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γ-CYCLES AND TRANSITIVITY BY MONOCHROMATIC PATHS IN ARC-COLOURED DIGRAPHS

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Abstract

We call the digraph D an *m-coloured* digraph if its arcs are coloured with m colours. If D is an m-coloured digraph and $a \in A(D)$, $colour(a)$ will denote the colour has been used on a. A path (or a cycle) is called *monochromatic* if all of its arcs are coloured alike. A *γ-cycle* in D is a sequence of vertices, say $\gamma = (u_0, u_1, \ldots, u_n)$, such that $u_i \neq u_j$ if $i \neq j$ and for every $i \in \{0, 1, \ldots, n\}$ there is a $u_i u_{i+1}$-monochromatic path in D and there is no $u_{i+1} u_i$-monochromatic path in D (the indices of the vertices will be taken mod $n+1$). A set $N \subseteq V(D)$ is said to be a *kernel by monochromatic paths* if it satisfies the following two conditions: (i) for every pair of different vertices $u, v \in N$ there is no monochromatic path between them and; (ii) for every vertex $x \in V(D) \setminus N$ there is a vertex $y \in N$ such that there is an xy-monochromatic path.

Let D be a finite m-coloured digraph. Suppose that $\{C_1, C_2\}$ is a partition of C, the set of colours of D, and D_i will be the spanning subdigraph of D such that $A(D_i) = \{a \in A(D) \mid colour(a) \in C_i\}$. In this paper, we give some sufficient conditions for the existence of a kernel by monochromatic paths in a digraph with the structure mentioned above. In particular we obtain an extension of the original result by B. Sands, N. Sauer and R. Woodrow that asserts: Every 2-coloured digraph has a kernel by monochromatic paths. Also, we extend other results obtained before where it is proved that under some conditions an m-coloured digraph has no γ-cycles.

Keywords: digraph, kernel, kernel by monochromatic paths, γ-cycle.

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1. Introduction

For general concepts we refer the reader to [1, 2]. Let D be a *digraph*, and let $V(D)$ and $A(D)$ denote the sets of *vertices* and *arcs* of D, respectively. We recall that a subdigraph D_1 of D is a *spanning* subdigraph if $V(D_1) = V(D)$. If S is a nonempty subset of $V(D)$ then the subdigraph of D *induced* by S, denoted by $D[S]$, is the digraph where $V(D[S]) = S$ and whose arcs are all those arcs of D joining vertices of S. An arc u_1u_2 of D will be called an S_1S_2-arc of D whenever $u_1 \in S_1$ and $u_2 \in S_2$.

A set $I \subseteq V(D)$ is *independent* if $A(D[I]) = \emptyset$. A *kernel* N of D is an independent set of vertices such that for each $z \in V(D) \setminus N$ there is an zN-arc in D, that is an arc from z toward some vertex in N. A digraph D is called a *kernel perfect* digraph when every induced subdigraph of D has a kernel. Sufficient conditions for the existence of kernels in a digraph have been investigated by several authors, Von Neumann and Morgenstern [26]; Duchet and Meyniel [8]; Duchet [6, 7] and Galeana-Sánchez and Neumann-Lara [14, 15]. The concept of kernel has found many applications, see for example [23, 24, 25].

In this paper all the walks, paths and cycles will be directed and we consider that each digraph has a (fixed) colouring of the arcs.

A path (or a cycle) is called *monochromatic* if all of its arcs are coloured alike. A cycle is called a *quasi-monochromatic* cycle if, with at most one exception, all of its arcs are coloured alike. A set $N \subseteq V(D)$ is said to be a *kernel by monochromatic paths* if it satisfies the following two conditions: (i) for every pair of different vertices $u, v \in N$ there is no monochromatic path between them (N is an *independent set by monochromatic paths*) and; (ii) for every vertex $x \in V(D) \setminus N$ there is a vertex $y \in N$ such that there is a xy-monochromatic path (N is an *absorbing set by monochromatic paths*).

The definition of kernel by monochromatic paths was introduced by Galeana-Sánchez [9], even though the research on kernels by monochromatic paths goes back to the classical paper of Sands *et al.* [27], kernel by monochromatic paths clearly, is a generalization of the concept of kernel. The *closure* of D, denoted by $\mathscr{C}(D)$ is the m-coloured multidigraph defined as follows: $V(\mathscr{C}(D)) = V(D)$, $A(\mathscr{C}(D)) = A(D) \cup \{(u, v)$ with colour $i \mid$ there is a uv-path coloured i contained in $D\}$. Notice that for any digraph D, $\mathscr{C}(\mathscr{C}(D)) \cong \mathscr{C}(D)$ and D has a kernel by monochromatic paths if and only if $\mathscr{C}(D)$ has a kernel.

In [27] Sands *et al.* have proven that any 2-coloured digraph D has a kernel by monochromatic paths; in particular they proved that any 2-coloured tournament T has a kernel by monochromatic paths. They also raised the following problem: Let T be a 3-coloured tournament such that every cycle of length 3 is a quasi-monochromatic cycle; must T have a kernel by monochromatic paths? (This question still remains open.)

In [28] Shen Minggang proved that if T is a m-coloured tournament such that every triangle (that is, a transitive tournament of order 3 or a cycle of length 3) is a quasimonochromatic subdigraph of T, then T has a kernel by monochromatic paths. He also proved that this result is the best possible for $m \geq 5$. In [16] H. Galeana-Sánchez and R. Rojas Monroy proved that the result of Shen Minggang is the best possible for $m \geq 4$.

In [13] H. Galeana-Sánchez, R. Rojas-Monroy and G. Gaytán-Gómez proved that if D is a finite m-coloured digraph that admits a partition $\{C_1, C_2\}$ of the set of colours of D such that for each $i \in \{1, 2\}$ every cycle in the subdigraph $D[C_i]$ spanned by the arcs with colours in C_i is monochromatic, $\mathscr{C}(D)$ does not contain neither *rainbow* triangles (all of its arcs have different colours) nor rainbow $\overrightarrow{P_3}$ (path of length 3) involving colours of both C_1 and C_2; then D has a kernel by monochromatic paths.

The known sufficient conditions for the existence of a kernel by monochromatic paths (*k.m.p.*) in m-coloured ($m \geq 3$) tournaments or nearly tournaments (such as digraphs obtained from a tournament by the deletion of a single arc, quasi-transitive digraphs, k-partite tournaments) ask for the monochromaticity or quasi-monochromaticity of small subdigraphs such as directed cycles or transitive tournaments of order 3. Other interesting results about the existence of k.m.p. in digraphs can be found in [9, 10, 11, 12, 17, 22, 29, 30].

If $\mathcal{W} = (z_0, z_1, \ldots, z_n)$ is a walk, we say that the length of $\mathcal{W}$ is n and we will denote it by $\ell(\mathcal{W})$. If $\mathcal{P}$ is a path and $z_i, z_j \in V(\mathcal{P})$ with $i \leq j$ we denote by $(z_i, \mathcal{P}, z_j)$ the $z_i z_j$-path contained in $\mathcal{P}$, and $\ell(z_i, \mathcal{P}, z_j)$ will denote its length.

We will need the following basic elementary results.

Lemma 1. *Let D be a digraph, $u, v \in V(D)$. Then every uv-monochromatic walk in D contains a uv-monochromatic path.*

Lemma 2. *Every closed walk in a digraph D contains a cycle.*

And the following theorem.

Theorem 3 [3]**.** *If D is a digraph such that every cycle of D has at least one symmetrical arc, then D is a kernel-perfect digraph.*

2. Main Results

Definition. Let D be a m-coloured digraph, a γ-*cycle* in D is a sequence of vertices $\gamma = (u_0, u_1, \ldots, u_n)$ such that

1. $u_i \neq u_j$ for each $i \neq j$,
2. for each $i \in \{0, 1, \ldots, n\}$ there is a $u_i u_{i+1}$-monochromatic path in D (the indices are taken $\bmod n+1$), and
3. for each $i \in \{0, 1, \ldots, n\}$ there is no $u_{i+1} u_i$-monochromatic path.

We will say that the *length* of γ is $\ell(\gamma) = n$.

A digraph D is called *transitive by monochromatic paths* if the existence of an xy-monochromatic path and a yz-monochromatic path in D imply that there is an xz-monochromatic path in D.

The following lemmas will be useful in the proof of our main result.

Lemma 4. *Let D be a m-coloured and transitive by monochromatic paths digraph, then D has no γ-cycles.*

Proof. Let $\mathcal{C} = (u_0, u_1, \ldots, u_{n-1}, u_0)$ be a sequence of vertices such that $u_i \neq u_j$ for each $i \neq j$, and for every $i \in \{0, 1, \ldots, n-1\}$ there is a $u_i u_{i+1}$-monochromatic path in D (the indices of the vertices will be taken $\bmod n$). We can prove, by induction and from transitivity by monochromatic paths that there exists a $u_0 u_k$-monochromatic path in D for each $k \in \{2, \ldots, n-1\}$.

Then, there is a $u_0 u_{n-1}$-monochromatic path in D. We conclude that D has no γ-cycles. ∎

Lemma 5. *Let D be a m-coloured digraph such that has no γ-cycles. Then there is no sequence of vertices $(x_0, x_1, x_2, \ldots)$ such that for every i there is an $x_i x_{i+1}$-monochromatic path in D and there is no $x_{i+1} x_i$-monochromatic path in D.*

Proof. It follows immediately from the finiteness of D. ∎

Definition. Let D be an m-coloured digraph. A set $S \subseteq V(D)$ is a *semikernel by monochromatic paths* of D if the following conditions are fulfilled:

1. S is an independent set by monochromatic paths, and
2. for each $z \in V(D) \setminus S$ such that there exists an Sz-monochromatic path, then there exists a zS-monochromatic path in D.

Lemma 6. *Let D be an m-coloured digraph such that has no γ-cycles. Then there exists $x_0 \in V(D)$ such that $\{x_0\}$ is a semikernel by monochromatic paths of D.*

Proof. If there exists no vertex that satisfies the affirmation of Lemma 6, it is straightforward to build a vertex sequence that contradicts Lemma 5. ■

From now on, D will denote a finite m-coloured digraph and $\{C_1, C_2\}$ will be a partition of C, the set of colours of D. Also, D_i will be the spanning subdigraph of D such that $A(D_i) = \{a \in A(D) \mid colour(a) \in C_i\}$. If $W = (u_0, \ldots, u_k = v_0, \ldots, v_m = w_0, \ldots, w_n = u_0)$ is a cycle, we say that W is a 3-*coloured* (C_1, C, C_2) *subdivision of* $\overrightarrow{C_3}$ (cycle of length 3) if $T_1 = (u_0, \ldots, u_k)$ is a monochromatic path of colour a and it is contained in D_1, $T_2 = (v_0, \ldots, v_m)$ is a monochromatic path of colour b and it is contained in D, and $T_3 = (w_0, \ldots, w_n)$ is a monochromatic path of colour c and it is contained in D_2 with $a \neq b$, $b \neq c$, and $a \neq c$. And, if $P = (u_0, \ldots, u_k = v_0, \ldots, v_m = w_0, \ldots, w_n)$ is a directed path, we say that P is a 3-*coloured* (C_1, C, C_2) *subdivision of* $\overrightarrow{P_3}$ if $T_1 = (u_0, \ldots, u_k)$ is a monochromatic path of colour a and it is contained in D_1, $T_2 = (v_0, \ldots, v_m)$ is a monochromatic path of colour b and it is contained in D, and $T_3 = (w_0, \ldots, w_n)$ is a monochromatic path of colour c and it is contained in D_2 with $a \neq b$, $b \neq c$, and $a \neq c$. In particular, we say that a cycle (u_0, u_1, u_2, u_0) is a 3-*coloured* $(C_1, C_1, C_2) - \overrightarrow{C_3}$ if $a = colour((u_0, u_1)) \in C_1$, $b = colour((u_1, u_2)) \in C_1$ and $c = colour((u_2, u_0)) \in C_2$ with $a \neq b$, $b \neq c$, and $a \neq c$. We say that a path (u_0, u_1, u_2, u_3) is a 3-*coloured* $(C_1, C_1, C_2) - \overrightarrow{P_3}$ if $a = colour((u_0, u_1)) \in C_1$, $b = colour((u_1, u_2)) \in C_1$ and $c = colour((u_2, u_0)) \in C_2$ with $a \neq b$, $b \neq c$, and $a \neq c$. We say that $v \in V(D)$ is a vertex with 3-*coloured* (C_1, C_1, C_2) *in-neighbourhood* if there exists w, x and z in $V(D)$ such that $\{(w, v), (x, v), (z, v)\} \subseteq A(D)$ and $a = colour((w, v)) \in C_1$, $b = colour((x, v)) \in C_1$ and $c = colour((z, v)) \in C_2$ with $a \neq b$, $b \neq c$, and $a \neq c$.

Definition. Let $S \subseteq V(D)$. We will say that S is a *semikernel by monochromatic paths modulo* D_2 *of* D if S is independent by monochromatic paths and for every $z \in V(D) \setminus S$, if there is an Sz-monochromatic path contained in D_1 then there is a zS-monochromatic path contained in D.

Lemma 7. *Suppose that* D_1 *has no* γ*-cycles. Then there exists* $x_0 \in V(D)$ *such that* $\{x_0\}$ *is a semikernel by monochromatic paths modulo* D_2 *of* D.

Proof. Since D_1 has no γ-cycles, then it follows from Lemma 6 that there exists $x_0 \in V(D_1)$ such that $\{x_0\}$ is a semikernel by monochromatic paths of D_1. From the definition of semikernel by monochromatic paths modulo D_2 of D, we have that $\{x_0\}$ is a semikernel by monochromatic paths modulo D_2 of D. ■

Let $\varsigma = \{\emptyset \neq S \subseteq V(D) \mid S$ is a semikernel by monochromatic paths modulo D_2 of $D\}$.

Whenever $\varsigma \neq \emptyset$, we will denote by D_ς the digraph defined as follows: $V(D_\varsigma) = \varsigma$ (i.e, for every element of ς we consider a vertex in D_ς) and $(S_1, S_2) \in A(D_\varsigma)$ if and only if for every $s_1 \in S_1$ there exists $s_2 \in S_2$ such that $s_1 = s_2$

or there is an s_1s_2-monochromatic path contained in D_2 and there is no s_2S_1-monochromatic path contained in D.

Lemma 8. *Suppose that:*
(1) *D_1 has no γ-cycles, and*
(2) *D_2 is transitive by monochromatic paths.*
Then D_ς is an acyclic digraph.

Proof. First we will prove that D_ς is transitive and anti-symmetric.

Transitive. Suppose that $(S,T) \in A(D_\varsigma)$ and $(T,W) \in A(D_\varsigma)$, and let $s \in S$. If $s \notin W$, we may suppose $s \notin T$ as well from $(T,W) \in A(D_\varsigma)$, and so there is a monochromatic path contained in D_2 from s to some $t \in T$. If $t \in W$, we are done. Otherwise there is a monochromatic path contained in D_2 from t to some $w \in W$. Then since D_2 is transitive by monochromatic paths there is a monochromatic path from s to w. Then $(S,W) \in A(D_\varsigma)$.

Anti-symmetric. Suppose that $(S,T) \in A(D_\varsigma)$ and $(T,S) \in A(D_\varsigma)$ we will prove that $S = T$. Proceeding by contradiction, suppose, without loss of generality, that $s \in S \setminus T$. Then there is a monochromatic path contained in D_2 from s to some $t \in T$ and there is no tS-monochromatic path contained in D. Since $(T,S) \in A(D_\varsigma)$ then t must belong to S, a contradiction because $s \in S$, $t \in S$ and S is independent by monochromatic paths. Then $S = T$.

Now assume, for a contradiction, that D_ς has a cycle, say $\mathcal{C} = (S_0, S_1, \ldots, S_{n-1}, S_0)$, with $n \geq 2$. Since $\mathcal{C}$ is a cycle, we have that $S_i \neq S_j$ if $i \neq j$. We can prove, by induction and from transitivity that $(S_{i+1}, S_i) \in A(D_\varsigma)$ for each $i \in \{0, 1, \ldots, n-1\}$ (the indices of the vertices will be taken $\bmod n$). Since D_ς is anti-symmetric we have $S_i = S_j$, a contradiction. We conclude that D_ς is an acyclic digraph. ■

Lemma 9. *Suppose that α_1 is a uz-monochromatic path in D_1, α_2 is a zw-monochromatic path in D_1 and α_3 is a wx-monochromatic path in D_2, such that colour$(\alpha_1) \neq$ colour(α_2), colour$(\alpha_1) \neq$ colour(α_3) and colour$(\alpha_2) \neq$ colour(α_3). Additionally, assume that D has no uw-monochromatic path, no zx-monochromatic path, and no zu-monochromatic path. Then each one of the two following conditions imply that there is a ux-path which is a* 3*-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{P_3}$ or there is a* 3*-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{C_3}$:*

(a) *Each cycle of D contained in D_1 is monochromatic and D_2 is transitive by monochromatic paths.*
(b) *D has no vertex with* 3*-coloured (C_1, C_1, C_2) in-neighbourhood.*

Proof. From the hypothesis, we have immediately the following assertions:
(1) $u \notin V(\alpha_2)$.

(2) $z \notin V(\alpha_3)$.
(3) $w \notin V(\alpha_1)$.
(4) $x \notin V(\alpha_2)$.

Case I. $V(\alpha_1) \cap V(\alpha_2) = \{z\}$.

Subcase I.1. $V(\alpha_2) \cap V(\alpha_3) = \{w\}$.

Subcase I.1.1. $V(\alpha_1) \cap V(\alpha_3) = \emptyset$. In this case, we have that $\alpha_1 \cup \alpha_2 \cup \alpha_3$ is a ux-path which is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{P_3}$.

Subcase I.1.2. $V(\alpha_1) \cap V(\alpha_3) \neq \emptyset$. Let y be the last vertex of α_1 which is in α_3. We have that $y \neq z$ and $w \neq y$ (from assertions 2 and 3). Then $(y, \alpha_1, z) \cup \alpha_2 \cup (w, \alpha_3, y)$ is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{C_3}$.

Subcase I.2. $(V(\alpha_2) \cap V(\alpha_3)) \setminus \{w\} \neq \emptyset$.

Subcase I.2.1. $V(\alpha_1) \cap V(\alpha_3) = \emptyset$. Let y be the first vertex of α_2 that is in α_3. We have that $y \neq z$ and $y \neq x$ (from assertions 2 and 4). Then $\alpha_1 \cup (z, \alpha_2, y) \cup (y, \alpha_3, x)$ is a ux-path which is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{P_3}$.

Subcase I.2.2. $V(\alpha_1) \cap V(\alpha_3) \neq \emptyset$. Let y be the first vertex of α_3 that is in α_1 or in α_2 and let e be the last vertex of α_3 which is in α_1 or in α_2. If $y \in V(\alpha_1)$ then we have that $y \neq z$ (from assertion 2) and we may suppose that $y \neq w$ (from assertion 3). Then $(y, \alpha_1, z) \cup \alpha_2 \cup (w, \alpha_3, y)$ is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{C_3}$. Suppose that $y \notin V(\alpha_1)$, then $y \in V(\alpha_2)$. If $e \in V(\alpha_1)$ then $y \neq e$. Let a be the last vertex of α_3 which is in α_2 and let b be the first vertex of (a, α_3, x) that is in α_1. We have that $b \neq z$ and $a \neq z$ (from assertion 2), and $a \neq x$ (from assertion 4), $a \neq u$ (from assertion 1) and $b \neq w$ (from assertion 3). Also, $a \neq b$, otherwise $(a, \alpha_1, z) \cup (z, \alpha_2, a)$ contains a non-monochromatic cycle contained in D_1 and a is a vertex with 3-coloured (C_1, C_1, C_2) in-neighbourhood, a contradiction. Then $(b, \alpha_1, z) \cup (z, \alpha_2, a) \cup (a, \alpha_3, b)$ is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{C_3}$. Now, assume that $e \in V(\alpha_2)$. We have that $z \neq e$ and $e \neq x$ (from assertions 2 and 4). Then $\alpha_1 \cup (z, \alpha_2, e) \cup (e, \alpha_3, x)$ is a ux-path which is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{P_3}$.

Case II. $(V(\alpha_1) \cap V(\alpha_2)) \setminus \{z\} \neq \emptyset$. Suppose that D satisfies (a), $(V(\alpha_1) \cap V(\alpha_2)) \setminus \{z\} \neq \emptyset$ implies that there is a non-monochromatic cycle contained in $\alpha_1 \cup \alpha_2 \subseteq D_1$, a contradiction. Therefore, D satisfies (b).

Subcase II.1. $V(\alpha_2) \cap V(\alpha_3) = \{w\}$.

Subcase II.1.1 $V(\alpha_1) \cap V(\alpha_3) = \emptyset$. Let y be the first vertex of α_1 that is in α_2. We have that $y \neq u$ and $y \neq w$ (from assertions 1 and 3). Then $(u, \alpha_1, y) \cup (y, \alpha_2, w) \cup \alpha_3$ is a 3-coloured ux-path which is a (C_1, C_1, C_2) subdivision of $\overrightarrow{P_3}$.

Subcase II.1.2. $V(\alpha_1) \cap V(\alpha_3) \neq \emptyset$. Let y be the first vertex of α_1 that is in α_2 or α_3 and let e be the last vertex of α_1 that is in α_2 or α_3. If $y \in V(\alpha_2)$ then we have that $u \neq y$ and $y \neq w$ (from assertions 1 and 3). Then $(u, \alpha_1, y) \cup (y, \alpha_2, w) \cup \alpha_3$ is a ux-path which is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{P_3}$. Suppose that $y \in V(\alpha_3)$. If $e \in V(\alpha_2)$, let a be the last vertex of α_1 that is in α_3 and let b be the first vertex of (a, α_1, z) that is in α_2. We have that $b \neq w$ and $a \neq w$ (from assertion 3), $a \neq b$ (because $V(\alpha_2) \cap V(\alpha_3) = \{w\}$), and $a \neq z$ (from assertion 2). Then $(a, \alpha_1, b) \cup (b, \alpha_2, w) \cup (w, \alpha_3, a)$ is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{C_3}$. So, assume that $e \in V(\alpha_3)$. We have that $e \neq z$ and $e \neq w$ (from assertions 2 and 3). Then $(e, \alpha_1, z) \cup \alpha_2 \cup (w, \alpha_3, e)$ is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{C_3}$.

Subcase II.2. $(V(\alpha_2) \cap V(\alpha_3)) \setminus \{w\} \neq \emptyset$.

Subcase II.2.1. $V(\alpha_1) \cap V(\alpha_3) = \emptyset$. Let y be the first vertex of α_2 that is in α_1 or α_3 and let e be the last vertex of α_2 that is in α_1 or α_3. If $y \in V(\alpha_3)$ then we have that $y \neq z$ and $y \neq x$ (from assertions 2 and 4). Then $\alpha_1 \cup (z, \alpha_2, y) \cup (y, \alpha_3, x)$ is a ux-path which is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{P_3}$. Suppose that $y \in V(\alpha_1)$. If $e \in V(\alpha_1)$ then $u \neq e$ and $e \neq w$ (from assertions 1 and 3). Then, $(u, \alpha_1, e) \cup (e, \alpha_2, w) \cup \alpha_3$ is a ux-path which is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{P_3}$. If $e \in V(\alpha_2)$, then let a be the last vertex of α_2 that is in α_1 and let b be the first vertex of (a, α_2, w) that is in α_3. We have that $u \neq a$ and $b \neq x$ (from assertions 1 and 4), and $a \neq b$ ($V(\alpha_1) \cap V(\alpha_3) = \emptyset$). Then $(u, \alpha_1, a) \cup (a, \alpha_2, b) \cup (b, \alpha_3, x)$ is a ux-path which is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{P_3}$.

Subcase II.2.2. $V(\alpha_1) \cap V(\alpha_3) \neq \emptyset$. Let a be the first vertex of α_1 that is in α_2 and let b be the first vertex of (a, α_2, w) which is in α_3. Then, we have that $u \neq a$ and $b \neq x$ (from assertions 1 and 4), $a \neq w$ (from assertion 3), and $b \neq z$ (from assertion 2). Also, $a \neq b$, otherwise $(a, \alpha_1, z) \cup (z, \alpha_2, b)$ contains a non-monochromatic cycle in D_1 and a is a vertex with 3-coloured (C_1, C_1, C_2) in-neighbourhood, a contradiction. Suppose that $[V((b, \alpha_3, x)) \cap V((u, \alpha_1, a)] = \emptyset$. Then, $(u, \alpha_1, a) \cup (a, \alpha_2, b) \cup (b, \alpha_3, x)$ is a ux-path which is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{P_3}$. If $[V((b, \alpha_3, x)) \cap V((u, \alpha_1, a)] \neq \emptyset$, let c be the first vertex of (b, α_3, x) that is in (u, α_1, a). Since $a \neq b$ then the definitions of a and b imply that $c \neq a$ and $c \neq b$. Then $(c, \alpha_1, a) \cup (a, \alpha_2, b) \cup (b, \alpha_3, c)$ is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{C_3}$. ■

Definition. We say that the digraph D satisfies the *property A* if:

(1) D_1 has no γ-cycles, and

(2) $\mathscr{C}(D)$ possesses the following two conditions:

(i) every 3-coloured $(C_1, C_1, C_2) - \overrightarrow{C_3}$ has at least two symmetrical arcs,

(ii) if (u, z, w, x) is a 3-coloured $(C_1, C_1, C_2) - \overrightarrow{P_3}$ then $(u, x) \in A(\mathscr{C}(D))$.

Definition. We say that the digraph D satisfies the *property B* if:

(1) Every cycle contained in D_1 is monochromatic,

(2) D contains no 3-coloured (C_1, C_1, C_2) subdivisions of $\overrightarrow{C_3}$, and

(3) If (u, z, w, x) is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{P_3}$ then there is a monochromatic path between u and x in D.

Definition. We say that the digraph D satisfies the *property C* if:

(1) D_1 has no γ-cycles,

(2) D has no vertices with 3-coloured (C_1, C_1, C_2) in-neighbourhood,

(3) D contains no 3-coloured (C_1, C_1, C_2) subdivisions of $\overrightarrow{C_3}$, and

(4) If (u, z, w, x) is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{P_3}$ then there is a monochromatic path between u and x in D.

Theorem 10. *Suppose that D_2 is transitive by monochromatic paths. If D satisfies one of the properties A, B or C, then D has a k.m.p.*

Proof. Consider the digraph D_ς. Note that if every cycle in a digraph is monochromatic then such digraph contains no γ-cycles. So, in any case D_1 has no γ-cycles. Thus, Lemma 8 implies that D_ς is acyclic. Then D_ς contains at least one vertex with zero outdegree. Let $S \in V(D_\varsigma)$ be such that $\delta^+_{D_\varsigma}(S) = 0$. We will prove, by contradiction, that S is a k.m.p. of D.

Since $S \in V(D_\varsigma)$, then S is independent by monochromatic paths. If S is not a k.m.p., then S is not absorbent by monochromatic paths. Let $X = \{z \in V(D) \mid$ there is no zS-monochromatic path in $D\}$. From our assumption we obtain $X \neq \emptyset$. Given that $D[X]$ is an induced subdigraph of D, we have that $D[X]$ satisfies the hypothesis of Theorem 10 and the subdigraph of D_1 contained in $D[X]$ satisfies the hypothesis of Lemma 7. It follows that there exists $x_0 \in X$ such that $\{x_0\}$ is a semikernel by monochromatic paths modulo D_2 of $D[X]$.

Let $T = \{z \in S \mid$ there is no zx_0-monochromatic path in $D_2\}$. From the definition of T, we have that for each $z \in S \setminus T$ there is a zx_0-monochromatic path contained in D_2.

Note that each monochromatic path of D is contained either in D_1 or in D_2.

Claim 1. *$T \cup \{x_0\}$ is independent by monochromatic paths.*

Proof. T is independent by monochromatic paths because $T \subseteq S$ and $S \in \varsigma$.

There is no Tx_0-monochromatic path contained in D. Otherwise, from the definition of T, such path must be contained in D_1. Since $T \subseteq S \in \varsigma$ then there is a x_0S-monochromatic path, but this contradicts the definition of X.

There is no x_0T-monochromatic path. It follows from the definition of X.

We conclude that $T \cup \{x_0\}$ is independent by monochromatic paths. $\square$

Claim 2. *If there is a* $(T \cup \{x_0\})z$*-monochromatic path contained in* D_1 *then there is a* $z(T \cup \{x_0\})$*-monochromatic path.*

Proof. We have two cases.

Case 1. There is a Tz-monochromatic path contained in D_1. Since $T \subseteq S$ and $S \in D_\varsigma$, it follows that there is a zS-monochromatic path contained in D. We may suppose that such path is a $z(S \setminus T)$-monochromatic path. Let α_1 be a uz-monochromatic path contained in D_1 with $u \in T$ and let α_2 be a zw-monochromatic path contained in D with $w \in S \setminus T$. Since $w \in S \setminus T$, the definition of T implies that there is a wx_0-monochromatic path contained in D_2, say α_3. First, suppose that $\alpha_2 \subseteq D_2$, since D_2 is transitive by monochromatic paths then there is a zx_0-monochromatic path contained in D_2. So, we may suppose that $\alpha_2 \subseteq D_1$. If $colour(\alpha_1) = colour(\alpha_2)$, then $\alpha_1 \cup \alpha_2$ contains a uw-monochromatic path, a contradiction as $\{u, w\} \subseteq S$ and $S \in \varsigma$. Hence, $colour(\alpha_1) \neq colour(\alpha_2)$. Moreover, $colour(\alpha_1) \neq colour(\alpha_3)$ ($\alpha_1 \subseteq D_1$ and $\alpha_3 \subseteq D_2$) and $colour(\alpha_2) \neq colour(\alpha_3)$ ($\alpha_2 \subseteq D_1$ and $\alpha_3 \subseteq D_2$).

If D satisfy the property A, then (u, z, w, x_0) is a path in $\mathscr{C}(D)$ which is a 3-coloured $(C_1, C_1, C_2) - \overrightarrow{P_3}$. By hypothesis $(u, x_0) \in A(\mathscr{C}(D))$, then, we have a ux_0-monochromatic path in D; a clear contradiction because $u \in T$ and $T \cup \{x_0\}$ is independent by monochromatic paths. So, assume that D satisfies one of the properties B or C.

Now, note that: There is no uw-monochromatic path. It follows from $\{u, w\} \subseteq S$ and S is independent by monochromatic paths.

We may suppose that there is no zx_0-monochromatic path and there is no zu-monochromatic path, otherwise there is a $z(T \cup \{x_0\})$-monochromatic path.

Then D, α_1, α_2 and α_3 satisfies the hypothesis of Lemma 9. In any case:

- there is a ux_0-path which is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{P_3}$ or
- there is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{C_3}$.

In the first case, we have that there is a monochromatic path between u and x_0 in D. But, this contradicts the fact that $T \cup \{x_0\}$ is independent by monochromatic paths. The second case is not possible since D contains no 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{C_3}$.

Case 2. There is an x_0z-monochromatic path contained in D_1. Let α_1 be an x_0z-monochromatic path contained in D_1. From the choice of x_0 we may suppose that $z \notin X$. Then, the definition of X implies that there is a zS-monochromatic path contained in D, say α_2. Suppose that α_2 ends in w. If $w \in T$ then α_2 is a $z(T \cup \{x_0\})$-monochromatic path in D. Then, suppose that $w \in S \setminus T$. From the definition of T it follows that there is a wx_0-monochromatic path contained in D_2, call α_3 such path. Assume that $\alpha_2 \subseteq D_2$, since D_2 is transitive by monochromatic paths then there is a zx_0-monochromatic path contained in D_2. So, we may suppose that $\alpha_2 \subseteq D_1$. If $colour(\alpha_1) = colour(\alpha_2)$ then $\alpha_1 \cup \alpha_2$ contains an x_0w-monochromatic path, a contradiction with the definition of X. Hence, $colour(\alpha_1) \neq colour(\alpha_2)$. Furthermore, $colour(\alpha_1) \neq colour(\alpha_3)$ ($\alpha_1 \subseteq D_1$ and $\alpha_3 \subseteq D_2$) and $colour(\alpha_2) \neq colour(\alpha_3)$ ($\alpha_2 \subseteq D_1$ and $\alpha_3 \subseteq D_2$).

Suppose that D satisfies the property A, then $\mathscr{C}(D)$ contains a 3-coloured $(C_1, C_1, C_2) - \overrightarrow{C_3}$ (to be explicit: (x_0, z, w, x_0)), then this $\overrightarrow{C_3}$ has at least two symmetrical arcs. Then $(z, x_0) \in A(\mathscr{C}(D))$ or $(x_0, w) \in A(\mathscr{C}(D))$. If $(z, x_0) \in A(\mathscr{C}(D))$, then we have a zx_0-monochromatic path in D and Claim 2 is proved. If $(x_0, w) \in A(\mathscr{C}(D))$, then we have a x_0w-monochromatic path in D, contradicting the definition of X.

Now, suppose that D satisfies one of the properties B or C. Let $u = x_0$, note that: there is no uw-monochromatic path. It follows from the definition of X.

We may suppose that there is no zu-monochromatic path.

Then D, α_1, α_2 and α_3 satisfies the hypothesis of Lemma 9. In any case: there is a ux_0-path which is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{P_3}$ or

There is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{C_3}$.

The first case is not possible as $u = x_0$. The second case is not possible since D contains no 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{C_3}$. $\square$

It follows from Claim 1 and Claim 2 that $(T \cup \{x_0\}) \in \varsigma$, so, $(T \cup \{x_0\}) \in V(D_\varsigma)$.

Now, since $T \subseteq S$, $x_0 \in X$ and for each $s \in S$ such that $s \notin T$ there is an sx_0-monochromatic path contained in D_2 and there is no x_0S-monochromatic path contained in D then $(S, T \cup \{x_0\}) \in A(D_\varsigma)$. We obtain a contradiction with the assumption $\delta^+_{D_\varsigma}(S) = 0$.

We conclude that S is a k.m.p. of D. $\blacksquare$

Remark 11. Notice that Theorem 10 generalizes the theorem of Sands, Sauer and Woodrow since:

(1) A 2-coloured digraph can be divided in two monochromatic spanning subdigraphs $D_1 = D[\{a \in A(D) \mid colour(a) = colour\ 1\}]$ and $D_2 = D[\{a \in A(D) \mid colour(a) = colour\ 2\}]$.

(2) Every directed cycle in D_1 is monochromatic since D_1 is monochromatic.

(3) D_1 has no γ-cycles since D_1 is monochromatic.

(4) D_2 is transitive by monochromatic paths since D_2 is monochromatic.

Now, since only two colours are used on D, then we have the following assertions.

(5) $\mathscr{C}(D)$ satisfies the following two conditions:
   (i) all 3-coloured $\overrightarrow{C_3} - (C_1, C_1, C_2)$ has at least two symmetrical arcs,
   (ii) if (u, v, w, x) is a 3-coloured $\overrightarrow{P_3} - (C_1, C_1, C_2)$ then $(u, x) \in A(\mathscr{C}(D))$.

(6) D has no vertices with 3-coloured (C_1, C_1, C_2) in-neighbourhood.

(7) D contains no 3-coloured (C_1, C_1, C_2) subdivisions of $\overrightarrow{C_3}$.

(8) if (u, v, w, x) is a 3-coloured (C_1, C_1, C_2) subdivision of $\overrightarrow{P_3}$ then there is a monochromatic path in D between u and x.

Therefore, every 2-coloured digraph D fulfils the hypotheses of our main theorem, furthermore it satisfies the three properties A, B and C. We conclude that theorem generalizes the theorem of Sands, Sauer and Woodrow.

With Theorem 10 we can generate new theorems, for example, let D_1 be a tournament that satisfies the hypothesis of Shen Minggang's theorem, then it is possible to prove that D_1 has no γ-cycles. Then we obtain the following new theorem.

Theorem 12. *Let D be an m-coloured digraph such that:*

(1) *D_1 is a tournament such that every triangle is a quasi-monochromatic subdigraph of D_1.*

(2) *D_2 is transitive by monochromatic paths.*

(3) *$\mathscr{C}(D)$ has the following two conditions:*

  (i) *every 3-coloured $(C_1, C_1, C_2) - \overrightarrow{C_3}$ has at least two symmetrical arcs,*

  (ii) *if (u, v, w, x) is a 3-coloured $(C_1, C_1, C_2) - \overrightarrow{P_3}$ then $(u, x) \in A(\mathscr{C}(D))$.*

Then D has a k.m.p.

Similarly, it is possible to generate new theorems if D_1 is one of the following digraphs:

- (H. Galeana-Sánchez and J.J. García-Ruvalcaba, [11]) An m-coloured digraph resulting from the deletion of the single arc (x, y) from some m-coloured tournament such that every triangle is quasi-monochromatic.

- (H. Galeana-Sánchez, R. Rojas Monroy, [17]) An m-coloured bipartite tournament such that every directed cycle of length 4 is monochromatic.

- (H. Galeana-Sánchez and R. Rojas Monroy, [19]) An m-coloured k-partite tournament with each cycle of length 3 and each cycle of length 4 monochromatic.

- (Gena Hahn, Pierre Ille and Robert E. Woodrow, [22]) A finite k-coloured tournament satisfying:

  - every tournament on 3 vertices is quasi-monochromatic, and
  - for $s \geq 4$, each cycle of length s is quasi-monochromatic and no cycle of length less than s has at least three colours on its arcs.

Other conditions which imply that an m-coloured digraph has no γ-cycles can be found in [4, 5, 18, 20, 21].

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