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UNIVERSALITY IN GRAPH PROPERTIES WITH DEGREE RESTRICTIONS

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Abstract

Rado constructed a (simple) denumerable graph R with the positive integers as vertex set with the following edges: For given m and n with $m < n$, m is adjacent to n if n has a 1 in the m'th position of its binary expansion. It is well known that R is a universal graph in the set $\mathcal{I}_c$ of all countable graphs (since every graph in $\mathcal{I}_c$ is isomorphic to an induced subgraph of R).

A brief overview of known universality results for some induced-hereditary subsets of $\mathcal{I}_c$ is provided. We then construct a k-degenerate graph which is universal for the induced-hereditary property of finite k-degenerate graphs. In order to attempt the corresponding problem for the property of countable graphs with colouring number at most $k+1$, the notion of a property with assignment is introduced and studied. Using this notion, we

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are able to construct a universal graph in this graph property and investigate its attributes.

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1. Introduction

For general graph theoretic notions, the notation and terminology of [7] will be used. In particular, for any two graphs G and $H = (V', E')$, we say that G is a *subgraph* of H, denoted by $G \subseteq H$, if there is a subset $V \subseteq V'$ and a subset $E \subseteq E'$ (with every edge $e \in E$ an adjacency between two vertices in V) such that (V, E) is a graph which is isomorphic to G. We call (V, E) itself an *internal subgraph* of H. G is an *induced subgraph* of H, denoted by $G \leq H$, if G is isomorphic to such a graph (V, E) of which E contains all and only the edges $xy \in E'$ for which $x, y \in V$. We shall also write $G \subset H$ ($G < H$) to denote the fact that G is a subgraph (an induced subgraph respectively) of H which is not isomorphic to H.

There is (up to isomorphism) clearly only one subgraph induced by a given subset W of the vertex set V of a graph $G = (V, E)$; this subgraph is denoted by $G[W]$ and called the *subgraph of G generated* (*or spanned*) *by W*.

All graphs considered here for investigation are simple, undirected, unlabelled and have countable vertex sets. When the vertex set is taken to be the set, or some subset, of the positive integers $\mathbf{N} = \{1, 2, \ldots\}$, number-theoretic properties of the integers may be employed in constructions and proofs. Otherwise, the vertex set of a graph may be indexed by $\mathbf{N}$ or one of its subsets.

For notions related to hereditary graph properties the notation and terminology of [1] will be used. For ease of reference we formulate some of the basic definitions in this paper too. A (*graph*) *property* is an isomorphism-closed subclass of the class of all countable graphs. Since we have for many purposes, in a graph property, no reason to distinguish between isomorphic copies of a graph, we consider the class of all (simple) graphs to be a set and we use the notation $\mathcal{I}_c$ to denote this set of (countable) graphs. One subset of a property $\mathcal{P}$ of countable graphs ($\mathcal{P} = \mathcal{P}_c$) is also important for us and we introduce notation for it too: $\mathcal{P}_f$ will denote the set of finite graphs in $\mathcal{P}$. We say that the graph property $\mathcal{P}$ is *of finite character* if whenever for a graph G we have that, for every finite $H \leq G$, $H \in \mathcal{P}_f$, then we have $G \in \mathcal{P}$ too. In this paper we will often have occasion to deal with two graphs that are isomorphic and, if they are, we shall refer to any one of them as a *clone* of the other.

A property $\mathcal{P}$ is *induced-hereditary* if, whenever $G \in \mathcal{P}$ and $H \leq G$, then $H \in \mathcal{P}$ too. Let $\mathcal{P}$ be a set of countable graphs. Following [7], we define a graph U to be a *universal graph for* $\mathcal{P}$ if every graph in $\mathcal{P}$ is an induced subgraph of U; it is a *universal graph in* $\mathcal{P}$ if $U \in \mathcal{P}$ too. (In Section 4 we shall point out that if the property $\mathcal{P}$ involves extrinsic structure linked globally to the graph and called "assignment"—a notion defined there—then some extra care is appropriate in the employment of these concepts, as in Section 5.) Since a universal graph U for $\mathcal{P}$ is allowed to be outside $\mathcal{P}$ and hence, presumably, to be uncountable, the existence of at least one such U becomes trivial: take U to be the disjoint union of one clone from each isomorphism class in $\mathcal{P}$ (i.e., of a "skeleton" of $\mathcal{P}$). The fact that this U is in general uncountable follows from Lemma 1 of [4]; a countable universal graph for any induced-hereditary graph property is constructed in that paper too.

Rado [17] constructed the following (simple) denumerable graph on $\mathbf{N}$: For given m and n with $m < n$, m is adjacent to n if n has a 1 in the m'th position of its binary expansion. We shall denote this graph by R. It is well known that R is a universal graph in the induced-hereditary property $\mathcal{I}_c$ of countable graphs. A very useful and, in fact, a characteristic property of R is that it has the *extension property*: For every two finite disjoint sets U and V of vertices of R there is a vertex not in $U \cup V$ which is adjacent to every vertex of U and to no vertex of V.

In Section 2 we give a brief overview of known universality results for some induced-hereditary subsets of $\mathcal{I}_c$. In Section 3 we then construct a denumerable k-degenerate graph which is universal for the induced-hereditary property of finite k-degenerate graphs. The construction is obtained by restricting the choice of vertices and the choice of edges used above in the construction of R.

In Section 4, graph properties with assignment are introduced. This concept is then utilised in our study in Section 5 of universality in the property of graphs with colouring number at most $k+1$. Section 6 describes some further results on the universal graphs discussed in Sections 3 and 5.

2. Universal Graphs for and in Induced-hereditary Properties

The table below summarises some of the published results on universal graphs for (or in) some induced-hereditary properties of countable graphs. (There is more on (induced-)hereditary properties in [1].) Throughout this table, k is a positive integer. In this table, $\mathcal{S}$ denotes a finite set of cycles and $\mathcal{S}_k$ denotes the set of odd cycles $\{C_3, C_5, \ldots, C_{2k+1}\}$.

Given any graph H, a graph G is *H-colourable* if and only if there exists an edge-preserving function $f : V(G) \to V(H)$. Such a function is called a *homomorphism* and is denoted by $f : G \to H$. For a given finite graph H and a

given set of connected finite graphs $\mathcal{T}$, the induced-hereditary graph properties $\to H$ and $-\mathcal{T}$ are defined by
$\to H = \{G \in \mathcal{I}_f :$ there is a homomorphism from G into $H\}$,
$-\mathcal{T} = \{G \in \mathcal{I}_f :$ for each $T \in \mathcal{T}$, T is not an induced subgraph of $G\}$.
In [11] it is shown that there is no universal graph in any set $Forb(G)$ of countable graphs obtained by taking a finite, 2-connected graph G which is not complete and requiring that the graphs in $Forb(G)$ are exactly those not containing G as a subgraph.

As mentioned, in [4] a construction of a countable universal graph for every induced-hereditary property of countable graphs (even without restricting us to simple graphs) is given; this result includes each of the properties in the table below and also many of the well-known properties mentioned in [1].

Property	Description	$U \in \mathcal{P}$?	Characterisation of U?	Reference(s)
$\mathcal{I}_c$	All graphs	Yes, the Rado graph $R \in \mathcal{I}_c$	$C \cong R$ iff C has the extension property	[17]
$\mathcal{P}_{fin}$	Graphs with all vertices of finite degree	Does not exist in $\mathcal{P}_{fin}$		[17] (accredited to N.G. de Bruijn)
$\mathcal{F}_{k,n}$	Graphs with at most n vertices and degree at most k	No, in general $U_{k,n}$, though finite, has more than n vertices and degree at most k		[9]
$\mathcal{L}_k$	Directed labelled graphs	Yes, the graph $L_k \in \mathcal{L}_k$	$C \cong L_k$ iff C has the k-extension property	[2]
$-\{K_{k+2}\}$	K_{k+2}-free graphs	Yes, the graph $G_k \in -\{K_{k+2}\}$	$C \cong G_k$ iff C has an adapted extension property	[10] and [13]
$-\{K_{m,n}\}$	$K_{m,n}$-free graphs	Exists if and only if $m = 1$ and $n \leq 3$		[14]
$-\{C_3\}$	C_3-free graphs	Yes, the graph $G_1 \in -\{C_3\}$	Same as K_3-free graphs above	
$-\{C_4\}$	C_4-free graphs	Does not exist in $-\{C_4\}$		[12]
$-\{C_n\}$, $n \geq 5$	C_n-free graphs	Does not exist in $-\{C_n\}$		[5]
$-\mathcal{S}$	Limited cycle-free graphs	Exists in $-\mathcal{S}$ if and only if $\mathcal{S} = \mathcal{S}_k$		[6]
$\to H$	Hom-property for finite H	Known to exist in $\to H$		[16]

When investigating universality for, and especially *in*, induced-hereditary graph properties, one stumbles across a subtle distinction between two types of properties. In the *first* type the property is defined in a *purely intrinsic* way. When then saying that "G has property $\mathcal{P}$", no mathematical structure outside of G itself as a graph is involved at all. Except for "Directed labelled graphs" and the "Hom-property for finite H", all the properties in the table above are of this first type, have purely intrinsic descriptions, e.g. in terms of degrees of vertices, or the exclusion of certain induced subgraphs. Another property of this first type, in the focus of Section 3, is k-degeneracy, defined purely intrinsically in terms of the minimum degree of finite induced subgraphs.

The *second* type of graph property of present relevance *involves extrinsic structure* (linked to the graph globally) in its definition. The property of "Directed labelled graphs" involves mathematical functions from the edge set to a set of directions, and from the edge and vertex sets to sets of labels. The hom-property "H-colourable" [3] involves a mathematical function, a homomorphism, from the vertex set into $V(H)$. The property "has colouring number at most $k+1$", in the focus of Section 5, is of the second type, involving the extrinsic structure of a mathematical function, labelling bijection, from the vertex set to either $[n] = \{1, 2, \ldots, n\}$ or $\mathbf{N} = \{1, 2, \ldots\}$, with certain properties. But before that, in Section 4, we shall explicate more fully the nature of properties of the second type and what this entails for universality.

3. Universality for Finite k-degenerate Graphs

We now investigate universality for k-degenerate graphs and start with the definitions we need.

Definition 1 [15]**.** A finite graph G is defined to be *k-degenerate* if the minimum degree $\delta(H)$ of each induced subgraph H of G satisfies $\delta(H) \leq k$.

We now turn our attention to countable graphs.

Definition 2. The property of *k-degenerate countable graphs* is defined by $\mathcal{D}_k = \{G \in \mathcal{I}_c$: the minimum degree of every finite induced subgraph H of G satisfies $\delta(H) \leq k\}$.

Note that $\mathcal{D}_k$ is an induced-hereditary graph property; so, of course, is $(\mathcal{D}_k)_f$. We now construct a universal graph F_k for $(\mathcal{D}_k)_f$. A corresponding construction of a universal graph in the property of graphs with colouring number at most $k+1$ and a discussion of its properties is contained in Sections 5 and 6. Throughout this and the next sections, we assume that k is a given positive integer. In the (finite) power series $n = \sum_{i=0}^{\infty} n_i 2^i$ we shall refer to $n_{i-1} (i \geq 1)$ as *the entry in the i'th position* of the *binary expansion* of the positive integer n; $n_{i-1} \in \{0, 1\}$.

The graph F_k is obtained by taking a denumerable subset of $\mathbf{N}$ as vertex set and restricting the choice of edges of the Rado graph R.

Definition 3. Let $\mathbf{N}_k$ denote the set of positive integers with at most $k+1$ ones in their binary expansion. The graph F_k has $\mathbf{N}_k$ as its vertex set and has the following edges: For given positive integers m and n in $\mathbf{N}_k$ with $m < n$, m is adjacent to n if n has a one in position m and a one in position x for some $x > m$ of its binary expansion.

Note that from these definitions it follows that if the vertices m and n of F_k are adjacent and $m < n$, then n has at most $k+1$ ones in its binary expansion; suppose there are $\ell+1$ such ones and they are in positions $m_1, m_2, \ldots, m_{\ell+1}$ with $m_1 < m_2 < \cdots < m_{\ell+1}$ and with $\ell \leq k$. Then there is an $i, 1 \leq i \leq \ell$ such that $m = m_i$, i.e., $m \in \{m_1, m_2, \ldots, m_\ell\}$. Hence n is adjacent to at most ℓ vertices with lesser value m and $\ell \leq k$. (The edge $m_{\ell+1}n \in E(R[\mathbf{N}_k])$ does not occur in F_k, even if incidentally $m_{\ell+1} \in \mathbf{N}_k$.) On the other hand, for each $k \geq 1$, every vertex m of F_k is adjacent to each vertex n of F_k of which the binary expansion has a one in position m and a one in some position x with $x > m$; and for such an n we necessarily have $n > m$. There are clearly infinitely many such vertices n in F_k and hence each vertex of each F_k with $k \geq 1$ has denumerable degree.

Theorem 4. *Let k be a positive integer. Then F_k is in $\mathcal{D}_k$ and universal for $(\mathcal{D}_k)_f$.*

Proof. We first show that $F_k \in \mathcal{D}_k$: Consider any finite induced subgraph H of F_k and let $\beta : H \to (V, E)$ be an isomorphism from H onto the internal induced subgraph (V, E) of F_k. Let n be the largest positive integer in $\beta(V(H))$, a finite subset of $\mathbf{N}_k$. The degree of n in $F_k[\beta(V(H))]$ is at most k. Hence $deg_H(\beta^{-1}(n)) \leq k$, i.e., $\delta(H) \leq k$ as required.

Next we shall prove by complete induction on the cardinality of its vertex set that every finite k-degenerate graph is an induced subgraph of F_k. This is clearly true for a graph with only one vertex; assume that it is true for all k-degenerate graphs with at most $p-1$ vertices and let G be a k-degenerate graph with p vertices. Then $\delta(G) \leq k$ so that G has a vertex v of degree ℓ with $\ell \leq k$. But then $G - v$ is a k-degenerate graph with $p-1$ vertices and hence $G - v$ is an induced subgraph of F_k; assume that $m_1, m_2, \ldots, m_\ell$ are the vertices of F_k corresponding to the ℓ neighbours of v in G under an isomorphism from $G - v$ onto some internal induced subgraph of F_k. Then we construct a number $n \in \mathbf{N}_k$ by choosing ℓ ones in its binary expansion in positions $m_1, m_2, \ldots, m_\ell$ and a one in some position x which is large enough to ensure that

(i) x is not one of the vertices of F_k corresponding to vertices of $G - v$;

(ii) $x > m_i$ for every i (which also ensures that $n > m_i$ for every i); and

(iii) n is not a vertex of F_k corresponding to any vertex of $G - v$;

there are zeros in all the other positions of the binary expansion of n. Clearly, this n is a vertex of F_k which can correspond to v in an isomorphism between G and an internal induced subgraph of F_k by just adding the pair (v, n) to the available isomorphism defined on $G - v$.

This completes the induction step, proving that any graph in $(\mathcal{D}_k)_f$ is isomorphic to an induced subgraph of F_k. ∎

We do not know if a denumerable graph which is universal in $\mathcal{D}_k$ exists, i.e., one into which also all the *denumerable* k-degenerate graphs—even those possibly outside $\mathcal{C}_k$ (see Section 5)—can be isomorphically embedded.

4. Graph Properties with Assignment

In the previous section we considered the graph property $\mathcal{D}_k$ of k-degenerate countable graphs and constructed a member F_k in this property which is universal for $(\mathcal{D}_k)_f$. In the next section we endeavour to reach a stronger outcome for the graph property $\mathcal{C}_k$ (to be described there) of countable graphs having colouring number at most $k+1$ by constructing a member of $\mathcal{C}_k$ which is universal for all of $\mathcal{C}_k$. The properties $\mathcal{C}_k$ and $\mathcal{D}_k$ will be seen to be closely related since both are defined by imposing a degree restriction (see Lemma 13). But (besides being somewhat stricter than $\mathcal{D}_k$) the property $\mathcal{C}_k$ has a special characteristic which entails that a number of graph-theoretic notions (like induced subgraph, homomorphism, and universality, as described in the Introduction), occur also in a second, stronger, form for graphs in $\mathcal{C}_k$, namely that they are used *with assignment*, which links *extraneous* mathematical structure to the graph *globally*.

Definition 5. Let $\mathcal{P}$ be a property of countable graphs. We say that $\mathcal{P}$ is a property *with assignment* (or which *has assignment*) when (a part of) the definition of $\mathcal{P}$ stipulates an instance of the following schema:
"For a graph G to be in $\mathcal{P}$ it is necessary that there exists a finite, non-empty set $A = \{f_1, f_2, \ldots, g_1, g_2, \ldots\}$, where each f_i is a function defined on $V(G)$ and each g_j is a function defined on $E(G)$, and these functions satisfy ... "

Here an A satisfying the stipulation will be called a $\mathcal{P}$*-assignment* (or an *assignment*, for short); and a $G \in \mathcal{P}$ with an assignment A will be denoted by (G, A). For a given property $\mathcal{P}$ (with assignment) all $\mathcal{P}$-assignments are similar, even across graphs, with the same number of f_i's and g_j's, each of which satisfies a similar condition, which we call assignments *of the same type*.

To illustrate the idea of properties with assignment we consider some examples:

Example 6. The property of directed graphs. G is called *directed* if there exists a function g_1 defined on $E(G)$ such that, for each edge (unordered pair of vertices $\{v, v'\} \in E(G)$), $g_1(\{v, v'\})$ is either the ordered pair (v, v'), or (v', v); i.e., $g_1 : E(G) \to V(G) \times V(G)$, with specific properties.

Example 7. Directed labelled graphs. Pick a fixed, finite set L_v of vertex-labels and a fixed, finite set L_e of edge-labels. A (countable) *directed labelled* graph G has an assignment $A = \{f_1, g_1, g_2\}$ where $f_1 : V(G) \to L_v$; $g_1 : E(G) \to V(G) \times V(G)$ (as in (i)); and $g_2 : E(G) \to L_e$. In [2] a denumerable directed labelled graph (with k labels for both vertices and edges), called there L_k, is constructed, which is universal in that property.

Example 8. H-colourable graphs. A countable graph G is *H-colourable* (i.e., belongs to the "hom-property" $\to H_c$) if and only if there exists an $f_1 : V(G) \to V(H)$ which is a homomorphism, denoted by $f_1 : G \to H$. In [3] a universal graph in $\to H_c$ is constructed.

Example 9. Graphs having colouring number at most $k + 1$, $\mathcal{C}_k$. As will be specified precisely in Definition 10 in the next section, a graph G is in $\mathcal{C}_k$ when it has an assignment $A = \{f_1\}$ with a bijection $f_1 : V(G) \to B$ where B is some well-ordered set, with specified properties involving degree restrictions.

The general graph-theoretical notions of induced subgraph, induced-hereditary property, and homomorphism (including isomorphism)—which co-determine universality in an induced-hereditary property—evoke kindred but slightly strengthened notions for graphs (G, A) with assignment, in those cases where the property has assignment. These alternative assignment-respecting notions, which in a very natural way dominate the next section on universality in $\mathcal{C}_k$, are defined for $\mathcal{C}_k$ in Definition 14.

5. Universality in Graphs with Colouring Number at most $k + 1$

Let V be any countable set. It is clear that the following three statements about V are equivalent:

(i) V is well-ordered with the order type of the natural number n, $n \geq 1$.

(ii) There exists a bijection from V onto $[n] = \{1, 2, \ldots, n\}$.

(iii) The elements of V can be labelled as a finite sequence $v_1, v_2, \ldots, v_n$.

Similarly, the next three statements are also equivalent:

(i') V is well-ordered with the order type of ω.

(ii') There exists a bijection from V onto $\mathbf{N}$.

(iii') The elements of V can be labelled as a denumerable sequence $v_1, v_2, \ldots$.

We mention this to convince the reader that our next definition, although inspired by the definition from Erdős and Hajnal [8], is a very special case of their definition.

Definition 10. (i) We say that a countable graph G has *finite colouring number* if there exist a positive integer q and a labelling of the vertices of G as $v_1, v_2, \ldots$ in such a way that for each positive integer ℓ, $1 \leq \ell \leq |V(G)|$, the degree of v_ℓ in $G[\{v_1, v_2, \ldots, v_\ell\}]$ is at most q.

(ii) We say that a countable graph G has *colouring number* $k+1$ (where k is a positive integer) if G has finite colouring number and k is the least element of the (non-empty) set of all those positive integers q for which there exists a labelling of $V(G)$ such that the pair (q, labelling) satisfies the stipulation in (i).

(iii) $\mathcal{C}_k^* := \{G \in \mathcal{I}_c : G$ has colouring number $k+1\}$.

(iv) $\mathcal{C}_k := \{G \in \mathcal{I}_c : G$ has colouring number at most $k+1\}$.

We note that the different $\mathcal{C}_k^*$ are pairwise disjoint; that $\mathcal{C}_k = \mathcal{C}_1^* \cup \mathcal{C}_2^* \cup \cdots \cup \mathcal{C}_k^*$; and that the union of all the $\mathcal{C}_k^*$ (or $\mathcal{C}_k$) is the induced-hereditary property (with assignment) of all countable graphs with finite colouring number. The latter is by no means the whole of $\mathcal{I}_c$ since, e.g., $K_{\aleph_0}$ lacks a finite colouring number.

It is easy to prove (by an inductive argument using the indices of the labelling of the vertices of such a graph) that each graph with colouring number at most $k+1$ has chromatic number at most $k+1$, explaining its naming. Furthermore, as is well-known (and can easily be seen), each graph with colouring number at most $k+1$ is k-degenerate, hence $\mathcal{C}_k \subseteq \mathcal{D}_k$. If the graphs under consideration are restricted to be finite, then we have (by the "elementary observation" made by Lick and White in Proposition 1 of [15]) that $(\mathcal{C}_k)_f = (\mathcal{D}_k)_f$, which, for the sake of self-containedness in the context of properties with assignment, we now prove.

Lemma 11. *A finite graph has colouring number at most* $k+1$ *if and only if it is* k*-degenerate.*

Proof. Let G be a finite graph with colouring number at most $k+1$ and suppose the vertices of G are labelled as $v_1, v_2, \ldots, v_n$ in such a way that for each positive integer $1 \leq \ell \leq n$ the degree of v_ℓ in the subgraph of G induced by $\{v_1, v_2, \ldots, v_\ell\}$ is at most k. Then, for each induced subgraph H of G, it follows that $\delta(H) \leq k$. Let namely $\beta : H \to G$ be an isomorphic embedding and v_m the vertex of G with the largest index (i.e. m) among the elements of $\beta(V(H))$. Then the degree of v_m in $G[\beta(V(H))]$ is at most k, and hence the degree of $\beta^{-1}(v_m)$ in H is at most k.

For the converse, suppose that G is a finite k-degenerate graph of order n. We define a labelling $v_1, v_2, \ldots, v_n$ of the vertices of G recursively (starting with the largest index and working downwards) by labelling any vertex of G with degree at most k as v_n—it exists by the definition of k-degenerate since G is an induced subgraph of itself. Now suppose that the labels $v_n, v_{n-1}, \ldots, v_m$ have been allocated in such a way that each v_j has degree at most k in the subgraph induced by $v_j, v_{j-1}, \ldots, v_m$ together with the as yet unlabelled vertices. Then, if $m > 1$, we consider the subgraph induced by the as yet unlabelled vertices and choose a vertex of degree at most k from it; this vertex is then labelled v_{m-1}. This process clearly produces the desired labelling which is needed to prove that G has colouring number at most $k+1$. ■

Hence we have by Theorem 4

Corollary 12. *Let k be a positive integer. Then F_k is universal for $(\mathcal{C}_k)_f$.*

Let us now delve somewhat deeper into the attributes of and the relations between the properties $\mathcal{C}_k$ and $\mathcal{D}_k$, here and again briefly at the start of Section 6. Lemma 11 already told us that $(\mathcal{C}_k)_f = (\mathcal{D}_k)_f$.

Lemma 13. (i) *$\mathcal{C}_1$ is a property of finite character; so is $\mathcal{D}_k$ for every $k \geq 1$.*

(ii) *$\mathcal{C}_1 = \mathcal{D}_1 = \{G \in \mathcal{I}_c : G$ is a forest$\}$.*

(iii) *For all $k \geq 2$, $\mathcal{C}_k$ is not of finite character and the strict inclusions $\mathcal{C}_k \subset \mathcal{D}_k \subset \mathcal{C}_{2k-1}$ hold.*

Proof. (i) The following statement is a special case of a theorem by Erdős and Hajnal ([8], p. 80, Theorem 9.1): If G is a countable graph with the property that every finite induced subgraph H of G has colouring number at most $k+1$ (but at least 2, i.e., $k \geq 1$ and $H \in \mathcal{C}_k$), then G itself has colouring number at most $2k$ (i.e., $G \in \mathcal{C}_{2k-1}$). The special case $k = 1$ of this statement then says that if every finite $H \leq G$ is in $\mathcal{C}_1$, then $G \in \mathcal{C}_1$, i.e., $\mathcal{C}_1$ is of finite character. That $\mathcal{D}_k$ is of finite character follows immediately from its definition.

(ii) Just before Lemma 11 we remarked that $\mathcal{C}_k \subseteq \mathcal{D}_k$; hence $\mathcal{C}_1 \subseteq \mathcal{D}_1$. Next we show that $\mathcal{D}_1$ lies within the property of countable forests. Let $G \in \mathcal{D}_1$; we have to prove that G has no subgraph which is a cycle, while we know that, for every finite $H \leq G$, $\delta(H) \leq 1$. Suppose (contrariwise) that G contains a cycle C. Then, since $\delta(C) = 2$, we obtain a contradiction. Finally, let G be any countable forest; we want to show that $G \in \mathcal{C}_1$. Let H be any finite induced subgraph of G. Then H is a finite forest so that $\delta(H) \leq 1$; hence $H \in (\mathcal{D}_1)_f$. By Lemma 11, $H \in (\mathcal{C}_1)_f$, and it follows by (i) that $G \in \mathcal{C}_1$.

(iii) Erdős and Hajnal's Theorem 9.2 ([8], p. 80) establishes that the result of their Theorem 9.1 (given above in (i)) is the best possible in the following sense: when $k \geq 2$ then $2k-1 > k$, and there exists a graph G such that every finite induced subgraph of G has colouring number at most $k+1$ (is in $(\mathcal{C}_k)_f$), while G itself is in $\mathcal{C}_{2k-1}$ (in accord with their Theorem 9.1), but *not* in $\mathcal{C}_{2k-2}$—and hence *a fortiori* not in $\mathcal{C}_k$. Clearly this G demonstrates that $\mathcal{C}_k$ is not of finite character and that $\mathcal{C}_k \subset \mathcal{C}_{2k-1}$. Also, by Lemma 11, every finite induced subgraph of G is k-degenerate, implying that $G \in \mathcal{D}_k$, which demonstrates that $\mathcal{C}_k \subset \mathcal{D}_k$. That $\mathcal{D}_k \subset \mathcal{C}_{2k-1}$ follows from Theorem 9.1 of [8] (which establishes that $\mathcal{D}_k \subseteq \mathcal{C}_{2k-1}$) together with the fact that $\mathcal{D}_k$ is of finite character, while $\mathcal{C}_{2k-1}$ is not. To nail this last proper inclusion down even more concretely, note that the complete graph $K_{2k+1} \in \mathcal{C}_{2k-1}$, but $K_{2k+1} \notin \mathcal{D}_k$. ∎

We note that part (iii) above establishes the following possibility: There exists an induced-hereditary property of finite character (e.g. $\mathcal{D}_k$) and a graph in that property ($F_k \in \mathcal{D}_k$) which is universal for the finite members of the property (Theorem 4) and yet not universal in the property. This is proven by the denumerable graph G mentioned in the proof of (iii) above which is in $\mathcal{D}_k$, but is not an induced subgraph of F_k since it has colouring number $2k$ while F_k has colouring number at most $k+1$—see part (i) of Theorem 15 below.

Suppose that the elements of a countable set V have been labelled $v_1, v_2, \ldots$, and that $W \subseteq V$. Then we may see W as "picking out" some elements of V: $W = v_{i_1}, v_{i_2}, \ldots$, with the sequence of indices $i_1 < i_2 < \cdots$ being a subsequence of the indices $1, 2, \ldots$. It is only natural to re-label the elements of W as $w_1, w_2, \ldots$, where $w_j = v_{i_j}$ for each j. We call $w_1, w_2, \ldots$ the labelling of W *inherited* from the labelling $v_1, v_2, \ldots$ of V. If A denotes the assignment which is the v_i-labelling of V, while B denotes the w_j-labelling of W inherited from A, then we shall write $B = A \restriction W$.

It is clear that the property $\mathcal{C}_k$ is (in the terminology of Section 4) a property with assignment, where an assignment to a graph G is a bijection $f_1 : V(G) \to \{1, 2, \ldots, n\}$ or $f_1 : V(G) \to \mathbf{N}$—or, equivalently, a labelling $v_1, v_2, \ldots$ of $V(G)$ in terms of which a certain degree restriction is imposed. Notions crucial for universality now occur more naturally in a form somewhat boosted from their standard definitions as given in Section 1. Here follow the strengthened definitions introducing the $\mathcal{C}_k$-assignment-respecting (or just assignment-respecting, "$\mathcal{C}_k$-a-r" or "a-r" for short) notions needed:

Definition 14. (i) Consider $(G, A), (H, B) \in \mathcal{C}_k$, where A contains only a labelling $v_1, v_2, \ldots$ of $V(G)$ and B a labelling $w_1, w_2, \ldots$ of $V(H)$ which ensure membership in $\mathcal{C}_k$. Let $\lambda : G \to H$ be a graph homomorphism. Then we call λ an *a-r homomorphism* from (G, A) to (H, B) and write $\lambda : (G, A) \to (H, B)$ if also, whenever $v_i, v_j \in V(G)$, $\lambda(v_i) = w_k$, $\lambda(v_j) = w_\ell$, and $i \le j$, we have $k \le \ell$, i.e., λ is also (well-)order-preserving. (Note that then we have a surjective a-r homomorphism $\lambda : (G, A) \to (H[\lambda(V(G))], B \restriction \lambda(V(G)))$.)

(ii) If in case (i) λ is a graph isomorphism, then λ is an *a-r isomorphism* from (G, A) to (into or onto) (H, B) when $i < j$ if and only if $k < \ell$.

(iii) Consider again $(G, A), (H, B) \in \mathcal{C}_k$ with the labellings as in (i), but now G is an induced subgraph of H, $G \le H$, $\beta : G \to H$ a graph isomorphism injecting G into H. If this β is an a-r isomorphism, $\beta : (G, A) \to (H, B)$, then we say that (G, A) is an *a-r induced subgraph* of (H, B) (*with respect to* β) and write $(G, A) \le_\beta (H, B)$.

We remark that if $(G, A) \le_\beta (H, B)$, then β is also an a-r isomorphism from (G, A) onto $(H[\beta(V(G))], B \restriction \beta(V(G)))$.

It is clear that $\mathcal{C}_k$ is an *a-r induced-hereditary property* in the sense that if $(H, B) \in \mathcal{C}_k$ and $(G, A) \le (H, B)$, then $(G, A) \in \mathcal{C}_k$. For an $(H, B) \in \mathcal{C}_k$ it is

only natural now to call (H, B) *a-r universal in* $\mathcal{C}_k$ if, for every $(G, A) \in \mathcal{C}_k$, $(G, A) \leq (H, B)$.
We now proceed to show that F_k is indeed an a-r universal graph in $\mathcal{C}_k$. The assignment that is needed to consider its properties is the bijection $\iota : \mathbf{N}_k \to \mathbf{N}$ defined by choosing, for any $v \in \mathbf{N}_k$, $\iota(v)$ as the positive integer ℓ if there are exactly $\ell - 1$ vertices in $\mathbf{N}_k$ less than v.

In our next result we show that (F_k, ι) is a-r universal in $\mathcal{C}_k$. In order to do so, it is convenient to adopt the label v_ℓ for the vertex $v \in \mathbf{N}_k$ for which $\iota(v) = \ell$ for each $\ell \geq 1$. We may call $v_1, v_2, \ldots$ the ι-labelling of $V(F_k)$.

Theorem 15. *Let k be a positive integer. Then*
(i) *F_k has colouring number at most $k + 1$ and*
(ii) *(F_k, ι) is an a-r universal graph in $\mathcal{C}_k$.*

Proof. (i) Let, for the proof that F_k has colouring number at most $k + 1$, $\ell \geq 2$ be a positive integer and consider the subgraph induced by $\{v_1, v_2, \ldots, v_\ell\}$. Each vertex in this subgraph which is adjacent to v_ℓ corresponds to a one in the binary expansion of v_ℓ. But, if there is at least one such adjacency, then, since v_ℓ has at most $k+1$ ones in its binary expansion of which at most k correspond to vertices adjacent to v_ℓ, the degree of v_ℓ is at most k in that subgraph. Hence $F_k \in \mathcal{C}_k$.

(ii) Next we shall prove for any pair (G, κ), where G is a countable graph with colouring number at most $k+1$ and κ is a labelling of $V(G)$ as $\{w_1, w_2, \ldots\}$ in such a way that for each positive integer ℓ with $1 \leq \ell \leq |V(G)|$ the degree of w_ℓ in the subgraph of G induced by $\{w_1, w_2, \ldots, w_\ell\}$ is at most k, that there is an a-r isomorphic embedding of (G, κ) into (F_k, ι). This is done by constructing, by recursion on the indices $1, 2, \ldots$ of the κ-labels of the vertices of G, an injection $\alpha : \{w_1, w_2, \ldots\} \to \{v_1, v_2, \ldots\}$ of $V(G)$ into $V(F_k)$ which is an a-r isomorphic embedding $\alpha : (G, \kappa) \to (F_k, \iota)$.

Define $\alpha(w_1) = v_1(= 1)$ and let $W_p = \{w_1, w_2, \ldots, w_p\}$ for any $p \geq 1$. Suppose also that $\alpha(w_1), \alpha(w_2), \ldots, \alpha(w_p)$ have already been defined in such a way that $\alpha : (G[W_p], \kappa|W_p) \cong_{a-r} (F_k[\alpha(W_p)], \iota \restriction \alpha(W_p))$. We must now define $\alpha(w_{p+1})$ in such a way that, for the extended function (also denoted by α), $\alpha : (G[W_{p+1}], \kappa|W_{p+1}) \cong_{a-r} (F_k[\alpha(W_{p+1})], \iota \restriction \alpha(W_{p+1}))$.

Let $\{u_1, u_2, \ldots, u_\ell\}$ be the set of those elements in $\{w_1, w_2, \ldots, w_p\}$ which are adjacent to w_{p+1} in G; we know that $\ell \leq k$. Then we construct a number $n \in \mathbf{N}_k = V(F_k)$ by choosing $\ell + 1$ ones in its binary expansion in the ℓ positions $\alpha(u_1), \alpha(u_2), \ldots, \alpha(u_\ell)$ and in some position x which is large enough to ensure that $x > \alpha(u_i)$ for each relevant i, and that $n > \alpha(w_j)$ for each $w_j \in W_p$ (including each u_i); there are zeros in all the other positions of the binary expansion of n. By defining $\alpha(w_{p+1}) = n$ we have extended α from W_p to W_{p+1} in the required way. The recursive step can be repeated throughout $V(G)$, whether the latter is finite or denumerable, to establish that $\alpha : (G, \kappa) \to (F_k, \iota)$ is an a-r isomorphic embedding. ∎

We do not know whether there exists a graph (G, η) which is not a-r isomorphic to (F_k, ι) but which is a-r universal in $\mathcal{C}_k$.

More properties of the graph F_k will be explicated in Section 6.

6. Further Properties of F_k

We start by noting that F_k, unlike R, is not self-complementary since F_k contains arbitrary large edgeless subgraphs so that $\overline{F_k}$ contains arbitrary large complete subgraphs, which is incompatible with having colouring number at most $k+1$. Next we remark that, for each $k \geq 2$, $\mathcal{D}_{k-1} \subseteq \mathcal{D}_k$ and $\mathcal{C}_{k-1} \subseteq \mathcal{C}_k$. Furthermore the complete graph K_{k+1}, which is k-regular, satisfies $K_{k+1} \in \mathcal{D}_k \cap \mathcal{C}_k$ and $K_{k+1} \notin \mathcal{D}_{k-1} \cup \mathcal{C}_{k-1}$. Hence $\mathcal{D}_{k-1} \subset \mathcal{D}_k$ and $\mathcal{C}_{k-1} \subset \mathcal{C}_k$ for each $k \geq 2$. Turning to F_k, one sees immediately that $\mathbf{N}_{k-1} \subset \mathbf{N}_k$ and that $F_k[\mathbf{N}_{k-1}] = F_{k-1}$. Hence, assuming the labellings, F_{k-1} is a proper a-r induced subgraph of F_k and, using the same argument as above, they are not isomorphic, not even if one ignores the requirement that an isomorphism should be a-r. The sequences $(\mathcal{D}_k)_{k\geq 1}$, $(\mathcal{C}_k)_{k\geq 1}$ and $(F_k)_{k\geq 1}$ are therefore all strictly increasing sequences.

The rest of this section investigates some further properties of F_k; note that most of these properties of F_k do not involve assignments.

Definition 16. We say that a graph C has the *k-adjoining property* if for every two finite disjoint sets U and V of vertices of C with $|U| \leq k$ there is a vertex not in $U \cup V$ which is adjacent to every vertex of U and to no vertex of V.

This property is clearly weaker than the extension property, the characterising property of the Rado graph R (referred to in Section 1).

The graph F_1 is not homogeneous and, for $k \geq 2$, F_k does not seem to be homogeneous (in the sense defined in [13]), i.e., not every isomorphism between two finite, isomorphic, internal induced subgraphs of F_k can be extended to an automorphism of F_k. It does, however, possess a lesser property which we now define.

Definition 17. We say that a graph C *allows iso-extensions* if C is denumerable and every isomorphism between two finite induced subgraphs of C has an extension to an isomorphism between two (not necessarily different) denumerable induced subgraphs of C.

Note that every denumerable homogeneous graph allows iso-extensions, but not conversely.

Lemma 18. *The graph F_k has the following properties:*

(i) *F_k has the k-adjoining property.*

(ii) *For any two finite, isomorphic, internal induced subgraphs F and G of F_k with vertex sets X and Y respectively, any vertex m of F_k not in F which is larger than each vertex of F to which it is adjacent and any isomorphism α from F onto G, there is a vertex n of F_k which is not in G such that the function $\alpha \cup \{(m, n)\}$ is an isomorphism from $F_k[X \cup \{m\}]$ to $F_k[Y \cup \{n\}]$.*

(iii) *F_k allows iso-extensions.*

Proof. (i) Consider any two finite disjoint sets U and V of vertices of F_k with $U = \{u_1, u_2, \ldots, u_\ell\}$ and with $\ell \leq k$. Then we construct a vertex n by choosing $\ell + 1$ ones in its binary expansion in positions $u_1, u_2, \ldots, u_\ell$ and one position x which is large enough to ensure that $x > u_i$ for each i, that $n > u_i$ for each i, and that n is not a vertex of V; there are zeros in all the other positions of the binary expansion of n. Clearly, this vertex has the required properties to prove the k-adjoining property for F_k.

(ii) Consider any two finite, isomorphic, internal induced subgraphs F and G of F_k with vertex sets X and Y respectively, any vertex m of F_k not in F which is larger than each vertex of F to which it is adjacent and any isomorphism $\alpha : X \to Y$ from F onto G.

Consider the partition of the vertex set X of F into the disjoint subsets U and V, which are, respectively, the subsets containing the vertices of F adjacent to and not adjacent to m in F_k. Then the required vertex n of F_k not in Y can be constructed using the k-adjoining property of F_k by letting n be a vertex of F_k which is adjacent to each vertex of $\alpha(U)$ and to no vertex of $\alpha(V)$. Then, clearly, n is the required vertex using which we can extend the isomorphism α with the pair (m, n).

(iii) Consider any two finite, isomorphic, induced subgraphs of F_k and in particular their clones internal to F_k. The required extension of any isomorphism between them can now clearly be built through a recursive process using part (ii) of this lemma. ■

We remark that, in Definition 16 of the k-adjoining property, the existence of denumerably many new vertices n could have been specified with preservation of part (i) of Lemma 18. This *strong* version of the k-adjoining property for F_k leads immediately to the next conclusion.

Corollary 19. *F_k has colouring number $k + 1$, i.e., $F_k \in \mathcal{C}_k^*$.*

Proof. By part (i) of Theorem 15, F_k has colouring number at most $k + 1$. Its colouring number is also at least $k+1$: Consider namely any labelling whatsoever,

$z_1, z_2, \ldots$, of $V(F_k)$. By the strong k-adjoining property of F_k it is possible to select a subsequence $z_{i_1}, z_{i_2}, \ldots, z_{i_k}, z_{i_{k+1}}$ of $k+1$ vertices with strictly increasing indices such that for any j with $2 \leq j \leq k+1$, z_{i_j} is adjacent in F_k to each of its predecessors in the subsequence. Hence $z_{i_{k+1}}$ is adjacent to k predecessors in the labelling; the degree of $z_{i_{k+1}}$ in $F_k[\{z_{i_1}, z_{i_2}, \ldots, z_{i_{k+1}}\}]$ is thus k. ■

7. Concluding Remark

The "Rado-type" of construction for universal graphs mentioned in Section 2 and employed in Section 3 is called a construction of type A in [4]. In that paper, countable universal graphs for induced-hereditary graph properties of (general) graphs are constructed recursively and that type of construction is named of type B. The two types of construction and their concomitant properties are also contrasted in [4].

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