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INTERVAL EDGE-COLORINGS OF CARTESIAN PRODUCTS OF GRAPHS I

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Abstract

A proper edge-coloring of a graph G with colors $1, \ldots, t$ is an interval t-coloring if all colors are used and the colors of edges incident to each vertex of G form an interval of integers. A graph G is interval colorable if it has an interval t-coloring for some positive integer t. Let $\mathfrak{N}$ be the set of all interval colorable graphs. For a graph $G \in \mathfrak{N}$, the least and the greatest values of t for which G has an interval t-coloring are denoted by $w(G)$ and $W(G)$, respectively. In this paper we first show that if G is an r-regular graph and $G \in \mathfrak{N}$, then $W(G \square P_m) \geq W(G) + W(P_m) + (m-1)r$ ($m \in \mathbb{N}$) and $W(G \square C_{2n}) \geq W(G) + W(C_{2n}) + nr$ ($n \geq 2$). Next, we investigate interval edge-colorings of grids, cylinders and tori. In particular, we prove that if $G \square H$ is planar and both factors have at least 3 vertices, then $G \square H \in \mathfrak{N}$ and $w(G \square H) \leq 6$. Finally, we confirm the first author's conjecture on the n-dimensional cube Q_n and show that Q_n has an interval t-coloring if and only if $n \leq t \leq \frac{n(n+1)}{2}$.

Keywords: edge-coloring, interval coloring, grid, cylinder, torus, n-dimensional cube.

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1. Introduction

A proper edge-coloring of a graph G with colors $1,\dots,t$ is an interval t-coloring if all colors are used and the colors of edges incident to each vertex of G form an interval of integers. A graph G is interval colorable if it has an interval t-coloring for some positive integer t. Let $\mathfrak{N}$ be the set of all interval colorable graphs [1, 14]. For a graph $G\in\mathfrak{N}$, the least and the greatest values of t for which G has an interval t-coloring are denoted by $w(G)$ and $W(G)$, respectively. The concept of interval edge-coloring was introduced by Asratian and Kamalian [1]. In [1], they proved the following:

Theorem 1. *Let G be a regular graph. Then*
(1) $G\in\mathfrak{N}$ *if and only if* $\chi'(G)=\Delta(G)$.
(2) *If* $G\in\mathfrak{N}$ *and* $w(G)\leq t\leq W(G)$, *then G has an interval t-coloring.*

In [2], Asratian and Kamalian investigated interval edge-colorings of connected graphs. In particular, they obtained the following two results.

Theorem 2. *If G is a connected graph and $G\in\mathfrak{N}$, then*

$$W(G)\leq(\mathrm{diam}(G)+1)\,(\Delta(G)-1)+1.$$

Theorem 3. *If G is a connected bipartite graph and $G\in\mathfrak{N}$, then*

$$W(G)\leq\mathrm{diam}(G)\,(\Delta(G)-1)+1.$$

Recently, Kamalian and the first author [16] showed that these upper bounds cannot be significantly improved.

In [13], Kamalian investigated interval colorings of complete bipartite graphs and trees. In particular, he proved the following:

Theorem 4. *For any $r,s\in\mathbb{N}$, the complete bipartite graph $K_{r,s}$ is interval colorable, and*
(1) $w\left(K_{r,s}\right)=r+s-\gcd(r,s)$,
(2) $W\left(K_{r,s}\right)=r+s-1$,
(3) *if* $w\left(K_{r,s}\right)\leq t\leq W\left(K_{r,s}\right)$, *then $K_{r,s}$ has an interval t-coloring.*

In [21], the first author investigated interval colorings of complete graphs and n-dimensional cubes. In particular, he obtained the following two results.

Theorem 5. *If $n = p2^q$, where p is odd and q is nonnegative, then*

$$W(K_{2n}) \geq 4n - 2 - p - q.$$

Theorem 6. *If $n \in \mathbb{N}$, then $W(Q_n) \geq \frac{n(n+1)}{2}$.*

The NP-completeness of the problem of the existence of an interval edge-coloring of an arbitrary bipartite graph was shown in [24]. A similar result for regular graphs was obtained in [1, 2]. In [19, 22, 23], interval edge-colorings of various products of graphs were investigated. Some interesting results on interval colorings were also obtained in [3, 4, 6, 7, 8, 9, 10, 14, 15, 16, 17, 18, 19, 20]. Surveys on this topic can be found in some books [3, 12, 19].

In this paper we focus only on interval edge-colorings of Cartesian products of graphs.

2. Notations, Definitions and Auxiliary Results

Throughout this paper all graphs are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of graph G, respectively. The degree of a vertex v in G is denoted by $d_G(v)$, the maximum degree of G by $\Delta(G)$, and the chromatic index of G by $\chi'(G)$. If G is a connected graph, then the distance between two vertices u and v in G, we denote by $d(u,v)$, and the diameter of G by $\mathrm{diam}(G)$. We use the standard notations P_n, C_n, K_n and Q_n for the path, cycle, complete graph on n vertices and the n-dimensional cube, respectively. A partial edge-coloring of a graph G is a coloring of some edges of G such that no two adjacent edges receive the same color. If α is a partial edge-coloring of G and $v \in V(G)$, then $S(v,\alpha)$ denotes the set of colors appearing on colored edges incident to v. Clearly, if α is a proper edge-coloring of a graph G, then $|S(v,\alpha)| = d_G(v)$ for every $v \in V(G)$.

Let $[t]$ denote the set of the first t natural numbers. Let $\lfloor a \rfloor$ ($\lceil a \rceil$) denote the largest (least) integer less (greater) than or equal to a. For two positive integers a and b with $a \leq b$, the set $\{a,\ldots,b\}$ is denoted by $[a,b]$. The terms and concepts that we do not define can be found in [25].

Let G and H be graphs. The Cartesian product $G\square H$ is defined as follows: $V(G\square H) = V(G) \times V(H)$, $E(G\square H) = \{(u_1,v_1)(u_2,v_2)\colon u_1 = u_2 \text{ and } v_1v_2 \in E(H) \text{ or } v_1 = v_2 \text{ and } u_1u_2 \in E(G)\}$.

Let $V(G) = \{u_1,\ldots,u_n\}$ and $V(H) = \{w_1,\ldots,w_m\}$. We use the following notation for vertex and edge sets of the Cartesian product $G\square H$: $V(G\square H) = \bigcup_{i=1}^{m} V^i$, where $V^i = \left\{v_j^{(i)} : 1 \leq j \leq n\right\}$ and $E(G\square H) = \bigcup_{i=1}^{m} E^i \cup \bigcup_{j=1}^{n} E_j$, where $E^i = \left\{v_j^{(i)}v_k^{(i)} : u_ju_k \in E(G)\right\}$ and $E_j = \left\{v_j^{(i)}v_j^{(k)} : w_iw_k \in E(H)\right\}$.

We define subgraphs G^i of G as follows: $G^i = (V^i, E^i)$. Clearly, G^i is isomorphic to G for $1 \leq i \leq m$.

Clearly, if G and H are connected graphs, then $G\square H$ is connected, too. Moreover, $\Delta(G\square H) = \Delta(G) + \Delta(H)$ and $\mathrm{diam}(G\square H) = \mathrm{diam}(G) + \mathrm{diam}(H)$.

The k-dimensional grid $G(n_1, \dots, n_k)$, $n_i \in \mathbb{N}$, is the Cartesian product of paths $P_{n_1}\square P_{n_2}\square \cdots \square P_{n_k}$. The cylinder $C(n_1, n_2)$ is the Cartesian product $P_{n_1}\square C_{n_2}$, and the torus $T(n_1, n_2)$ is the Cartesian product of cycles $C_{n_1}\square C_{n_2}$.

We also need the following two lemmas.

Lemma 7. *If α is an edge-coloring of a connected graph G with colors $1, \dots, t$ such that the edges incident to each vertex $v \in V(G)$ are colored by distinct and consecutive colors, and $\min_{e\in E(G)}\{\alpha(e)\} = 1$, $\max_{e\in E(G)}\{\alpha(e)\} = t$, then α is an interval t-coloring of G.*

Proof. For the proof of the lemma, it suffices to show that all colors are used in the coloring α of G.

Let u and w be vertices such that $1 \in S(u, \alpha)$ and $t \in S(w, \alpha)$. Also, let $P = v_1, \dots, v_k$, where $u = v_1$ and $v_k = w$ be a u, w-path in G. If $k = 1$, then $t \in S(u, \alpha)$ and all colors appear on edges incident to u. Assume that $k \geq 2$. The sets $S(v_i, \alpha)$ for $v_i \in V(P)$ are intervals, and for $2 \leq i \leq k$, intervals $S(v_{i-1}, \alpha)$ and $S(v_i, \alpha)$ share a color. Thus, the sets $S(v_1, \alpha), \dots, S(v_k, \alpha)$ cover $[1, t]$. ∎

The next lemma was proved by Behzad and Mahmoodian in [5].

Lemma 8. *If both G and H have at least 3 vertices, then the Cartesian product $G\square H$ is planar if and only if $G\square H = G(m, n)$ or $G\square H = C(m, n)$.*

3. The Cartesian Product of Regular Graphs

Interval edge-colorings of Cartesian products of graphs were first investigated by Giaro and Kubale in [7], where they proved the following:

Theorem 9. *If $G \in \mathfrak{N}$, then $G\square P_m \in \mathfrak{N}$ $(m \in \mathbb{N})$ and $G\square C_{2n} \in \mathfrak{N}$ $(n \geq 2)$.*

It is well-known that $P_m, C_{2n} \in \mathfrak{N}$ and $W(P_m) = m - 1$, $W(C_{2n}) = n + 1$ for $m \in \mathbb{N}$ and $n \geq 2$. Later, Giaro and Kubale [9, 19] proved a more general result.

Theorem 10. *If $G, H \in \mathfrak{N}$, then $G\square H \in \mathfrak{N}$. Moreover, $w(G\square H) \leq w(G) + w(H)$ and $W(G\square H) \geq W(G) + W(H)$.*

Let us note that if $G \in \mathfrak{N}$ and $H = P_m$ or $H = C_{2n}$, then, by Theorem 10, we obtain $w(G\square H) \leq w(G) + 2$ and $W(G\square P_m) \geq W(G) + m - 1$, $W(G\square C_{2n}) \geq W(G) + n + 1$. Now we improve the lower bound in Theorem 10 for $W(G\square P_m)$ and $W(G\square C_{2n})$ when G is a regular graph and $G \in \mathfrak{N}$. More precisely, we show that the following two theorems hold.

Theorem 11. *If G is an r-regular graph and $G \in \mathfrak{N}$, then $G \square P_m \in \mathfrak{N}$ $(m \in \mathbb{N})$ and $W(G \square P_m) \geq W(G) + W(P_m) + (m-1)r$.*

Proof. For the proof, we construct an edge-coloring of the graph $G \square P_m$ that satisfies the specified conditions.

Since $G \in \mathfrak{N}$, there exists an interval $W(G)$-coloring α of G. Now we define an edge-coloring β of the subgraphs $G^1, \ldots, G^m$. For $1 \leq i \leq m$ and for every edge $v_j^{(i)} v_k^{(i)} \in E(G^i)$, let

$$\beta\left(v_j^{(i)} v_k^{(i)}\right) = \alpha(v_j v_k) + (i-1)(r+1).$$

It is easy to see that the color of each edge of the subgraph G^i is obtained by shifting the color of the associated edge of G by $(i-1)(r+1)$. Thus the set $S\left(v_j^{(i)}, \beta\right)$ is an interval for each vertex $v_j^{(i)} \in V(G^i)$, where $1 \leq i \leq m$, $1 \leq j \leq n$. Now we define an edge-coloring γ of the graph $G \square P_m$. For every $e \in E(G \square P_m)$, let

$$\gamma(e) = \begin{cases} \beta(e), & \text{if } e \in E(G^i), \\ \max S\left(v_j^{(i)}, \beta\right) + 1, & \text{if } e = v_j^{(i)} v_j^{(i+1)} \in E_j, \end{cases}$$

where $1 \leq i \leq m, 1 \leq j \leq n$.

Let us prove that γ is an interval $(W(G) + W(P_m) + (m-1)r)$-coloring of the graph $G \square P_m$ for $m \in \mathbb{N}$.

First we prove that the set $S\left(v_j^{(i)}, \gamma\right)$ is an interval for each vertex $v_j^{(i)} \in V(G \square P_m)$, where $1 \leq i \leq m, 1 \leq j \leq n$.

For each vertex $v_j^{(i)} \in V(G \square P_m)$, the set $S\left(v_j^{(i)}, \gamma\right)$ can be represented as a union of three sets, $S\left(v_j^{(i)}, \gamma\right) = A_j^{(i)} \cup B_j^{(i)} \cup C_j^{(i)}$, where $A_j^{(i)}$ corresponds to the edges of i-th layer, $B_j^{(i)}$ corresponds to the edges from the vertices of lower layer and $C_j^{(i)}$ corresponds to the edges from the vertices of higher layer. More specifically, for $1 \leq i \leq m$, $1 \leq j \leq n$, define sets $A_j^{(i)}$, $B_j^{(i)}$ and $C_j^{(i)}$ as follows:

$$A_j^{(i)} = \left\{\gamma\left(v_j^{(i)} u\right) : v_j^{(i)} u \in E^i\right\},$$

$$B_j^{(i)} = \begin{cases} \emptyset, & \text{if } i = 1, \\ \left\{\gamma\left(v_j^{(i)} u\right) : v_j^{(i)} u \in E_j,\ u \in V^{i-1}\right\}, & \text{if } 2 \leq i \leq m, \end{cases}$$

$$C_j^{(i)} = \begin{cases} \left\{\gamma\left(v_j^{(i)} u\right) : v_j^{(i)} u \in E_j,\ u \in V^{i+1}\right\}, & \text{if } 1 \leq i \leq m-1, \\ \emptyset, & \text{if } i = m. \end{cases}$$

By the definition of γ, we have that for $1 \leq i \leq m$, $1 \leq j \leq n$,

$$A_j^{(i)} = [\min S(v_j,\alpha) + (i-1)(r+1), \max S(v_j,\alpha) + (i-1)(r+1)],$$

for $2 \leq i \leq m$, $1 \leq j \leq n$,

$$B_j^{(i)} = \{\max S(v_j,\alpha) + (i-2)(r+1) + 1\},$$

and for $1 \leq i \leq m-1$, $1 \leq j \leq n$,

$$C_j^{(i)} = \{\max S(v_j,\alpha) + (i-1)(r+1) + 1\}.$$

By this and taking into account that $\max S(v_j,\alpha) - \min S(v_j,\alpha) = r-1$ for $1 \leq j \leq n$, we have that $A_j^{(i)} \cup B_j^{(i)} \cup C_j^{(i)}$ is an interval for each vertex $v_j^{(i)} \in V(G^i)$, where $1 \leq i \leq m$, $1 \leq j \leq n$.

Next we show that in the coloring γ all colors are used. Clearly, there exists an edge $v_{j_0}^{(1)} v_{k_0}^{(1)} \in E(G^1)$ such that $\gamma\left(v_{j_0}^{(1)} v_{k_0}^{(1)}\right) = 1$, since in the coloring α there exists an edge $v_{j_0}v_{k_0}$ with $\alpha(v_{j_0}v_{k_0}) = 1$ and $\gamma\left(v_{j_0}^{(1)} v_{k_0}^{(1)}\right) = \beta\left(v_{j_0}^{(1)} v_{k_0}^{(1)}\right) = \alpha(v_{j_0}v_{k_0})$. Similarly, there exists an edge $v_{j_1}^{(m)} v_{k_1}^{(m)} \in E(G^m)$ such that $\gamma\left(v_{j_1}^{(m)} v_{k_1}^{(m)}\right) = W(G) + (m-1)(r+1) = W(G) + W(P_m) + (m-1)r$, since in the coloring α there exists an edge $v_{j_1}v_{k_1}$ with $\alpha(v_{j_1}v_{k_1}) = W(G)$ and $\gamma\left(v_{j_1}^{(m)} v_{k_1}^{(m)}\right) = \beta\left(v_{j_1}^{(m)} v_{k_1}^{(m)}\right) = \alpha(v_{j_1}v_{k_1}) + (m-1)(r+1)$.

Now, by Lemma 7, we have that γ is an interval $(W(G) + W(P_m) + (m-1)r)$ -coloring of the graph $G\square P_m$ for $m \in \mathbb{N}$. ∎

Corollary 12. *If G is an r-regular graph and $G \in \mathfrak{N}$, then $G\square Q_n \in \mathfrak{N}$ $(n \in \mathbb{N})$ and*

$$W(G\square Q_n) \geq W(G) + \tfrac{n(n+2r+1)}{2}.$$

Proof. By Theorem 11 and using associativity of the Cartesian product, we get

$$W(G\square Q_n) = W(\cdots((G\square K_2)\square K_2)\square\cdots\square K_2) \geq W(G) + \tfrac{n(n+2r+1)}{2}.$$

∎

Theorem 13. *If G is an r-regular graph and $G \in \mathfrak{N}$, then $G\square C_{2n} \in \mathfrak{N}$ $(n \geq 2)$ and $W(G\square C_{2n}) \geq W(G) + W(C_{2n}) + nr$.*

Proof. For the proof, we construct an edge-coloring of the graph $G\square C_{2n}$ that satisfies the specified conditions.

Since $G \in \mathfrak{N}$, there exists an interval $W(G)$-coloring α of G. Now we define an edge-coloring β of the subgraphs $G^1,\ldots,G^{2n}$.

For $1 \leq i \leq 2n$ and for every edge $v_j^{(i)} v_k^{(i)} \in E(G^i)$, let

$$\beta\left(v_j^{(i)}v_k^{(i)}\right)=\begin{cases}\alpha\left(v_jv_k\right), & \text{if } i=1,\\ \alpha\left(v_jv_k\right)+(i-1)(r+1)+1, & \text{if } 2\leq i\leq n+1,\\ \alpha\left(v_jv_k\right)+(2n+1-i)(r+1), & \text{if } n+2\leq i\leq 2n.\end{cases}$$

It is easy to see that the color of each edge of the subgraph G^i is obtained by shifting the color of the associated edge of G by $(i-1)(r+1)+1$ for $2\leq i\leq n+1$, and by $(2n-i+1)(r+1)$ for $n+2\leq i\leq 2n$, thus the set $S\left(v_j^{(i)},\beta\right)$ is an interval for each vertex $v_j^{(i)}\in V(G^i)$, where $1\leq i\leq 2n$, $1\leq j\leq p$. Now we define an edge-coloring γ of the graph $G\Box C_{2n}$.

For every $e\in E\left(G\Box C_{2n}\right)$, let

$$\gamma(e)=\begin{cases}\beta(e), & \text{if } e\in E(G^i),\\ \max S\left(v_j^{(1)},\beta\right)+1, & \text{if } e=v_j^{(1)}v_j^{(2n)}\in E_j,\\ \max S\left(v_j^{(1)},\beta\right)+2, & \text{if } e=v_j^{(1)}v_j^{(2)}\in E_j,\\ \max S\left(v_j^{(i)},\beta\right)+1, & \text{if } e=v_j^{(i)}v_j^{(i+1)}\in E_j, 2\leq i\leq n,\\ \max S\left(v_j^{(i)},\beta\right)+1, & \text{if } e=v_j^{(i-1)}v_j^{(i)}\in E_j, n+2\leq i\leq 2n,\end{cases}$$

where $1\leq i\leq 2n, 1\leq j\leq p$.

Let us prove that γ is an interval $(W(G)+W\left(C_{2n}\right)+nr)$-coloring of the graph $G\Box C_{2n}$ for $n\geq 2$.

First we prove that the set $S\left(v_j^{(i)},\gamma\right)$ is an interval for each vertex $v_j^{(i)}\in V\left(G\Box C_{2n}\right)$, where $1\leq i\leq 2n, 1\leq j\leq p$.

Case 1. $i=1$, $1\leq j\leq p$. By the definition of γ and taking into account that $\max S\left(v_j,\alpha\right)-\min S\left(v_j,\alpha\right)=r-1$ for $1\leq j\leq p$, we have

$$\begin{aligned}S\left(v_j^{(1)},\gamma\right) &= \left\{\min S\left(v_j,\alpha\right),\ldots,\max S\left(v_j,\alpha\right)\right\}\cup\left\{\max S\left(v_j,\alpha\right)+2\right\}\\ &\cup \left\{\max S\left(v_j,\alpha\right)+1\right\}=\left[\min S\left(v_j,\alpha\right),\max S\left(v_j,\alpha\right)+2\right].\end{aligned}$$

Case 2. $2\leq i\leq n$, $1\leq j\leq p$. By the definition of γ and taking into account that $\max S\left(v_j,\alpha\right)-\min S\left(v_j,\alpha\right)=r-1$ for $1\leq j\leq p$, we have

$$\begin{aligned}S\left(v_j^{(i)},\gamma\right) &= \left\{\min S\left(v_j,\alpha\right)+(i-1)(r+1)+1,\ldots,\max S\left(v_j,\alpha\right)\right.\\ &+ \left.(i-1)(r+1)+1\right\}\cup\left\{\max S\left(v_j,\alpha\right)+(i-2)(r+1)+2\right\}\\ &\cup \left\{\max S\left(v_j,\alpha\right)+(i-1)(r+1)+2\right\}\\ &= \left[\min S\left(v_j,\alpha\right)+(i-1)(r+1),\max S\left(v_j,\alpha\right)+(i-1)(r+1)+2\right].\end{aligned}$$

Case 3. $i=n+1$, $1\leq j\leq p$. By the definition of γ and taking into account that $\max S\left(v_j,\alpha\right)-\min S\left(v_j,\alpha\right)=r-1$ for $1\leq j\leq p$, we have

$$\begin{aligned} S\left(v_j^{(n+1)},\gamma\right) &= \{\min S(v_j,\alpha)+n(r+1)+1,\dots,\max S(v_j,\alpha)+n(r+1)+1\} \\ &\cup \{\max S(v_j,\alpha)+(n-1)(r+1)+2\} \\ &\cup \{\max S(v_j,\alpha)+(n-1)(r+1)+1\} \\ &= [\min S(v_j,\alpha)+n(r+1)-1,\max S(v_j,\alpha)+n(r+1)+1]. \end{aligned}$$

Case 4. $n+2 \leq i \leq 2n$, $1 \leq j \leq p$. By the definition of γ and taking into account that $\max S(v_j,\alpha) - \min S(v_j,\alpha) = r-1$ for $1 \leq j \leq p$, we have

$$\begin{aligned} S\left(v_j^{(i)},\gamma\right) &= \{\min S(v_j,\alpha)+(2n+1-i)(r+1),\dots,\max S(v_j,\alpha) \\ &+ (2n+1-i)(r+1)\} \cup \{\max S(v_j,\alpha)+(2n+1-i)(r+1)+1\} \\ &\cup \{\max S(v_j,\alpha)+(2n-i)(r+1)+1\} = [\min S(v_j,\alpha) \\ &+ (2n-i+1)(r+1)-1,\max S(v_j,\alpha)+(2n-i+1)(r+1)+1]. \end{aligned}$$

Next we show that in the coloring γ all colors are used. Clearly, there exists an edge $v_{j_0}^{(1)}v_{k_0}^{(1)} \in E(G^1)$ such that $\gamma\left(v_{j_0}^{(1)}v_{k_0}^{(1)}\right) = 1$, since in the coloring α there exists an edge $v_{j_0}v_{k_0}$ with $\alpha(v_{j_0}v_{k_0}) = 1$ and $\gamma\left(v_{j_0}^{(1)}v_{k_0}^{(1)}\right) = \beta\left(v_{j_0}^{(1)}v_{k_0}^{(1)}\right) = \alpha(v_{j_0}v_{k_0})$. Similarly, there exists an edge $v_{j_1}^{(n+1)}v_{k_1}^{(n+1)} \in E(G^{n+1})$ such that $\gamma\left(v_{j_1}^{(n+1)}v_{k_1}^{(n+1)}\right) = W(G)+n(r+1)+1 = W(G)+W(C_{2n})+nr$, since in the coloring α there exists an edge $v_{j_1}v_{k_1}$ with $\alpha(v_{j_1}v_{k_1}) = W(G)$ and $\gamma\left(v_{j_1}^{(n+1)}v_{k_1}^{(n+1)}\right) = \beta\left(v_{j_1}^{(n+1)}v_{k_1}^{(n+1)}\right) = \alpha(v_{j_1}v_{k_1})+n(r+1)+1$.

Now, by Lemma 7, we have that γ is an interval $(W(G)+W(C_{2n})+nr)$-coloring of the graph $G\Box C_{2n}$ for $n \geq 2$. ■

From Theorems 5 and 13, we have:

Corollary 14. *If $n = p2^q$, where p is odd and q is nonnegative, then*

$$W(K_{2n}\Box C_{2n}) \geq 2n^2+4n-1-p-q.$$

Note that the lower bound in Corollary 14 is close to the upper bound for $W(K_{2n}\Box C_{2n})$, since $\Delta(K_{2n}\Box C_{2n}) = 2n+1$ and $\mathrm{diam}(K_{2n}\Box C_{2n}) = n+1$, by Theorem 2, we have $W(K_{2n}\Box C_{2n}) \leq 2n^2+4n+1$.

4. Grids, Cylinders and Tori

Interval edge-colorings of grids, cylinders and tori were first considered by Giaro and Kubale in [7], where they proved the following:

Theorem 15. *If $G = G(n_1, \dots, n_k)$ or $G = C(m, 2n)$, $m \in \mathbb{N}, n \geq 2$, or $G = T(2m, 2n)$, $m, n \geq 2$, then $G \in \mathfrak{N}$ and $w(G) = \Delta(G)$.*

For the greatest possible number of colors in interval colorings of grid graphs, the first author and Karapetyan [20] proved the following theorems:

Theorem 16. *For any $m \in \mathbb{N}, n \geq 2$, we have $W(C(m, 2n)) \geq 3m + n - 2$.*

Theorem 17. *For any $m, n \geq 2$, we have $W(T(2m, 2n)) \geq \max\{3m+n, 3n+m\}$.*

First we consider grids. It is easy to see that $W(G(2, n)) = 2n - 1$ for any $n \in \mathbb{N}$. Now we provide a lower bound for $W(G(m, n))$ when $m, n \geq 2$.

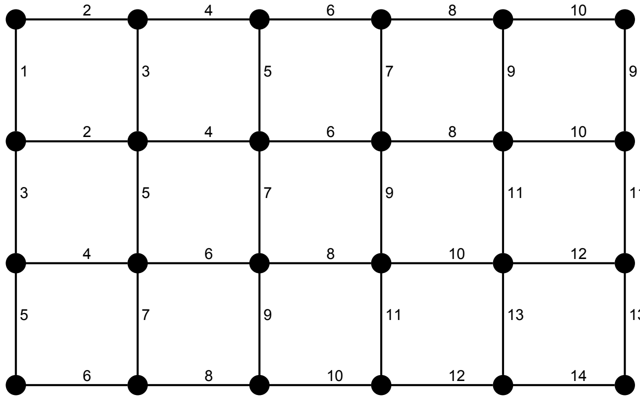

Figure 1. Interval 14-coloring of the graph $G(4, 6)$.

Theorem 18. *For any $m, n \geq 2$, we have $W(G(m, n)) \geq 2(m + n - 3)$.*

Proof. For the proof, we are going to construct an edge-coloring of the graph $G(m, n)$ that satisfies the specified conditions.

Define an edge-coloring α of $G(m, n)$ as follows:

(1) for $i = 1, \dots, m - 1$, $j = 1, \dots, n - 1$, let

$$\alpha\left(v_j^{(i)} v_j^{(i+1)}\right) = 2(i + j) - 3;$$

(2) for $i = 1, \dots, m - 1$, let

$$\alpha\left(v_n^{(i)} v_n^{(i+1)}\right) = 2(n + i) - 5;$$

(3) for $j = 1, \dots, n - 1$, let

$$\alpha\left(v_j^{(1)}v_{j+1}^{(1)}\right) = 2j;$$

(4) for $i = 2, \dots, m$, $j = 1, \dots, n-1$, let

$$\alpha\left(v_j^{(i)}v_{j+1}^{(i)}\right) = 2(i+j) - 4.$$

It is easy to see that α is an interval $(2(m+n-3))$-coloring of $G(m,n)$ when $m, n \geq 2$. ■

Figure 1 shows the interval 14-coloring α of the graph $G(4,6)$ described in the proof of Theorem 18.

Note that the lower bound in Theorem 18 is not far from the upper bound for $W(G(m,n))$, since $G(m,n)$ is bipartite, $2 \leq \Delta(G(m,n)) \leq 4$ and diam $(G(m,n)) = m+n-2$, by Theorem 3, we have $W(G(m,n)) \leq 3(m+n-2)+1$.

From Theorems 10 and 18, we have:

Corollary 19. *If $n_1 \geq \cdots \geq n_{2k} \geq 2$ $(k \in \mathbb{N})$, then*

$$W(G(n_1, \dots, n_{2k})) \geq 2\sum_{i=1}^{2k} n_i - 6k,$$

and if $n_1 \geq \cdots \geq n_{2k+1} \geq 2$ $(k \in \mathbb{N})$, then

$$W(G(n_1, \dots, n_{2k+1})) \geq 2\sum_{i=1}^{2k} n_i + n_{2k+1} - 6k - 1.$$

Next we consider cylinders. In [18], Khchoyan proved the following:

Theorem 20. *For any $n \geq 3$, we have*
(1) $C(2,n) \in \mathfrak{N}$,
(2) $w(C(2,n)) = 3$,
(3) $W(C(2,n)) = n+2$,
(4) *if $w(C(2,n)) \leq t \leq W(C(2,n))$, then $C(2,n)$ has an interval t-coloring.*

Now we prove some general results on cylinders.

Theorem 21. *For any $m \geq 3, n \in \mathbb{N}$, we have $C(m, 2n+1) \in \mathfrak{N}$ and*

$$w(C(m,2n+1)) = \begin{cases} 4, & \text{if } m \text{ is even}, \\ 6, & \text{if } m \text{ is odd}. \end{cases}$$

Proof. First we show that if m is even, then $C(m, 2n+1)$ has an interval 4-coloring. For $1 \leq i \leq \frac{m}{2}$, define a subgraph C^i of the graph $C(m, 2n+1)$ as follows:

$$C^i = \left(V^{2i-1} \cup V^{2i}, E^{2i-1} \cup E^{2i} \cup \left\{v_j^{(2i-1)}v_j^{(2i)} \colon 1 \leq j \leq 2n+1\right\}\right).$$

Clearly, C^i is isomorphic to $C(2, 2n+1)$ for $1 \le i \le \frac{m}{2}$. By Theorem 20, $C(2, 2n+1) \in \mathfrak{N}$ and there exists an interval 3-coloring α of $C(2, 2n+1)$. Now we define an edge-coloring β of $C(m, 2n+1)$. First we color the edges of C^i according to α for $1 \le i \le \frac{m}{2}$. Then we color the edges $v_j^{(2i)} v_j^{(2i+1)} \in E_j$ with color 4 for $1 \le i \le \frac{m}{2} - 1, 1 \le j \le 2n+1$. It is easy to see that β is an interval 4-coloring of $C(m, 2n+1)$. This shows that $C(m, 2n+1) \in \mathfrak{N}$ and $w(C(m, 2n+1)) \le 4$. On the other hand, $w(C(m, 2n+1)) \ge \Delta(C(m, 2n+1)) = 4$; thus $w(C(m, 2n+1)) = 4$ for even m.

Now assume that m is odd. First we show that $C(3, 2n+1)$ has an interval 6-coloring. Define an edge-coloring γ of $C(3, 2n+1)$ as follows:

(1) $\gamma\left(v_1^{(1)} v_1^{(2)}\right) = 6$ and for $j = 2, \dots, 2\left\lfloor \frac{n+1}{2} \right\rfloor$, let $\gamma\left(v_j^{(1)} v_j^{(2)}\right) = 4$;

(2) $\gamma\left(v^{(1)}_{2\left\lfloor \frac{n+1}{2} \right\rfloor+1} v^{(2)}_{2\left\lfloor \frac{n+1}{2} \right\rfloor+1}\right) = 2$ and for $j = 2\left\lfloor \frac{n+1}{2} \right\rfloor + 2, \dots, 2n+1$, let

$$\gamma\left(v_j^{(1)} v_j^{(2)}\right) = 3;$$

(3) $\gamma\left(v_1^{(2)} v_1^{(3)}\right) = 3$ and for $j = 2, \dots, 2\left\lfloor \frac{n+1}{2} \right\rfloor$, let $\gamma\left(v_j^{(2)} v_j^{(3)}\right) = 2$;

(4) for $j = 2\left\lfloor \frac{n+1}{2} \right\rfloor + 1, \dots, 2n+1$, let $\gamma\left(v_j^{(2)} v_j^{(3)}\right) = 1$;

(5) $j = 1, \dots, \left\lfloor \frac{n+1}{2} \right\rfloor$, let

$$\gamma\left(v_{2j-1}^{(1)} v_{2j}^{(1)}\right) = \gamma\left(v_{2j-1}^{(2)} v_{2j}^{(2)}\right) = 5 \text{ and } \gamma\left(v_{2j}^{(1)} v_{2j+1}^{(1)}\right) = \gamma\left(v_{2j}^{(2)} v_{2j+1}^{(2)}\right) = 3;$$

(6) for $j = \left\lfloor \frac{n+1}{2} \right\rfloor + 1, \dots, n$, let

$$\gamma\left(v_{2j-1}^{(1)} v_{2j}^{(1)}\right) = \gamma\left(v_{2j-1}^{(2)} v_{2j}^{(2)}\right) = 4 \text{ and } \gamma\left(v_1^{(1)} v_{2n+1}^{(1)}\right) = \gamma\left(v_1^{(2)} v_{2n+1}^{(2)}\right) = 4;$$

(7) for $j = \left\lfloor \frac{n+1}{2} \right\rfloor + 1, \dots, n$, let $\gamma\left(v_{2j}^{(1)} v_{2j+1}^{(1)}\right) = \gamma\left(v_{2j}^{(2)} v_{2j+1}^{(2)}\right) = 2$;

(8) for $j = 1, \dots, \left\lfloor \frac{n+1}{2} \right\rfloor$, let $\gamma\left(v_{2j-1}^{(3)} v_{2j}^{(3)}\right) = 1$ and $\gamma\left(v_{2j}^{(3)} v_{2j+1}^{(3)}\right) = 3$;

(9) for $j = \left\lfloor \frac{n+1}{2} \right\rfloor + 1, \dots, n$, let $\gamma\left(v_{2j-1}^{(3)} v_{2j}^{(3)}\right) = 2$ and $\gamma\left(v_1^{(3)} v_{2n+1}^{(3)}\right) = 2$;

(10) for $j = \left\lfloor \frac{n+1}{2} \right\rfloor + 1, \dots, n$, let $\gamma\left(v_{2j}^{(3)} v_{2j+1}^{(3)}\right) = 3$.

It is not difficult to see that γ is an interval 6-coloring of $C(3, 2n+1)$ for which $S(v_j^{(3)}, \gamma) = [1, 3]$ when $1 \leq j \leq 2n+1$.

Next we define an edge-coloring ϕ of $C(m, 2n+1)$ as follows: first we color the edges of the subgraph $C(3, 2n+1)$ of $C(m, 2n+1)$ according to γ. Secondly, we color the edges of the remaining subgraph $C(m-3, 2n+1)$ of $C(m, 2n+1)$ according to β, and finally, we color the edges $v_j^{(3)} v_j^{(4)} \in E_j$ with color 4 for $1 \leq j \leq 2n+1$. It is easy to see that ϕ is an interval 6-coloring of $C(m, 2n+1)$. This shows that $C(m, 2n+1) \in \mathfrak{N}$ and $w(C(m, 2n+1)) \leq 6$.

Now we prove that $w(C(m, 2n+1)) \geq 6$ for odd m. Let ψ be an interval $w(C(m, 2n+1))$-coloring of $C(m, 2n+1)$ and $w(C(m, 2n+1)) \leq 5$. Consider the set $S\left(v_j^{(i)}, \psi\right)$ for $1 \leq i \leq m, 1 \leq j \leq 2n+1$. It is easy to see that if $d\left(v_j^{(i)}\right) = 3$, then $1 \leq \min S\left(v_j^{(i)}, \psi\right) \leq 3$, and if $d\left(v_j^{(i)}\right) = 4$, then $1 \leq \min S\left(v_j^{(i)}, \psi\right) \leq 2$. Hence, $3 \in S\left(v_j^{(i)}, \psi\right)$ for $1 \leq i \leq m, 1 \leq j \leq 2n+1$, but this implies that the edges with color 3 form a perfect matching in $C(m, 2n+1)$, which contradicts the fact that $C(m, 2n+1)$ does not have one. Thus $w(C(m, 2n+1)) = 6$ for odd m. ■

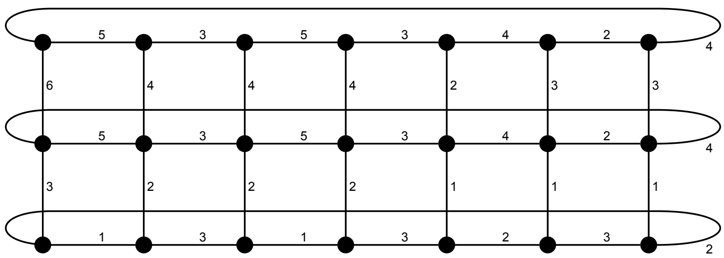

Figure 2. Interval 6-coloring of the graph $C(3, 7)$.

Figure 2 shows the interval 6-coloring γ of the graph $C(3, 7)$ described in the proof of Theorem 21.

Before we derive lower bounds for $W(C(2m, 2n))$ and $W(C(2m, 2n+1))$, let us note that Lemma 8, Theorems 15 and 21 imply the following:

Corollary 22. *If $G\square H$ is planar and both factors have at least* 3 *vertices, then $G\square H \in \mathfrak{N}$ and $w(G\square H) \leq 6$.*

Theorem 23. *If $m \in \mathbb{N}, n \geq 2$, then $W(C(2m, 2n)) \geq 4m + 2n - 2$, and if $m, n \in \mathbb{N}$, then $W(C(2m, 2n+1)) \geq 4m + 2n - 1$.*

Proof. For the proof of the theorem, it suffices to construct edge-colorings that satisfies the specified conditions. First we construct an interval $(4m+2n-2)$-coloring of $C(2m, 2n)$ when $m \in \mathbb{N}, n \geq 2$.

Define an edge-coloring α of $C(2m, 2n)$ as follows:

(1) for $i = 1, \ldots, m$, $j = 1, \ldots, n$, let

$$\alpha\left(v_j^{(2i-1)} v_{j+1}^{(2i-1)}\right) = \alpha\left(v_j^{(2i)} v_{j+1}^{(2i)}\right) = 4i + 2j - 4;$$

(2) for $i = 1, \ldots, m$, $j = n+1, \ldots, 2n-1$, let

$$\alpha\left(v_j^{(2i-1)} v_{j+1}^{(2i-1)}\right) = \alpha\left(v_j^{(2i)} v_{j+1}^{(2i)}\right) = 4i - 2j + 4n - 1;$$

(3) for $i = 1, \ldots, m$, let

$$\alpha\left(v_1^{(2i-1)} v_{2n}^{(2i-1)}\right) = \alpha\left(v_1^{(2i)} v_{2n}^{(2i)}\right) = 4i - 1;$$

(4) for $i = 1, \ldots, m$, $j = 1, \ldots, n$, let

$$\alpha\left(v_j^{(2i-1)} v_j^{(2i)}\right) = 4i + 2j - 5;$$

(5) for $i = 1, \ldots, m$, $j = n+1, \ldots, 2n$, let

$$\alpha\left(v_j^{(2i-1)} v_j^{(2i)}\right) = 4i - 2j + 4n;$$

(6) for $i = 1, \ldots, m-1$, $j = 2, \ldots, n+1$, let

$$\alpha\left(v_j^{(2i)} v_j^{(2i+1)}\right) = 4i + 2j - 3;$$

(7) for $i = 1, \ldots, m-1$, $j = n+2, \ldots, 2n$, let

$$\alpha\left(v_j^{(2i)} v_j^{(2i+1)}\right) = 4i - 2j + 4n + 2;$$

(8) for $i = 1, \ldots, m-1$, let

$$\alpha\left(v_1^{(2i)} v_1^{(2i+1)}\right) = 4i.$$

Next we construct an interval $(4m + 2n - 1)$-coloring of $C(2m, 2n+1)$ when $m, n \in \mathbb{N}$. Define an edge-coloring β of $C(2m, 2n+1)$ as follows:

(1) for $i = 1, \ldots, m$, $j = 1, \ldots, n+1$, let

$$\beta\left(u_j^{(2i-1)} u_{j+1}^{(2i-1)}\right) = \beta\left(u_j^{(2i)} u_{j+1}^{(2i)}\right) = 4i + 2j - 4;$$

(2) for $i = 1, \ldots, m$, $j = n+2, \ldots, 2n$, let

$$\beta\left(u_j^{(2i-1)}u_{j+1}^{(2i-1)}\right)=\beta\left(u_j^{(2i)}u_{j+1}^{(2i)}\right)=4i-2j+4n+1;$$

(3) for $i=1,\ldots,m$, let

$$\beta\left(u_1^{(2i-1)}u_{2n+1}^{(2i-1)}\right)=\beta\left(u_1^{(2i)}u_{2n+1}^{(2i)}\right)=4i-1;$$

(4) for $i=1,\ldots,m$, $j=1,\ldots,n+2$, let

$$\beta\left(u_j^{(2i-1)}u_j^{(2i)}\right)=4i+2j-5;$$

(5) for $i=1,\ldots,m$, $j=n+3,\ldots,2n+1$, let

$$\beta\left(u_j^{(2i-1)}u_j^{(2i)}\right)=4i-2j+4n+2;$$

(6) for $i=1,\ldots,m-1$, $j=2,\ldots,n+1$, let

$$\beta\left(u_j^{(2i)}u_j^{(2i+1)}\right)=4i+2j-3;$$

(7) for $i=1,\ldots,m-1$, $j=n+2,\ldots,2n+1$, let

$$\beta\left(u_j^{(2i)}u_j^{(2i+1)}\right)=4i-2j+4n+4;$$

(8) for $i=1,\ldots,m-1$, let

$$\beta\left(u_1^{(2i)}u_1^{(2i+1)}\right)=4i.$$

It is straightforward to check that α is an interval $(4m+2n-2)$-coloring of $C(2m,2n)$ when $m\in\mathbb{N},n\geq 2$, and β is an interval $(4m+2n-1)$-coloring of $C(2m,2n+1)$ when $m,n\in\mathbb{N}$. ■

Note that the lower bound in Theorem 23 is not so far from the upper bound for $W\left(C(m,n)\right)$. Indeed, since $C(2m,2n)$ is bipartite, $3\leq\Delta\left(C(2m,2n)\right)\leq 4$ and $\mathrm{diam}\left(C(2m,2n)\right)=2m+n-1$, by Theorem 3, we have $W\left(C(2m,2n)\right)\leq 3(2m+n-1)+1$. Similarly, since $3\leq\Delta\left(C(2m,2n+1)\right)\leq 4$ and $\mathrm{diam}\left(C(2m,2n+1)\right)=2m+n-1$, by Theorem 2, we have $W\left(C(2m,2n+1)\right)\leq 3(2m+n)+1$. Next we would like to compare obtained lower bounds for $W(C(m,n))$. If m is even and $m<n$, then the lower bound in Theorem 23 is better than in Theorem 16, if m is even and $m>n$, then the lower bound in Theorem 16 is better than in Theorem 23, and if m is even and $m=n$, then we obtain the same lower bound in both theorems.

In the following we consider tori. In [22], the first author proved that the torus $T(m,n)\in\mathfrak{N}$ if and only if mn is even. Since $T(m,n)$ is 4-regular, by Theorem 1, we obtain that $w(T(m,n))=4$ when mn is even. Now we derive a new lower bound for $W(T(m,n))$ when mn is even.

Theorem 24. *For any $m, n \geq 2$, we have $W(T(2m,2n)) \geq \max\{3m+n+2, 3n+m+2\}$, and for any $m \geq 2$, $n \in \mathbb{N}$, we have*

$$W\left(T(2m,2n+1)\right) \geq \begin{cases} 2m+2n+2, & \textit{if } m \textit{ is odd,} \\ 2m+2n+3, & \textit{if } m \textit{ is even.} \end{cases}$$

Proof. First note that the lower bound for $W(T(2m,2n))$ $(m,n \geq 2)$ follows from Theorem 13. For the proof of a second part of the theorem, it suffices to construct an edge-coloring of $T(2m,2n+1)$ that satisfies the specified conditions.

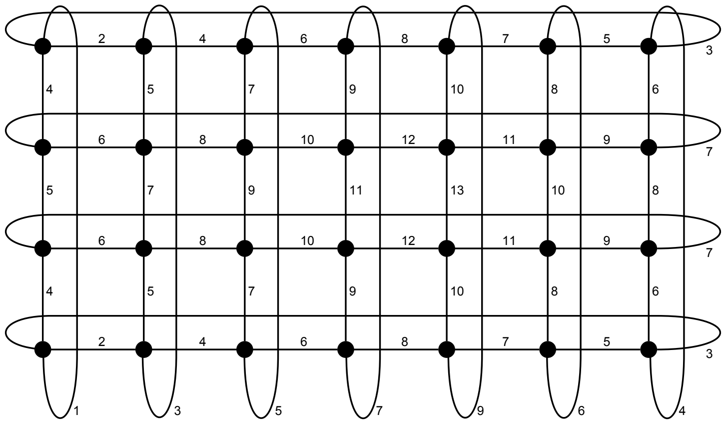

Figure 3. Interval 13-coloring of the graph $T(4,7)$.

Define an edge-coloring α of $T(2m,2n+1)$ as follows:

(1) for $j = 1, \ldots, n+1$, let

$$\alpha\left(v_j^{(1)} v_{j+1}^{(1)}\right) = \alpha\left(v_j^{(2m)} v_{j+1}^{(2m)}\right) = 2j;$$

(2) for $j = n+2, \ldots, 2n$, let

$$\alpha\left(v_j^{(1)} v_{j+1}^{(1)}\right) = \alpha\left(v_j^{(2m)} v_{j+1}^{(2m)}\right) = 2(2n+1-j)+3$$

and

$$\alpha\left(v_1^{(1)} v_{2n+1}^{(1)}\right) = \alpha\left(v_1^{(2m)} v_{2n+1}^{(2m)}\right) = 3;$$

(3) for $j = 1, \ldots, n+2$, let

$$\alpha\left(v_j^{(1)} v_j^{(2m)}\right) = 2j-1;$$

(4) for $j = n+3, \ldots, 2n+1$, let

$$\alpha\left(v_j^{(1)}v_j^{(2m)}\right)=2(2n+3-j);$$

(5) for $i=1,\dots,\left\lfloor\frac{m}{2}\right\rfloor$, $j=1,\dots,n+1$, let

$$\alpha\left(v_j^{(2i)}v_{j+1}^{(2i)}\right)=\alpha\left(v_j^{(2i+1)}v_{j+1}^{(2i+1)}\right)$$

$$=\alpha\left(v_j^{(2m-2i)}v_{j+1}^{(2m-2i)}\right)=\alpha\left(v_j^{(2m-2i+1)}v_{j+1}^{(2m-2i+1)}\right)=4i+2j;$$

(6) for $i=1,\dots,\left\lfloor\frac{m}{2}\right\rfloor$, $j=n+2,\dots,2n$, let

$$\alpha\left(v_j^{(2i)}v_{j+1}^{(2i)}\right)=\alpha\left(v_j^{(2i+1)}v_{j+1}^{(2i+1)}\right)=\alpha\left(v_j^{(2m-2i)}v_{j+1}^{(2m-2i)}\right)$$

$$=\alpha\left(v_j^{(2m-2i+1)}v_{j+1}^{(2m-2i+1)}\right)=4i+2(2n+1-j)+3$$

and

$$\alpha\left(v_1^{(2i)}v_{2n+1}^{(2i)}\right)=\alpha\left(v_1^{(2i+1)}v_{2n+1}^{(2i+1)}\right)$$

$$=\alpha\left(v_1^{(2m-2i)}v_{2n+1}^{(2m-2i)}\right)=\alpha\left(v_1^{(2m-2i+1)}v_{2n+1}^{(2m-2i+1)}\right)=4i+3;$$

(7) for $i=1,\dots,\left\lceil\frac{m}{2}\right\rceil$, $j=2,\dots,n+1$, let

$$\alpha\left(v_j^{(2i-1)}v_j^{(2i)}\right)=\alpha\left(v_j^{(2m-2i+1)}v_j^{(2m-2i+2)}\right)=4i+2j-3;$$

(8) for $i=1,\dots,\left\lceil\frac{m}{2}\right\rceil$, $j=n+2,\dots,2n+1$, let

$$\alpha\left(v_j^{(2i-1)}v_j^{(2i)}\right)=\alpha\left(v_j^{(2m-2i+1)}v_j^{(2m-2i+2)}\right)=4(n+1+i)-2j;$$

(9) for $i=1,\dots,\left\lceil\frac{m}{2}\right\rceil$, let

$$\alpha\left(v_1^{(2i-1)}v_1^{(2i)}\right)=\alpha\left(v_1^{(2m-2i+1)}v_1^{(2m-2i+2)}\right)=4i;$$

(10) for $i=1,\dots,\left\lfloor\frac{m}{2}\right\rfloor$, $j=1,\dots,n+2$, let

$$\alpha\left(v_j^{(2i)}v_j^{(2i+1)}\right)=\alpha\left(v_j^{(2m-2i)}v_j^{(2m-2i+1)}\right)=4i+2j-1;$$

(11) for $i=1,\dots,\left\lfloor\frac{m}{2}\right\rfloor$, $j=n+3,\dots,2n+1$, let

$$\alpha\left(v_j^{(2i)}v_j^{(2i+1)}\right)=\alpha\left(v_j^{(2m-2i)}v_j^{(2m-2i+1)}\right)=4i+2(2n+3-j).$$

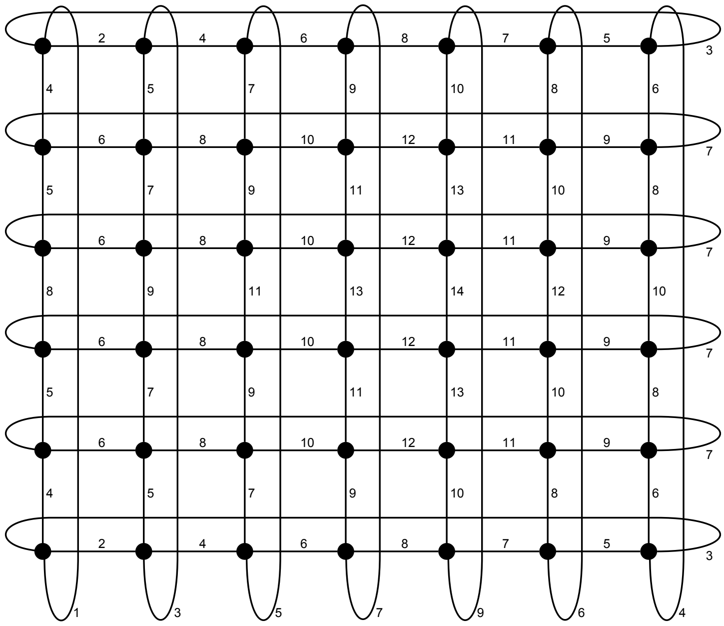

Figure 4. Interval 14-coloring of the graph $T(6,7)$.

Let us show that the edges incident to any vertex of $T(2m,2n+1)$ are colored by four consecutive colors. For example, let $2 \leq i \leq \lfloor \frac{m}{2} \rfloor$ and $2 \leq j \leq n+1$. By the points (5), (7) and (10) of the definition of α, for $2 \leq i \leq \lfloor \frac{m}{2} \rfloor$, $2 \leq j \leq n+1$, we have

$$\begin{aligned} S\left(v_j^{(2i)},\alpha\right) &= S\left(v_j^{(2m-2i)},\alpha\right) = \{4i+2j-2, 4i+2j\} \\ &\cup \{4i+2j-3\} \cup \{4i+2j-1\} = [4i+2j-3, 4i+2j]. \end{aligned}$$

Similarly, it can be verified that the edges incident to other vertices of $T(2m,2n+1)$ are also colored by four consecutive colors. It is easy to see that $\alpha\left(v_1^{(1)}v_1^{(2m)}\right) = 1$. Now if m is odd, then $\alpha\left(v_{n+2}^{(m)}v_{n+2}^{(m+1)}\right) = 2m+2n+2$ and, by Lemma 7, α is an interval $(2m+2n+2)$-coloring of $T(2m,2n+1)$ when m is odd. If m is even, then $\alpha\left(v_{n+2}^{(m)}v_{n+2}^{(m+1)}\right) = 2m+2n+3$ and, by Lemma 7, α is an interval $(2m+2n+3)$-coloring of $T(2m,2n+1)$ when m is even. ■

Figure 3 and 4 show the interval colorings of the graphs $T(4,7)$ and $T(6,7)$ described in the proof of Theorem 24.

From Theorems 1, 15 and 24, we have:

Corollary 25. *If* $G = T(2m,2n)$ $(m,n \geq 2)$ *and* $4 \leq t \leq \max\{3m+n+2, 3n+m+2\}$, *then* G *has an interval* t*-coloring. Also, if* $H = T(2m,2n+1)$ $(m \geq$

$2, n \in \mathbb{N}$), *m is odd and* $4 \le t \le 2m+2n+2$, *then* H *has an interval* t*-coloring, and if* $H = T(2m, 2n+1)$ ($m \ge 2, n \in \mathbb{N}$), *m is even and* $4 \le t \le 2m+2n+3$, *then* H *has an interval* t*-coloring.*

Let us note that the lower bound in Theorem 24 is not so far from the upper bound for $W(T(m,n))$. Indeed, since $T(2m,2n)$ is bipartite, $\Delta(T(2m,2n)) = 4$ and $\operatorname{diam}(C(2m,2n)) = m+n$, by Theorem 3, we have $W(T(2m,2n)) \le 3(m+n)+1$. Similarly, since $\Delta(T(2m,2n+1)) = 4$ and $\operatorname{diam}(T(2m,2n+1)) = m+n$, by Theorem 2, we have $W(T(2m,2n+1)) \le 3(m+n+1)+1$.

5. n-dimensional Cubes

It is well-known that the n-dimensional cube Q_n is the Cartesian product of n copies of K_2. In [21], the first author investigated interval colorings of n-dimensional cubes and proved that $w(Q_n) = n$ and $W(Q_n) \ge \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$. In the same paper he also conjectured that $W(Q_n) = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$. Here, we prove this conjecture.

Let $e, e' \in E(Q_n)$ and $e = u_1u_2$, $e' = v_1v_2$. The distance between two edges e and e' in Q_n, we define as follows:

$$d(e,e') = \min_{1 \le i \le 2, 1 \le j \le 2} \{d(u_i, v_j)\}.$$

Let α be an interval t-coloring of Q_n. Define an edge span $\mathrm{sp}_\alpha(e,e')$ of edges e and e' ($e, e' \in E(Q_n)$) in coloring α as follows:

$$\mathrm{sp}_\alpha(e,e') = |\alpha(e) - \alpha(e')|.$$

For any $k, 0 \le k \le n-1$, define an edge span at distance k $\mathrm{sp}_{\alpha,k}$ in coloring α as follows:

$$\mathrm{sp}_{\alpha,k} = \max\{\mathrm{sp}_\alpha(e,e') : e, e' \in E(Q_n) \text{ and } d(e,e') = k\}.$$

Clearly, $\mathrm{sp}_{\alpha,0} = n-1$.

Theorem 26. *If* $n \in \mathbb{N}$, *then* $W(Q_n) \le \frac{n(n+1)}{2}$.

Proof. Let α be an interval $W(Q_n)$-coloring of Q_n. First we show that if $1 \le k \le n-1$, then $\mathrm{sp}_{\alpha,k} \le \mathrm{sp}_{\alpha,k-1} + n - k$.

Let $e, e' \in E(Q_n)$ be any two edges of Q_n with $d(e,e') = k$. Without loss of generality, we may assume that $\alpha(e) \ge \alpha(e')$. Since $d(e,e') = k$, there exist u and v vertices such that $u \in e$ and $v \in e'$ and $d(u,v) = k$. There are $v_1, v_2, \dots, v_k$ ($v_i \ne v_j$ when $i \ne j$) vertices such that $d(u, v_i) = k-1$ and $vv_i \in E(Q_n)$ for $i = 1, \dots, k$. Since Q_n is n-regular, we have

$$(*) \qquad \min_{1 \le i \le k} \{\alpha(v_iv)\} \le \alpha(e') + n - k.$$

Let $\alpha(e'') = \min_{1 \le i \le k}\{\alpha(v_iv)\}$. By $(*)$, we obtain

$$\alpha(e') \geq \alpha(e'') - (n-k) \text{ and } d(e, e'') = k - 1.$$

Thus,

$$\begin{aligned} \mathrm{sp}_\alpha(e, e') &= |\alpha(e) - \alpha(e')| \leq |\alpha(e) - \alpha(e'') + n - k| \leq |\alpha(e) - \alpha(e'')| + n - k \\ &\leq \mathrm{sp}_{\alpha,k-1} + n - k. \end{aligned}$$

Since e and e' were arbitrary edges with $d(e, e') = k$, we obtain $\mathrm{sp}_{\alpha,k} \leq \mathrm{sp}_{\alpha,k-1} + n - k$. Now by induction on k with $\mathrm{sp}_{\alpha,0} = n - 1$, we obtain $\mathrm{sp}_{\alpha,n-1} \leq \frac{n(n+1)}{2} - 1$. From this and taking into account that $d(e, e') \leq n - 1$ for all $e, e' \in E(Q_n)$, we get $W(Q_n) \leq \frac{n(n+1)}{2}$. ■

By Theorems 6 and 26, we obtain $W(Q_n) = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$. Moreover, by Theorem 1, we have that Q_n has an interval t-coloring if and only if $n \leq t \leq \frac{n(n+1)}{2}$.

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References

[1] A.S. Asratian and R.R. Kamalian, *Interval colorings of edges of a multigraph*, Appl. Math. **5** (1987) 25–34 (in Russian).

[2] A.S. Asratian and R.R. Kamalian, *Investigation on interval edge-colorings of graphs*, J. Combin. Theory (B) **62** (1994) 34–43. doi:10.1006/jctb.1994.1053

[3] A.S. Asratian, T.M.J. Denley and R. Häggkvist, Bipartite Graphs and their Applications (Cambridge University Press, Cambridge, 1998). doi:10.1017/CBO9780511984068

[4] M.A. Axenovich, *On interval colorings of planar graphs*, Congr. Numer. **159** (2002) 77–94.

[5] M. Behzad and E.S. Mahmoodian, *On topological invariants of the product of graphs*, Canad. Math. Bull. **12** (1969) 157–166. doi:10.4153/CMB-1969-015-9

[6] Y. Feng and Q. Huang, *Consecutive edge-coloring of the generalized θ-graph*, Discrete Appl. Math. **155** (2007) 2321–2327. doi:10.1016/j.dam.2007.06.010

[7] K. Giaro and M. Kubale, *Consecutive edge-colorings of complete and incomplete Cartesian products of graphs*, Congr. Numer. **128** (1997) 143–149.

[8] K. Giaro, M. Kubale and M. Małafiejski, *Consecutive colorings of the edges of general graphs*, Discrete Math. **236** (2001) 131–143. doi:10.1016/S0012-365X(00)00437-4

[9] K. Giaro and M. Kubale, *Compact scheduling of zero-one time operations in multi-stage systems*, Discrete Appl. Math. **145** (2004) 95–103. doi:10.1016/j.dam.2003.09.010

[10] D. Hanson, C.O.M. Loten and B. Toft, *On interval colorings of bi-regular bipartite graphs*, Ars Combin. **50** (1998) 23–32.

[11] R. Hammack, W. Imrich, S. Klavžar, Handbook of Product Graphs, Second Edition (CRC Press, 2011).

[12] T.R. Jensen, B. Toft, Graph Coloring Problems (Wiley Interscience Series in Discrete Mathematics and Optimization, 1995).

[13] R.R. Kamalian, *Interval colorings of complete bipartite graphs and trees, preprint*, Comp. Cen. Acad. Sci. Armenian SSR, Erevan (1989) (in Russian).

[14] R.R. Kamalian, Interval edge colorings of graphs (Doctoral Thesis, Novosibirsk, 1990).

[15] R.R. Kamalian and A.N. Mirumian, *Interval edge colorings of bipartite graphs of some class*, Dokl. NAN RA **97** (1997) 3–5 (in Russian).

[16] R.R. Kamalian and P.A. Petrosyan, *A note on upper bounds for the maximum span in interval edge-colorings of graphs*, Discrete Math. **312** (2012) 1393–1399.

[17] R.R. Kamalian and P.A. Petrosyan, *A note on interval edge-colorings of graphs*, Mathematical Problems of Computer Science **36** (2012) 13–16.

[18] A. Khchoyan, *Interval edge-colorings of subcubic graphs and multigraphs*, Yerevan State University, BS Thesis (2010) 30p.

[19] M. Kubale, Graph Colorings (American Mathematical Society, 2004).

[20] P.A. Petrosyan and G.H. Karapetyan, *Lower bounds for the greatest possible number of colors in interval edge colorings of bipartite cylinders and bipartite tori*, in: Proceedings of the CSIT Conference (2007) 86–88.

[21] P.A. Petrosyan, *Interval edge-colorings of complete graphs and n-dimensional cubes*, Discrete Math. **310** (2010) 1580–1587. doi:10.1016/j.disc.2010.02.001

[22] P.A. Petrosyan, *Interval edge colorings of some products of graphs*, Discuss. Math. Graph Theory **31** (2011) 357–373. doi:10.7151/dmgt.1551

[23] P.A. Petrosyan, H.H. Khachatrian, L.E. Yepremyan and H.G. Tananyan, *Interval edge-colorings of graph products*, in: Proceedings of the CSIT Conference (2011) 89–92.

[24] S.V. Sevast'janov, *Interval colorability of the edges of a bipartite graph*, Metody Diskret. Analiza **50** (1990) 61–72 (in Russian).

[25] D.B. West, Introduction to Graph Theory (Prentice-Hall, New Jersey, 2001).

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