Discussiones Mathematicae
Graph Theory 33 (2013) 665–676
doi:10.7151/dmgt.1697

GENERALIZED FRACTIONAL TOTAL COLORINGS OF COMPLETE GRAPHS[1]

Gabriela Karafová

Institute of Mathematics,
P.J. Šafárik University, Jesenná 5,
040 01 Košice, Slovakia

e-mail: gabriela.karafova@student.upjs.sk

Abstract

An additive and hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphism. Let $\mathcal{P}$ and $\mathcal{Q}$ be two additive and hereditary graph properties and let r, s be integers such that $r \geq s$. Then an $\frac{r}{s}$*-fractional* $(\mathcal{P}, \mathcal{Q})$*-total coloring* of a finite graph $G = (V, E)$ is a mapping f, which assigns an s-element subset of the set $\{1, 2, \ldots, r\}$ to each vertex and each edge, moreover, for any color i all vertices of color i induce a subgraph of property $\mathcal{P}$, all edges of color i induce a subgraph of property $\mathcal{Q}$ and vertices and incident edges have assigned disjoint sets of colors. The minimum ratio $\frac{r}{s}$ of an $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of G is called *fractional* $(\mathcal{P}, \mathcal{Q})$*-total chromatic number* $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \frac{r}{s}$. Let $k = \sup\{i : K_{i+1} \in \mathcal{P}\}$ and $l = \sup\{i : K_{i+1} \in \mathcal{Q}\}$. We show for a complete graph K_n that if $l \geq k + 2$ then $\chi''_{f,\mathcal{P},\mathcal{Q}}(K_n) = \frac{n}{k+1}$ for a sufficiently large n.

Keywords: fractional coloring, total coloring, complete graphs.

2010 Mathematics Subject Classification: 05C15.

1. Introduction

Let $G = (V, E)$ be a finite simple graph with the vertex set V and edge set E. We denote by n the number of vertices of G and by m the number of edges of G. By *elements* we will mean the vertices and the edges of a graph G.

A *total coloring* of a graph G is a coloring of the vertices and the edges, such that each two elements which are adjacent or incident obtain distinct colors. The

[1]This work was supported by the Slovak VEGA Grant under the contract No. VEGA 1/0428/10 and VVGS PF 46/2010/M.

minimum number of colors of a total coloring of G is called *total chromatic number* $\chi''(G)$ of G. There are some papers about this topic, for example [1, 3, 6].

The following conjecture is known as the Total Colouring Conjecture and was formulated independently in the 1960s by Behzad [1] and Vizing [9]. It has been verified for several special classes of graphs, including for example complete graphs (see [2, 6] for surveys).

Conjecture 1. *If G is a graph with maximum degree $\Delta(G)$, then $\chi''(G) \leq \Delta(G)+2$.*

An $\frac{r}{s}$*-fractional total coloring* of a graph G is a coloring of the vertices and the edges such that each vertex and each edge has been assigned an s-element subset of the set $\{1, 2, \ldots, r\}$ and each two adjacent or incident elements receive disjoint sets of colors. The *fractional total chromatic number* $\chi''_f(G)$ of G is the infimum ratio $\frac{r}{s}$ of an $\frac{r}{s}$-fractional total coloring of G. Kilakos and Reed [8] proved that $\chi''_f(G) \leq \Delta(G) + 2$ for any graph G.

In this paper we deal with generalized fractional total colorings of graphs. We denote the class of all finite simple graphs by $\mathcal{I}$. A *graph property* $\mathcal{P}$ is a non-empty isomorphism-closed subclass of $\mathcal{I}$. A property $\mathcal{P}$ is called *additive* if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$. A property $\mathcal{P}$ is called *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$.
We use the following standard notations for specific hereditary properties:

$\mathcal{O} = \{G \in \mathcal{I} : E(G) = \emptyset\}$,
$\mathcal{O}^k = \{G \in \mathcal{I} : \chi(G) \leq k\}$,
$\mathcal{D}_k = \{G \in \mathcal{I} :$ each subgraph of G contains a vertex of degree at most $k\}$,
$\mathcal{T} = \{G \in \mathcal{I} : G$ is a planar graph$\}$,
$\mathcal{I}_k = \{G \in \mathcal{I} : G$ does not contain $K_{k+2}\}$,
$\mathcal{O}_k = \{G \in \mathcal{I} :$ each component of G has at most $k+1$ vertices$\}$,
$\mathcal{S}_k = \{G \in \mathcal{I} : \Delta(G) \leq k\}$,

where $\chi(G)$ is the *chromatic number* and $\Delta(G)$ the *maximum degree* of the graph G.

In the following we will use only additive and hereditary graph properties.

Let $c(\mathcal{P}) = \sup\{i : K_{i+1} \in \mathcal{P}\}$ be the *completeness* of the property $\mathcal{P}$. Note that $c(\mathcal{O}^k) = c(\mathcal{I}_k) = c(\mathcal{O}_k) = c(\mathcal{D}_k) = c(\mathcal{S}_k) = k$ and $c(\mathcal{T}) = 3$.

Borowiecki and Mihók [5] dealt with graph properties and showed that the set of all additive and hereditary properties is a complete distributive lattice $(\mathbb{L}^a, \subseteq)$, where $\mathcal{O}$ is the smallest element of it and $\mathcal{I}$ is the greatest one. The set of properties $\mathcal{P} \in \mathbb{L}^a$ with $c(\mathcal{P}) = k, k \in \mathbb{N}$, is also a complete distributive lattice $(\mathbb{L}^a_k, \subseteq)$ with the smallest element $\mathcal{O}_k$ and the greatest element $\mathcal{I}_k$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two graph properties. We consider a total coloring of a graph G such that adjacent elements can obtain the same color but we require that subgraphs of G induced by the set of vertices of the same color to be of

property $\mathcal{P}$ and subgraphs of G induced by the set of edges of the same color to be of property $\mathcal{Q}$ and incident elements cannot have assigned the same color. For example, if $\mathcal{P} = \mathcal{O}$ and $\mathcal{Q} = \mathcal{O}_1$, then it is an ordinary total coloring of a graph G. Such properties have also been studied by Borowiecki *et al.* in [3, 5].

An $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of a finite graph $G = (V, E)$ is a mapping f, which assigns an s-element subset of the set $\{1, 2, \ldots, r\}$ to each vertex and each edge $\left(f : V \cup E \to \binom{\{1,2,\ldots,r\}}{s}\right)$, moreover, for any color i all vertices of the color i induce a subgraph of property $\mathcal{P}$, all edges of the color i induce a subgraph of property $\mathcal{Q}$ and vertices and incident edges have assigned disjoint sets of colors. The infimum ratio $\frac{r}{s}$ of an $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of G is called the fractional $(\mathcal{P}, \mathcal{Q})$-total chromatic number $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \frac{r}{s}$.

We deal with $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total colorings of complete graphs by using linear programming and the simplex method. The fractional $(\mathcal{P}, \mathcal{Q})$-total chromatic number can be obtained as a solution of a linear program with $|V| + |E|$ inequalities. The main result is that this linear program for complete graphs is equivalent with another one, which has only two inequalities and we can easily solve this problem by the simplex method.

2. General Graphs

As we know, there are two equivalent definitions of the fractional coloring of a graph. This is also true for generalized $(\mathcal{P}, \mathcal{Q})$-total coloring of G which has been shown in [7], and so we state here both these definitions.

Definition 1. Let G be a simple graph. Let $r, s \in \mathbb{N}$ and $s \leq r$. An $\frac{r}{s}$-*fractional* $(\mathcal{P},\mathcal{Q})$-*total coloring* of G is a mapping $f : V \cup E \to \binom{\{1,2,\ldots,r\}}{s}$ such that for each color i all vertices of color i induce a subgraph of property $\mathcal{P}$, all edges of color i induce a subgraph of property $\mathcal{Q}$, moreover, each incident vertex and edge have assigned disjoint sets of colors. The fractional $(\mathcal{P},\mathcal{Q})$-total chromatic number is $\chi''_{1,f,\mathcal{P},\mathcal{Q}}(G) = \inf\{\frac{r}{s} :\ G \text{ has an } \frac{r}{s}\text{-fractional } (\mathcal{P}, \mathcal{Q})\text{-total coloring}\}$.

Note that if $E \neq \emptyset$, then $r \geq 2s$ in every $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of a graph G.

Definition 2. A $(\mathcal{P},\mathcal{Q})$-*independent set* is a subset of $V \cup E$ such that the vertices in this set induce a graph of property $\mathcal{P}$, the edges induce a graph of property $\mathcal{Q}$ and, moreover, vertices and edges are not incident.

Definition 3. Let $I_1, I_2, \ldots, I_t,\ \ t \in \mathbb{N}$ be all (maximal) $(\mathcal{P},\mathcal{Q})$-independent sets in G. A *fractional* $(\mathcal{P},\mathcal{Q})$-*total coloring* of G is a mapping g, which assigns to each set $I_j,\ \ j = 1, \ldots, t$ a non-negative weight $g(I_j)$ such that $\sum_{u \in I_j} g(I_j) \geq 1$ for each

element $u \in V \cup E$. The fractional $(\mathcal{P},\mathcal{Q})$-total chromatic number $\chi''_{2,f,\mathcal{P},\mathcal{Q}}(G)$ of G is the least total weight of the fractional $(\mathcal{P},\mathcal{Q})$-total coloring of G.

As we mentioned above, Definitions 1 and 3 of the fractional total chromatic number $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ are equivalent.

For determining $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ according to Definition 3, we have to solve the following linear program:

$$\begin{aligned} &\textstyle\sum_{j=1}^{t} f(I_j) \to \min \\ &\textstyle\sum_{u\in I_j} f(I_j) \geq 1, \text{ for each } u \in V \cup E \\ &f(I_j) \geq 0, \text{ for each } j = 1, \ldots, t. \end{aligned} \tag{1}$$

Let $\mathcal{P} \supseteq \mathcal{O}$ and $\mathcal{Q} \supseteq \mathcal{O}_1$ be two additive and hereditary graph properties and G be a graph with n vertices and m edges. Let $k = c(\mathcal{P})$. Take an $\frac{r}{s}$-fractional $(\mathcal{P},\mathcal{Q})$-total coloring $\varphi(G)$ of a graph G, where r is the number of used colors, from which we choose an s-element subset for every element of the graph G. We denote the subset of vertices colored with color c by $V_c(G)$ and the number of colors used for exactly i vertices by x_i, $i \in \{0, \ldots, k+1\}$ ($x_i = |\{c : |V_c| = i\}|$). Consider an induced subraph of G on the vertices without color c and denote it by $G[V(G) \setminus V_c(G)]$. We choose a subgraph of $G\,[V(G) \setminus V_c(G)]$ of a property $\mathcal{Q}$ with the maximum number of edges and denote this number by $a_i(G, \varphi(G))$, because it depends on the graph G and the coloring $\varphi(G)$ of G and it holds that $a_i(G, \varphi(G)) \leq a_i(K_n, \varphi(K_n)) = a(n-i, \mathcal{Q}) = a_i$. It means that we can use the same color i we used for x_i vertices at most for a_i edges in G. It is easy to see that the sequence $(a_i)_{i=0}^{k+1}$ is decreasing. In the following we will often use only the notation a_i.

The cardinality of the multiset of all colors used for a fractional total coloring of a graph G is $(n+m)s$, otherwise, we do not have sufficient number of multicolors (colors with their multiplicities) for a correct coloring. Therefore we need ns multicolors for the vertices and ms for the edges. We get the following two inequalities sufficient for determining r for fixed s.

$$\begin{aligned} &h : \textstyle\sum_{i=0}^{k+1} x_i \to \min \sum_{i=0}^{k+1} i x_i \geq ns \\ &\textstyle\sum_{i=0}^{k+1} a_i x_i \geq ms \\ &x_i \in \mathbb{N}_0, \quad \text{for each } i = 0, \ldots, k+1. \end{aligned} \tag{2}$$

It means that the value $\frac{r}{s}$ is a lower bound for $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$. Now let $x'_i = \frac{x_i}{s}$. Then we can reformulate the problem (2) as follows:

$$
(3) \qquad \begin{aligned}
&h' : \textstyle\sum_{i=0}^{k+1} x'_i \to \min \\
&\textstyle\sum_{i=0}^{k+1} i x'_i \geq n \\
&\textstyle\sum_{i=0}^{k+1} a_i x'_i \geq m \\
&x'_i \in \mathbb{Q}_0^+, \quad \text{for each } i = 0, \ldots, k+1.
\end{aligned}
$$

The linear programs (2) and (3) are equivalent in the following sense: If $(x'_0, x'_1, \ldots, x'_{k+1})$ is an optimal solution of (3) and s is the least common multiple of the denominators of x'_i, $i = 1, 2, \ldots, k+1$, then $\sum_{i=0}^{k+1} x'_i = \frac{r}{s}$ with $r \in \mathbb{N}_0$ and $(x'_0.s, x'_1.s, \ldots, x'_{k+1}.s)$ is an optimal solution of (2) for fixed s and $\sum_{i=0}^{k+1} x'_i.s = \sum_{i=0}^{k+1} x_i = r$. Moreover, since for each s $\frac{r}{s}$ from the solution of (2) is a lower bound for $\chi''_{f,\mathcal{P},\mathcal{Q}}$, then a solution of (3) is also a lower bound for $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$. Finally we want to show that this lower bound is the exact value of $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$. Now we show that (1) implies (3) for all graphs and in the following section we show that for complete graphs also (3) implies (1).

Theorem 2. *Let G be a simple graph, $\mathcal{P}$, $\mathcal{Q}$ be two additive and hereditary graph properties and $k = c(\mathcal{P})$. Then for each optimal solution of the linear program* (1) $\boldsymbol{f(I)} = (f(I_1), f(I_2), \ldots, f(I_t))$, $t \in \mathbb{N}$ *there exists a feasible solution* $\boldsymbol{x'} = (x'_0, x'_1, \ldots, x'_{k+1})$ *of the linear program* (3)*, moreover,* $h'(\boldsymbol{x'}) = h(\boldsymbol{f(I)})$.

Proof. Let $I_1, \ldots, I_t$, $t \in \mathbb{N}$ be all $(\mathcal{P}, \mathcal{Q})$-independent subsets of $V \cup E$. Suppose that there exists an optimal solution of problem (1) $\mathbf{f(I)} = (f(I_1), f(I_2), \ldots, f(I_t))$, $t \in \mathbb{N}$. Let $x'_i = \sum_{|I_j \cap V| = i} f(I_j)$. According to the assumptions it holds that for each $u \in V \cup E$

$$\sum\nolimits_{u \in I_j} f(I_j) \geq \text{ and } f(I_j) \geq 0,$$

from which we obtain the following inequality

$$\sum\nolimits_{v \in V} \sum\nolimits_{I_j \ni v} f(I_j) \geq n.$$

Then the first inequality that we need holds:

$$\sum\nolimits_{i=0}^{k+1} i x'_i = \sum\nolimits_{i=0}^{k+1} i \sum\nolimits_{|I_j \cap V| = i} f(I_j) = \sum\nolimits_{v \in V} \sum\nolimits_{I_j \ni v} f(I_j) \geq n.$$

The last equality holds because on both sides there is the sum of all weights over all vertices.

Analogously we show the inequality for the edges where we have the following constraints from the previous:

$$\sum\nolimits_{e \in E} \sum\nolimits_{I_j \ni e} f(I_j) \geq m.$$

Let I^i be a set of all $(\mathcal{P},\mathcal{Q})$-independent sets I_j with exactly i vertices and we denote its elements by I^i_b, where $1 \le b \le \left|I^i\right|$. The definition of a_i implies $a_i \ge \left|E(I^i_b)\right|$ for each $b \in \left[1, \left|I^i\right|\right]$

$$\begin{aligned}\sum\nolimits_{i=0}^{k+1} a_i x'_i &= \sum\nolimits_{i=0}^{k+1} a_i \sum\nolimits_{|I_j\cap V|=i} f(I_j) = \sum\nolimits_{i=0}^{k+1} a_i \sum\nolimits_{b=1}^{|I^i|} f(I^i_b)\\ &\ge \sum\nolimits_{i=0}^{k+1} \sum\nolimits_{b=1}^{|I^i|} \left|E(I^i_b)\right| f(I^i_b) = \sum\nolimits_{e\in E} \sum\nolimits_{I_j\ni e} f(I_j) \ge m.\end{aligned}$$

The values of the objective functions of problems (1) and (3) are equal:

$$\begin{aligned}h(\mathbf{f(I)}) &= \sum\nolimits_{j=1}^{t} f(I_j) = \sum\nolimits_{i=0}^{k+1} \sum\nolimits_{|I_j\cap V|=i} f(I_j)\\ &= \sum\nolimits_{i=0}^{k+1} f(I^i) = \sum\nolimits_{i=0}^{k+1} x'_i = h'(\mathbf{x}').\end{aligned}$$

■

3. Complete Graphs

The results in the Theorem 2 and the following theorem mean that it is sufficient to solve problem (3) with only two inequalities to determine the fractional $(\mathcal{P},\mathcal{Q})$-total chromatic number of a complete graph.

Theorem 3. *Let G be a complete graph, $\mathcal{P}$, $\mathcal{Q}$ be two additive and hereditary graph properties and $k = c(\mathcal{P})$. Then for each optimal solution of the linear program (3) $\boldsymbol{x'} = (x'_0, x'_1, \ldots, x'_{k+1})$ there exists a feasible solution of the linear program (1) $\boldsymbol{f(I)} = (f(I_1), f(I_2), \ldots, f(I_t))$, $t \in \mathbb{N}$, moreover, $h'(\boldsymbol{x'}) = h(\boldsymbol{f(I)})$.*

Proof. Suppose that we have an optimal solution of problem (3) $\mathbf{x}' = (x'_0, x'_1, \ldots, x'_{k+1})$ and according to the previous we know that $\sum_{i=0}^{k+1} i x'_i \ge n$ and $\sum_{i=0}^{k+1} x'_i a_i \ge \binom{n}{2}$.

Denote $I^i := \{I_j : |I_j \cap V| = i \wedge |I_j \cap E| = a_i\}$, $m_i := \left|I^i\right|$ and let

$$f(I_j) = \begin{cases} \frac{x'_i}{m_i}, & \text{if } I_j \in I^i,\\ 0, & \text{otherwise.}\end{cases}$$

Then the following statement holds for each $v \in V$:

$$\sum\nolimits_{I_j\ni v} f(I_j) = \sum\nolimits_{i=0}^{k+1} \sum\nolimits_{I_j\in I^i : I_j\ni v} \frac{x'_i}{m_i} = \sum\nolimits_{i=0}^{k+1} \frac{x'_i}{m_i}\cdot\frac{im_i}{n} \ge 1.$$

The last equality holds, because when we count all vertices over all I_j from I^i we get a number im_i and independent sets in I^i are symmetric. Therefore we know that each vertex $v \in V$ belongs to $\frac{im_i}{n}$ subsets $I_j \in I^i$ for each $i \in \{0, 1, \ldots, k+1\}$.

Analogously we count all edges over all sets $I_j \in I^i$ with non-zero weights and again we use the fact that these sets are symmetric. Consequently each edge belongs to $\frac{a_i m_i}{\binom{n}{2}}$ subsets $I_j \in I^i$ for each $i \in \{0, 1, \ldots, k+1\}$ and so the following statement is also satisfied for each $e \in E$:

$$\sum\nolimits_{I_j \ni e} f(I_j) = \sum\nolimits_{i=0}^{k+1} \sum\nolimits_{I_j \in I^i : I_j \ni e} \frac{x'_i}{m_i} = \sum\nolimits_{i=0}^{k+1} \frac{x'_i}{m_i} \cdot \frac{a_i m_i}{\binom{n}{2}} \geq 1.$$

The values of the objective functions are equal:

$$h'(\mathbf{x}') = \sum\nolimits_{i=0}^{k+1} x'_i = \sum\nolimits_{i=0}^{k+1} f(I^i) = \sum_{i=0}^{k+1} \sum\nolimits_{|I_j \cap V| = i} f(I_j) = \sum\nolimits_{j=1}^{t} f(I_j) = h(\mathbf{f(I)}).$$

■

We should consider all maximal independent sets I_j in one direction of the last proof, but we can see that it is sufficient to consider only all maximum from maximal sets.

Obviously $a_i \geq \lfloor \frac{n-i}{l+1} \rfloor \binom{l+1}{2} + \binom{n-i-\lfloor \frac{n-i}{l+1} \rfloor (l+1)}{2}$ for complete graphs, because when we use color c for i vertices, we get $n-i$ vertices without color c. Divide these $n-i$ vertices according to the property $\mathcal{Q}$ into $\lfloor \frac{n-i}{l+1} \rfloor$ sets with $l+1$ vertices. These arising subgraphs have at most $\binom{l+1}{2}$ edges. So all subgraphs consist of at most $\lfloor \frac{n-i}{l+1} \rfloor \binom{l+1}{2}$ edges, which induce a subgraph belonging to $\mathcal{Q}$. The number of the residual edges is $\binom{n-i-\lfloor \frac{n-i}{l+1} \rfloor (l+1)}{2}$. The equality holds for $\mathcal{Q} = \mathcal{O}_l$ and $\mathcal{O}_l$ is the smallest graph property for edges. It means that for other properties strict inequality holds.

In the following two theorems we show that for an arbitrary additive and hereditary property $\mathcal{P}$ it holds that $\chi''_{f,\mathcal{P},\mathcal{Q}}(K_n) = \frac{n}{c(\mathcal{P})+1}$ if $\mathcal{Q} = \mathcal{I}_l$ or if $\mathcal{Q}$ is also arbitrary and hereditary property with $c(\mathcal{Q}) \geq c(\mathcal{P}) + 2$. We use the previous observations about a_i for complete graphs.

Theorem 4. *Let $\mathcal{P}, \mathcal{Q}$ be two additive and hereditary properties such that $c(\mathcal{Q}) \geq c(\mathcal{P}) + 2$. Then there exists a $T(\mathcal{P}, \mathcal{Q})$ such that $\chi''_{f,\mathcal{P},\mathcal{Q}}(K_n) = \frac{n}{c(\mathcal{P})+1}$ holds for each $n \geq T(\mathcal{P}, \mathcal{Q})$.*

Proof. Let $k = c(\mathcal{P})$ and $l = c(\mathcal{Q})$. According to the previous theorem it is sufficient to solve problem (3) in order to determine $\chi''_{f,\mathcal{P},\mathcal{Q}}(K_n)$ and therefore to prove this theorem. We rewrite this linear program to the standard form. We

need to use two slack variables $p_1, p_2 \geq 0$:

$$
\begin{aligned}
& h' : \textstyle\sum_{i=0}^{k+1} x'_i \to \min \\
& -\textstyle\sum_{i=0}^{k+1} i x'_i + p_1 = -n \\
& -\textstyle\sum_{i=0}^{k+1} a_i x'_i + p_2 = -\frac{n(n-1)}{2} \\
& p_1, p_2, x'_i \geq 0, \quad \text{for each } i = 0, \ldots, k+1.
\end{aligned} \tag{4}
$$

We can see that the variables p_1 and p_2 are in the basis and we are able to solve the dual problem:

	x'_0	x'_1	x'_2	$\ldots$	x'_{k-1}	x'_k	x'_{k+1}	p_1	p_2	
	1	1	1	$\ldots$	1	1	1	0	0	0
p_1	0	-1	-2	$\ldots$	$-(k-1)$	$-k$	$-(k+1)$	1	0	$-n$
p_2	$-a_0$	$-a_1$	$-a_2$	$\ldots$	$-a_{k-1}$	$-a_k$	$-a_{k+1}$	0	1	$-\frac{n(n-1)}{2}$

A pivot in the first row of this table is $-(k+1)$. We need to get a number 1 instead of the pivot and zeros instead of other numbers in the column with pivot.

	x'_0	x'_1	$\ldots$	x'_k	x'_{k+1}	p_1	p_2	
	1	$\frac{k}{k+1}$	$\ldots$	$\frac{1}{k+1}$	0	$\frac{1}{k+1}$	0	$-\frac{n}{k+1}$
x'_{k+1}	0	$\frac{1}{k+1}$	$\ldots$	$\frac{k}{k+1}$	1	$-\frac{1}{k+1}$	0	$\frac{n}{k+1} \geq 0$
p_2	$-a_0$	$\frac{a_{k+1}}{k+1} - a_1$	$\ldots$	$\frac{ka_{k+1}}{k+1} - a_k$	0	$-\frac{a_{k+1}}{k+1}$	1	$\frac{na_{k+1}}{k+1} - \frac{n(n-1)}{2}$

If $\frac{na_{k+1}}{k+1} - \frac{n(n-1)}{2} \geq 0$ then the optimal value is $h'(x'^*_0, \ldots, x'^*_{k+1}) = h'(0, \ldots, 0, \frac{n}{k+1})$ $= \frac{n}{k+1}$. Now we want to find a relation between $c(\mathcal{P}) = k$ and $c(\mathcal{Q}) = l$ such that $a_{k+1} \geq \frac{(k+1)(n-1)}{2}$. We know that

$$\binom{n-(k+1) - \left\lfloor \frac{n-(k+1)}{l+1} \right\rfloor (l+1)}{2} \geq 0,$$

$$
\begin{aligned}
a_{k+1} &\geq \left\lfloor \frac{n-(k+1)}{l+1} \right\rfloor \binom{l+1}{2} + \binom{n-(k+1) - \left\lfloor \frac{n-(k+1)}{l+1} \right\rfloor (l+1)}{2} \\
&\geq \left\lfloor \frac{n-(k+1)}{l+1} \right\rfloor \binom{l+1}{2} > \left(\frac{n-(k+1)}{l+1} - 1 \right) \binom{l+1}{2},
\end{aligned}
$$

therefore

$$a_{k+1} > \frac{(n-(k+1)-(l+1))l}{2}.$$

So we want to show that $\frac{(n-(k+1)-(l+1))l}{2} \geq \frac{(k+1)(n-1)}{2}$ if $l \geq k+2$:

$$\frac{(n-(k+1)-(l+1))l}{2} \geq \frac{(k+1)(n-1)}{2} \Leftrightarrow n(l-k-1)-l(k+2)-l^2+k+1 \geq 0.$$

If $l \geq k+2$, i.e. $l-k-1>0$ then we get $n \geq \frac{l^2+l(k+2)-k-1}{l-k-1}$. We can take this lower bound as $T(\mathcal{P},\mathcal{Q})$ but the exact value of $T(\mathcal{P},\mathcal{Q})$ can be lower. If $l-k-1<0$ we get $n < \frac{l^2+l(k+2)-k-1}{l-k-1} < 0$ and so this case cannot occur. If $l-k-1=0$ then the last inequality is equivalent to the expression $-2l^2 \geq 0$ for each n therefore this case cannot occur, too. ■

We get $(k+2)$-tuple $h'(x_0'^*,\ldots,x_{k+1}'^*) = h'(0,\ldots,0,\frac{n}{k+1})$ as an optimal solution of linear program in the previous proof. It means that we use each color for exactly $k+1$ vertices and exactly a_{k+1} edges. If we do not get any optimal solution after the first step of the simplex method, we get two nonzero variables $x_{k+1}'^* = \frac{n}{k+1}$ and some $x_i'^* > 0$. In this case we use only colors, which are used exactly for $k+1$ vertices and colors that are used for exactly i vertices. If we consider $\frac{r}{s}$-coloring, the number of used colors exactly for i vertices is $x_i'^*.s$.

Corollary 5. *Let $\mathcal{P},\mathcal{Q}$ be two additive and hereditary properties such that $c(\mathcal{Q}) \geq c(\mathcal{P})+2$. Then $\chi''_{f,\mathcal{P},\mathcal{Q}}(K_n) = \frac{n}{c(\mathcal{P})+1}$ if and only if there exists a graph of property $\mathcal{Q}$ on $n-(c(\mathcal{P})+1)$ vertices with at least $\frac{(n-1)(c(\mathcal{P})+1)}{2}$ edges.*

The result in the following theorem was proved by A. Kemnitz *et al.* in [7] by the construction of coloring. We prove this result by using the results from this paper.

Theorem 6. *Let $\mathcal{P}$ be additive and hereditary property and $\mathcal{Q}=\mathcal{I}_l$. There exists $T(\mathcal{P},\mathcal{I}_l)$ such that for each $n \geq T(\mathcal{P},\mathcal{I}_l)$ it holds that $\chi''_{f,\mathcal{P},\mathcal{I}_l}(K_n) = \frac{n}{c(\mathcal{P})+1}$.*

Proof. Denote $k=c(\mathcal{P})$. This proof is similar to the proof of Theorem 4, but we know the precise value of a_i for $0 \leq i \leq k+1$ by using the well known Turan's theorem. Whereas each $(l+1)-$partite graph does not contain any complete graph on $l+2$ vertices as a subgraph, we can divide all vertices into $l+1$ equable sets. We have $n-i-(l+1)\left\lfloor\frac{n-i}{l+1}\right\rfloor < l+1$ sets with $\left\lceil\frac{n-i}{l+1}\right\rceil$ vertices and $l+1-\left(n-i-(l+1)\left\lfloor\frac{n-i}{l+1}\right\rfloor\right)$ sets with $\left\lfloor\frac{n-i}{l+1}\right\rfloor$ vertices. Therefore a_i is the number of edges in such complete $(l+1)$- partite graph:

$$a_i = \left\lceil\frac{n-i}{l+1}\right\rceil^{n-i-(l+1)\left\lfloor\frac{n-i}{l+1}\right\rfloor} \cdot \left\lfloor\frac{n-i}{l+1}\right\rfloor^{l+1-(n-i-(l+1)\left\lfloor\frac{n-i}{l+1}\right\rfloor)}.$$

According to the previous proof we need to find out whether $\frac{na_{k+1}}{k+1} - \frac{n(n-1)}{2} \geq 0$. We want to know whether there exists $T(\mathcal{P},\mathcal{I}_l)$ for each l and k such that for each $n \geq T(\mathcal{P},\mathcal{I}_l)$ the following inequality is satisfied:

$$\left\lceil \frac{n-(k+1)}{l+1} \right\rceil^{n-(k+1)-(l+1)\left\lfloor \frac{n-(k+1)}{l+1} \right\rfloor} \cdot \left\lfloor \frac{n-(k+1)}{l+1} \right\rfloor^{l+1-\left(n-(k+1)-(l+1)\left\lfloor \frac{n-(k+1)}{l+1} \right\rfloor\right)}$$
$$\geq \frac{(n-1)(k+1)}{2}.$$

We know that the following inequalities hold:

$$\left\lceil \frac{n-(k+1)}{l+1} \right\rceil^{n-(k+1)-(l+1)\left\lfloor \frac{n-(k+1)}{l+1} \right\rfloor} \cdot \left\lfloor \frac{n-(k+1)}{l+1} \right\rfloor^{l+1-\left(n-(k+1)-(l+1)\left\lfloor \frac{n-(k+1)}{l+1} \right\rfloor\right)}$$
$$\geq \left\lfloor \frac{n-(k+1)}{l+1} \right\rfloor^{l+1} \geq \left(\frac{n-(k+1)}{l+1} - 1 \right)^{l+1} = \left(\frac{n-(k+1)-(l+1)}{l+1} \right)^{l+1}$$
$$\geq \frac{(n-1)(k+1)}{2}.$$

The last inequality is obvious, because there exists $T(\mathcal{P}, \mathcal{I}_l)$ such that for each $n \geq T(\mathcal{P}, \mathcal{I}_l)$ this polynomial of unknown n is non-negative since $l \geq 1$. ■

A. Kemnitz *et al.* in [7] proved that $chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(K_n) = \frac{n(n+1)}{2(n-1)}$ for odd n and they determined the lower and upper bound for even n. In the following theorem we show that it is the same number for $n \geq 3$.

Theorem 7. $\chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(K_n) = \frac{n(n+1)}{2(n-1)}$ *for each integer* $n \geq 3$.

Proof. Here we have $a_i = (n-i) - 1$ for $i = 0, 1, 2$, because the graph property $\mathcal{D}_1$ is a class of forests. Every a_i is non-negative. According to the previous we solve the following linear program:

$$\begin{aligned}
&x'_0 + x'_1 + x'_2 \to \min \\
&x'_1 + 2x'_2 \geq n \\
&(n-1)x'_0 + (n-2)x'_1 + (n-3)x'_2 \geq \tfrac{n(n-1)}{2} \\
&x'_0, x'_1, x'_2 \geq 0.
\end{aligned}$$

	x'_0	x'_1	x'_2	p_1	p_2	
	1	1	1	0	0	0
p_1	0	-1	**-2**	1	0	-n
p_2	$1-n$	$2-n$	$3-n$	0	1	$-\frac{n(n-1)}{2}$

After pivoting according to -2 in the p_1-row:

	x'_0	x'_1	x'_2	p_1	p_2	
	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	-$\frac{n}{2}$
x'_2	0	$\frac{1}{2}$	1	-$\frac{1}{2}$	0	$\frac{n}{2}$
p_2	$1-n$	$\frac{1-n}{2}$	0	$\frac{3-n}{2}$	1	-n

After second pivoting according to $1-n$ in p_2-row:

	x'_0	x'_1	x'_2	p_1	p_2	
	0	0	0	$\frac{1}{n-1}$	$\frac{1}{n-1}$	$-\frac{n(n+1)}{2(n-1)}$
x'_2	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	$\frac{n}{2}$
x'_0	1	$\frac{1}{2}$	0	$\frac{3-n}{2(1-n)}$	$\frac{1}{1-n}$	$\frac{n}{n-1}$

For $n \geq 3$ all conditions, which are required for the table in optimum, are met. It is easy to see that $\chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(K_n) = \frac{n(n-1)}{2(n-1)}$. It means that we can choose $s = 2(n-1)$. Furthermore, from the last column in this table, we can say that we need $\frac{n}{2}s = n(n-1)$ colors, which will be used for exactly two vertices and $a_2 = n-3$ edges and $\frac{n}{n-1}s = 2n$ colors for no vertex and $a_0 = n-1$ edges. ■

References

[1] M. Behzad, Graphs and their chromatic numbers, Doctoral Thesis (Michigan state University, 1965).

[2] M. Behzad, *The total chromatic number of a graph*, in: Combinatorial Mathematics and its Applications, D.J.A.Welsh, Ed., (Academic Press, London, 1971) 1–10.

[3] M. Borowiecki, I. Broere, M. Frick, P. Mihók and G. Semanišin, *A survey of hereditary properties of graphs*, Discuss. Math. Graph Theory **17** (1997) 5–50. doi:10.7151/dmgt.1037

[4] M. Borowiecki, A. Kemnitz, M. Marangio and P. Mihók, *Generalized total colorings of graphs*, Discuss. Math. Graph Theory **31** (2011) 209–222. doi:10.7151/dmgt.1540

[5] M. Borowiecki and P. Mihók, *Hereditary properties of graphs*, in: Advances in Graph Theory, V.R. Kulli, Ed., (Vishwa International Publication, Gulbarga, 1991) 41–68.

[6] A. Chetwynd, *Total colourings*, in: Graphs Colourings, Pitman Research Notes in Mathematics No.218, R. Nelson and R.J. Wilson Eds., (London, 1990) 65–77.

[7] A. Kemnitz, M. Marangio, P. Mihók, J. Oravcová and R. Soták, *Generalized fractional and circular total colorings of graphs*, (2010), preprint.

[8] K. Kilakos and B. Reed, *Fractionally colouring total graphs*, Combinatorica **13** (1993) 435–440. doi:10.1007/BF01303515

[9] V.G. Vizing, *Some unsolved problems in graph theory*, Russian Math. Surveys **23** (1968) 125–141.
doi:10.1070/RM1968v023n06ABEH001252

Received 25 May 2011
Revised 6 August 2012
Accepted 6 August 2012