H_1, \lambda)(\lambda^{q-1} - 1)(\lambda^{p-1} - 1)$;

$f(u) = \lambda_0$ and $f(v) \neq \lambda_0$, the number is $\varphi(H_1, \lambda)(\lambda - 1)(\lambda^{p+q-2} - 1)$;

$f(u) \neq \lambda_0$ and $f(v) = \lambda_0$, we get $\varphi(H_1, \lambda)(\lambda - 1)(\lambda^{p-1} - 1)(\lambda^{q-1} - 1)$;

if $f(u) \neq \lambda_0$ and $f(v) \neq \lambda_0$, then the number of λ-colorings of F equals $\varphi(H_1, \lambda)(\lambda - 1)^2\lambda^{p+q-2}$.

b) If A and B have different colors then the number of λ-colorings of F is equal to $\varphi(H_1, \lambda)(\lambda - 1)\lambda^p(\lambda^q - 1)$.

By summing up these values we deduce that $\varphi(F, \lambda) = \varphi(H, \lambda)$, which completes the proof that $P(F, \lambda) = P(H, \lambda)$.

We have proved the result for any hypergraph having a pendant path of length 2; it is clear that it also holds for hypergraphs containing pendant paths of length at least 2. ■

For $m = 2$ the path $P_2^{h,r}$ is a sunflower hypergraph and its chromaticity follows from Theorem 3. If $m \geq 3$ the previous theorem has the following Corollary:

Corollary 5. *$P_m^{h,r}$ is not chromatically unique for every $m \geq 3$, $r \geq 1$ and $h \geq 2r + 1$.*

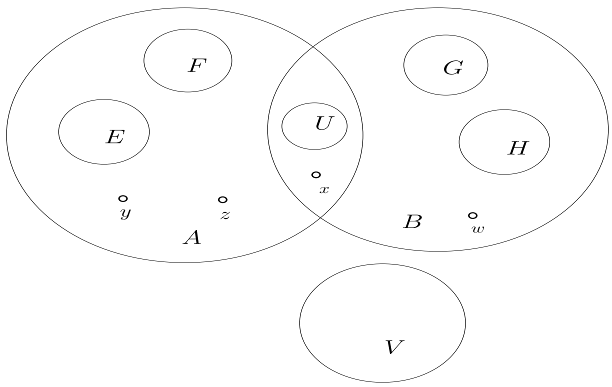

Figure 3. Structure of hypergraph X.

Theorem 6. *Cycle $C_3^{h,r}$ is not chromatically unique if $r \geq 2$.*

Proof. Denote $p = h - r$, where $h \geq p + 2$; $h \geq 2r$ is equivalent to $h \leq 2p$. By (1) we deduce $P(C_3^{h,r}, \lambda) = \lambda^{3p} - 3\lambda^{3p-h+1} + 3\lambda^{2p-h+1} - \lambda$. We shall define a hypergraph X which is not h-uniform and is chromatically equivalent to $C_3^{h,r}$.

For this consider the sets in Figure 3: E, F, G, H, U, V are pairwise disjoint, vertices x, y, z, w are distinct and $A = E \cup F \cup U \cup \{x, y, z\}$, $B = G \cup H \cup U \cup \{x, w\}$. $|A| = |B| = h$, $|V| = 2p - h - 1$, $|E| = 2p - h - 1$, $|F| = h - p$, $|U| = h - p - 2$, $|G| = h - p$, $|H| = 2p - h$. X is defined as follows: $V(X) = A \cup B \cup V$ and $E(X) = \{A, B, C, D\}$, where $C = F \cup G \cup H$ and $D = F \cup G \cup U \cup V \cup \{y, z, w\}$. It follows that $|V(X)| = 3p$, $|C| = h$ and $|D| = 2h - p$. Using (1) we get $P(X, \lambda) = \lambda^{3p} - 3\lambda^{3p-h+1} - \lambda^{c(D)} + \lambda^{c(A,B)} + \lambda^{c(A,C)} + \lambda^{c(A,D)} + \lambda^{c(B,C)} + \lambda^{c(B,D)} + \lambda^{c(C,D)} - \lambda^{c(A,B,C)} - \lambda^{c(A,B,D)} - \lambda^{c(A,C,D)} - \lambda^{c(B,C,D)} + \lambda^{c(A,B,C,D)} = P(C_3^{h,r}, \lambda)$, since: $\lambda^{c(D)} = \lambda^{c(B,C)} = \lambda^{4p-2h+1}, \lambda^{c(A,B)} = \lambda^{c(A,B,C)} = \lambda^{2p-h}, \lambda^{c(B,D)} = \lambda^{c(B,C,D)} = \lambda^{2p-h}, \lambda^{c(A,B,D)} = \lambda^{c(A,C,D)} = \lambda^{c(A,B,C,D)} = \lambda, \lambda^{c(A,C)} = \lambda^{c(A,D)} = \lambda^{c(C,D)} = \lambda^{2p-h+1}$. ■

Bokhary, Tomescu and Bhatti [2] proved that h-uniform linear elementary cycles $C_m^{h,1}$ of length m are not chromatically unique for every $m, h \geq 3$. This result and the previous theorem support the following conjecture:

Conjecture 7. *Cycles $C_m^{h,r}$ are not chromatically unique for every $m, h \geq 3$ and $r \geq 1$.*

It is not difficult to show that for small values of $m \geq 3$ and for every $r \geq 2$ paths $P_m^{h,r}$ and cycles $C_m^{h,r}$ are h-chromatically unique. This observation leads to the following:

Conjecture 8. *For every $m \geq 3$ and $r \geq 2$ paths $P_m^{h,r}$ and cycles $C_m^{h,r}$ are h-chromatically unique.*

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