

INTERVAL EDGE-COLORINGS OF CARTESIAN PRODUCTS OF GRAPHS I

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Abstract

A proper edge-coloring of a graph G with colors $1, \dots, t$ is an interval t -coloring if all colors are used and the colors of edges incident to each vertex of G form an interval of integers. A graph G is interval colorable if it has an interval t -coloring for some positive integer t . Let \mathfrak{N} be the set of all interval colorable graphs. For a graph $G \in \mathfrak{N}$, the least and the greatest values of t for which G has an interval t -coloring are denoted by $w(G)$ and $W(G)$, respectively. In this paper we first show that if G is an r -regular graph and $G \in \mathfrak{N}$, then $W(G \square P_m) \geq W(G) + W(P_m) + (m - 1)r$ ($m \in \mathbb{N}$) and $W(G \square C_{2n}) \geq W(G) + W(C_{2n}) + nr$ ($n \geq 2$). Next, we investigate interval edge-colorings of grids, cylinders and tori. In particular, we prove that if $G \square H$ is planar and both factors have at least 3 vertices, then $G \square H \in \mathfrak{N}$ and $w(G \square H) \leq 6$. Finally, we confirm the first author's conjecture on the n -dimensional cube Q_n and show that Q_n has an interval t -coloring if and only if $n \leq t \leq \frac{n(n+1)}{2}$.

Keywords: edge-coloring, interval coloring, grid, cylinder, torus, n -dimensional cube.

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1. INTRODUCTION

A proper edge-coloring of a graph G with colors $1, \dots, t$ is an interval t -coloring if all colors are used and the colors of edges incident to each vertex of G form an interval of integers. A graph G is interval colorable if it has an interval t -coloring for some positive integer t . Let \mathfrak{N} be the set of all interval colorable graphs [1, 14]. For a graph $G \in \mathfrak{N}$, the least and the greatest values of t for which G has an interval t -coloring are denoted by $w(G)$ and $W(G)$, respectively. The concept of interval edge-coloring was introduced by Asratian and Kamalian [1]. In [1], they proved the following:

Theorem 1. *Let G be a regular graph. Then*

- (1) $G \in \mathfrak{N}$ if and only if $\chi'(G) = \Delta(G)$.
- (2) If $G \in \mathfrak{N}$ and $w(G) \leq t \leq W(G)$, then G has an interval t -coloring.

In [2], Asratian and Kamalian investigated interval edge-colorings of connected graphs. In particular, they obtained the following two results.

Theorem 2. *If G is a connected graph and $G \in \mathfrak{N}$, then*

$$W(G) \leq (\text{diam}(G) + 1)(\Delta(G) - 1) + 1.$$

Theorem 3. *If G is a connected bipartite graph and $G \in \mathfrak{N}$, then*

$$W(G) \leq \text{diam}(G)(\Delta(G) - 1) + 1.$$

Recently, Kamalian and the first author [16] showed that these upper bounds cannot be significantly improved.

In [13], Kamalian investigated interval colorings of complete bipartite graphs and trees. In particular, he proved the following:

Theorem 4. *For any $r, s \in \mathbb{N}$, the complete bipartite graph $K_{r,s}$ is interval colorable, and*

- (1) $w(K_{r,s}) = r + s - \gcd(r, s)$,
- (2) $W(K_{r,s}) = r + s - 1$,
- (3) if $w(K_{r,s}) \leq t \leq W(K_{r,s})$, then $K_{r,s}$ has an interval t -coloring.

In [21], the first author investigated interval colorings of complete graphs and n -dimensional cubes. In particular, he obtained the following two results.

Theorem 5. *If $n = p2^q$, where p is odd and q is nonnegative, then*

$$W(K_{2n}) \geq 4n - 2 - p - q.$$

Theorem 6. *If $n \in \mathbb{N}$, then $W(Q_n) \geq \frac{n(n+1)}{2}$.*

The NP -completeness of the problem of the existence of an interval edge-coloring of an arbitrary bipartite graph was shown in [24]. A similar result for regular graphs was obtained in [1, 2]. In [19, 22, 23], interval edge-colorings of various products of graphs were investigated. Some interesting results on interval colorings were also obtained in [3, 4, 6, 7, 8, 9, 10, 14, 15, 16, 17, 18, 19, 20]. Surveys on this topic can be found in some books [3, 12, 19].

In this paper we focus only on interval edge-colorings of Cartesian products of graphs.

2. NOTATIONS, DEFINITIONS AND AUXILIARY RESULTS

Throughout this paper all graphs are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of graph G , respectively. The degree of a vertex v in G is denoted by $d_G(v)$, the maximum degree of G by $\Delta(G)$, and the chromatic index of G by $\chi'(G)$. If G is a connected graph, then the distance between two vertices u and v in G , we denote by $d(u, v)$, and the diameter of G by $\text{diam}(G)$. We use the standard notations P_n , C_n , K_n and Q_n for the path, cycle, complete graph on n vertices and the n -dimensional cube, respectively. A partial edge-coloring of a graph G is a coloring of some edges of G such that no two adjacent edges receive the same color. If α is a partial edge-coloring of G and $v \in V(G)$, then $S(v, \alpha)$ denotes the set of colors appearing on colored edges incident to v . Clearly, if α is a proper edge-coloring of a graph G , then $|S(v, \alpha)| = d_G(v)$ for every $v \in V(G)$.

Let $[t]$ denote the set of the first t natural numbers. Let $\lfloor a \rfloor$ ($\lceil a \rceil$) denote the largest (least) integer less (greater) than or equal to a . For two positive integers a and b with $a \leq b$, the set $\{a, \dots, b\}$ is denoted by $[a, b]$. The terms and concepts that we do not define can be found in [25].

Let G and H be graphs. The Cartesian product $G \square H$ is defined as follows: $V(G \square H) = V(G) \times V(H)$, $E(G \square H) = \{(u_1, v_1)(u_2, v_2) : u_1 = u_2 \text{ and } v_1 v_2 \in E(H) \text{ or } v_1 = v_2 \text{ and } u_1 u_2 \in E(G)\}$.

Let $V(G) = \{u_1, \dots, u_n\}$ and $V(H) = \{w_1, \dots, w_m\}$. We use the following notation for vertex and edge sets of the Cartesian product $G \square H$: $V(G \square H) = \bigcup_{i=1}^m V^i$, where $V^i = \{v_j^{(i)} : 1 \leq j \leq n\}$ and $E(G \square H) = \bigcup_{i=1}^m E^i \cup \bigcup_{j=1}^n E_j$, where $E^i = \{v_j^{(i)} v_k^{(i)} : u_j u_k \in E(G)\}$ and $E_j = \{v_j^{(i)} v_j^{(k)} : w_i w_k \in E(H)\}$.

We define subgraphs G^i of G as follows: $G^i = (V^i, E^i)$. Clearly, G^i is isomorphic to G for $1 \leq i \leq m$.

Clearly, if G and H are connected graphs, then $G \square H$ is connected, too. Moreover, $\Delta(G \square H) = \Delta(G) + \Delta(H)$ and $\text{diam}(G \square H) = \text{diam}(G) + \text{diam}(H)$.

The k -dimensional grid $G(n_1, \dots, n_k)$, $n_i \in \mathbb{N}$, is the Cartesian product of paths $P_{n_1} \square P_{n_2} \square \dots \square P_{n_k}$. The cylinder $C(n_1, n_2)$ is the Cartesian product $P_{n_1} \square C_{n_2}$, and the torus $T(n_1, n_2)$ is the Cartesian product of cycles $C_{n_1} \square C_{n_2}$.

We also need the following two lemmas.

Lemma 7. *If α is an edge-coloring of a connected graph G with colors $1, \dots, t$ such that the edges incident to each vertex $v \in V(G)$ are colored by distinct and consecutive colors, and $\min_{e \in E(G)} \{\alpha(e)\} = 1$, $\max_{e \in E(G)} \{\alpha(e)\} = t$, then α is an interval t -coloring of G .*

Proof. For the proof of the lemma, it suffices to show that all colors are used in the coloring α of G .

Let u and w be vertices such that $1 \in S(u, \alpha)$ and $t \in S(w, \alpha)$. Also, let $P = v_1, \dots, v_k$, where $u = v_1$ and $v_k = w$ be a u, w -path in G . If $k = 1$, then $t \in S(u, \alpha)$ and all colors appear on edges incident to u . Assume that $k \geq 2$. The sets $S(v_i, \alpha)$ for $v_i \in V(P)$ are intervals, and for $2 \leq i \leq k$, intervals $S(v_{i-1}, \alpha)$ and $S(v_i, \alpha)$ share a color. Thus, the sets $S(v_1, \alpha), \dots, S(v_k, \alpha)$ cover $[1, t]$. ■

The next lemma was proved by Behzad and Mahmoodian in [5].

Lemma 8. *If both G and H have at least 3 vertices, then the Cartesian product $G \square H$ is planar if and only if $G \square H = G(m, n)$ or $G \square H = C(m, n)$.*

3. THE CARTESIAN PRODUCT OF REGULAR GRAPHS

Interval edge-colorings of Cartesian products of graphs were first investigated by Giaro and Kubale in [7], where they proved the following:

Theorem 9. *If $G \in \mathfrak{N}$, then $G \square P_m \in \mathfrak{N}$ ($m \in \mathbb{N}$) and $G \square C_{2n} \in \mathfrak{N}$ ($n \geq 2$).*

It is well-known that $P_m, C_{2n} \in \mathfrak{N}$ and $W(P_m) = m - 1$, $W(C_{2n}) = n + 1$ for $m \in \mathbb{N}$ and $n \geq 2$. Later, Giaro and Kubale [9, 19] proved a more general result.

Theorem 10. *If $G, H \in \mathfrak{N}$, then $G \square H \in \mathfrak{N}$. Moreover, $w(G \square H) \leq w(G) + w(H)$ and $W(G \square H) \geq W(G) + W(H)$.*

Let us note that if $G \in \mathfrak{N}$ and $H = P_m$ or $H = C_{2n}$, then, by Theorem 10, we obtain $w(G \square H) \leq w(G) + 2$ and $W(G \square P_m) \geq W(G) + m - 1$, $W(G \square C_{2n}) \geq W(G) + n + 1$. Now we improve the lower bound in Theorem 10 for $W(G \square P_m)$ and $W(G \square C_{2n})$ when G is a regular graph and $G \in \mathfrak{N}$. More precisely, we show that the following two theorems hold.

Theorem 11. *If G is an r -regular graph and $G \in \mathfrak{N}$, then $G \square P_m \in \mathfrak{N}$ ($m \in \mathbb{N}$) and $W(G \square P_m) \geq W(G) + W(P_m) + (m-1)r$.*

Proof. For the proof, we construct an edge-coloring of the graph $G \square P_m$ that satisfies the specified conditions.

Since $G \in \mathfrak{N}$, there exists an interval $W(G)$ -coloring α of G . Now we define an edge-coloring β of the subgraphs G^1, \dots, G^m . For $1 \leq i \leq m$ and for every edge $v_j^{(i)} v_k^{(i)} \in E(G^i)$, let

$$\beta(v_j^{(i)} v_k^{(i)}) = \alpha(v_j v_k) + (i-1)(r+1).$$

It is easy to see that the color of each edge of the subgraph G^i is obtained by shifting the color of the associated edge of G by $(i-1)(r+1)$. Thus the set $S(v_j^{(i)}, \beta)$ is an interval for each vertex $v_j^{(i)} \in V(G^i)$, where $1 \leq i \leq m$, $1 \leq j \leq n$. Now we define an edge-coloring γ of the graph $G \square P_m$. For every $e \in E(G \square P_m)$, let

$$\gamma(e) = \begin{cases} \beta(e), & \text{if } e \in E(G^i), \\ \max S(v_j^{(i)}, \beta) + 1, & \text{if } e = v_j^{(i)} v_j^{(i+1)} \in E_j, \end{cases}$$

where $1 \leq i \leq m, 1 \leq j \leq n$.

Let us prove that γ is an interval $(W(G) + W(P_m) + (m-1)r)$ -coloring of the graph $G \square P_m$ for $m \in \mathbb{N}$.

First we prove that the set $S(v_j^{(i)}, \gamma)$ is an interval for each vertex $v_j^{(i)} \in V(G \square P_m)$, where $1 \leq i \leq m, 1 \leq j \leq n$.

For each vertex $v_j^{(i)} \in V(G \square P_m)$, the set $S(v_j^{(i)}, \gamma)$ can be represented as a union of three sets, $S(v_j^{(i)}, \gamma) = A_j^{(i)} \cup B_j^{(i)} \cup C_j^{(i)}$, where $A_j^{(i)}$ corresponds to the edges of i -th layer, $B_j^{(i)}$ corresponds to the edges from the vertices of lower layer and $C_j^{(i)}$ corresponds to the edges from the vertices of higher layer. More specifically, for $1 \leq i \leq m, 1 \leq j \leq n$, define sets $A_j^{(i)}, B_j^{(i)}$ and $C_j^{(i)}$ as follows:

$$\begin{aligned} A_j^{(i)} &= \left\{ \gamma(v_j^{(i)} u) : v_j^{(i)} u \in E^i \right\}, \\ B_j^{(i)} &= \begin{cases} \emptyset, & \text{if } i = 1, \\ \left\{ \gamma(v_j^{(i)} u) : v_j^{(i)} u \in E_j, u \in V^{i-1} \right\}, & \text{if } 2 \leq i \leq m, \end{cases} \\ C_j^{(i)} &= \begin{cases} \left\{ \gamma(v_j^{(i)} u) : v_j^{(i)} u \in E_j, u \in V^{i+1} \right\}, & \text{if } 1 \leq i \leq m-1, \\ \emptyset, & \text{if } i = m. \end{cases} \end{aligned}$$

By the definition of γ , we have that for $1 \leq i \leq m, 1 \leq j \leq n$,

$$A_j^{(i)} = [\min S(v_j, \alpha) + (i-1)(r+1), \max S(v_j, \alpha) + (i-1)(r+1)],$$

$$\text{for } 2 \leq i \leq m, 1 \leq j \leq n,$$

$$B_j^{(i)} = \{\max S(v_j, \alpha) + (i-2)(r+1) + 1\},$$

$$\text{and for } 1 \leq i \leq m-1, 1 \leq j \leq n,$$

$$C_j^{(i)} = \{\max S(v_j, \alpha) + (i-1)(r+1) + 1\}.$$

By this and taking into account that $\max S(v_j, \alpha) - \min S(v_j, \alpha) = r-1$ for $1 \leq j \leq n$, we have that $A_j^{(i)} \cup B_j^{(i)} \cup C_j^{(i)}$ is an interval for each vertex $v_j^{(i)} \in V(G^i)$, where $1 \leq i \leq m, 1 \leq j \leq n$.

Next we show that in the coloring γ all colors are used. Clearly, there exists an edge $v_{j_0}^{(1)} v_{k_0}^{(1)} \in E(G^1)$ such that $\gamma(v_{j_0}^{(1)} v_{k_0}^{(1)}) = 1$, since in the coloring α there exists an edge $v_{j_0} v_{k_0}$ with $\alpha(v_{j_0} v_{k_0}) = 1$ and $\gamma(v_{j_0}^{(1)} v_{k_0}^{(1)}) = \beta(v_{j_0}^{(1)} v_{k_0}^{(1)}) = \alpha(v_{j_0} v_{k_0})$. Similarly, there exists an edge $v_{j_1}^{(m)} v_{k_1}^{(m)} \in E(G^m)$ such that $\gamma(v_{j_1}^{(m)} v_{k_1}^{(m)}) = W(G) + (m-1)(r+1) = W(G) + W(P_m) + (m-1)r$, since in the coloring α there exists an edge $v_{j_1} v_{k_1}$ with $\alpha(v_{j_1} v_{k_1}) = W(G)$ and $\gamma(v_{j_1}^{(m)} v_{k_1}^{(m)}) = \beta(v_{j_1}^{(m)} v_{k_1}^{(m)}) = \alpha(v_{j_1} v_{k_1}) + (m-1)(r+1)$.

Now, by Lemma 7, we have that γ is an interval $(W(G) + W(P_m) + (m-1)r)$ -coloring of the graph $G \square P_m$ for $m \in \mathbb{N}$. ■

Corollary 12. *If G is an r -regular graph and $G \in \mathfrak{N}$, then $G \square Q_n \in \mathfrak{N}$ ($n \in \mathbb{N}$) and*

$$W(G \square Q_n) \geq W(G) + \frac{n(n+2r+1)}{2}.$$

Proof. By Theorem 11 and using associativity of the Cartesian product, we get

$$W(G \square Q_n) = W(\cdots ((G \square K_2) \square K_2) \square \cdots \square K_2) \geq W(G) + \frac{n(n+2r+1)}{2}. \quad \blacksquare$$

Theorem 13. *If G is an r -regular graph and $G \in \mathfrak{N}$, then $G \square C_{2n} \in \mathfrak{N}$ ($n \geq 2$) and $W(G \square C_{2n}) \geq W(G) + W(C_{2n}) + nr$.*

Proof. For the proof, we construct an edge-coloring of the graph $G \square C_{2n}$ that satisfies the specified conditions.

Since $G \in \mathfrak{N}$, there exists an interval $W(G)$ -coloring α of G . Now we define an edge-coloring β of the subgraphs G^1, \dots, G^{2n} .

For $1 \leq i \leq 2n$ and for every edge $v_j^{(i)} v_k^{(i)} \in E(G^i)$, let

$$\beta(v_j^{(i)}v_k^{(i)}) = \begin{cases} \alpha(v_jv_k), & \text{if } i = 1, \\ \alpha(v_jv_k) + (i-1)(r+1) + 1, & \text{if } 2 \leq i \leq n+1, \\ \alpha(v_jv_k) + (2n+1-i)(r+1), & \text{if } n+2 \leq i \leq 2n. \end{cases}$$

It is easy to see that the color of each edge of the subgraph G^i is obtained by shifting the color of the associated edge of G by $(i-1)(r+1)+1$ for $2 \leq i \leq n+1$, and by $(2n-i+1)(r+1)$ for $n+2 \leq i \leq 2n$, thus the set $S(v_j^{(i)}, \beta)$ is an interval for each vertex $v_j^{(i)} \in V(G^i)$, where $1 \leq i \leq 2n$, $1 \leq j \leq p$. Now we define an edge-coloring γ of the graph $G \square C_{2n}$.

For every $e \in E(G \square C_{2n})$, let

$$\gamma(e) = \begin{cases} \beta(e), & \text{if } e \in E(G^i), \\ \max S(v_j^{(1)}, \beta) + 1, & \text{if } e = v_j^{(1)}v_j^{(2n)} \in E_j, \\ \max S(v_j^{(1)}, \beta) + 2, & \text{if } e = v_j^{(1)}v_j^{(2)} \in E_j, \\ \max S(v_j^{(i)}, \beta) + 1, & \text{if } e = v_j^{(i)}v_j^{(i+1)} \in E_j, 2 \leq i \leq n, \\ \max S(v_j^{(i)}, \beta) + 1, & \text{if } e = v_j^{(i-1)}v_j^{(i)} \in E_j, n+2 \leq i \leq 2n, \end{cases}$$

where $1 \leq i \leq 2n$, $1 \leq j \leq p$.

Let us prove that γ is an interval $(W(G) + W(C_{2n}) + nr)$ -coloring of the graph $G \square C_{2n}$ for $n \geq 2$.

First we prove that the set $S(v_j^{(i)}, \gamma)$ is an interval for each vertex $v_j^{(i)} \in V(G \square C_{2n})$, where $1 \leq i \leq 2n$, $1 \leq j \leq p$.

Case 1. $i = 1$, $1 \leq j \leq p$. By the definition of γ and taking into account that $\max S(v_j, \alpha) - \min S(v_j, \alpha) = r - 1$ for $1 \leq j \leq p$, we have

$$\begin{aligned} S(v_j^{(1)}, \gamma) &= \{\min S(v_j, \alpha), \dots, \max S(v_j, \alpha)\} \cup \{\max S(v_j, \alpha) + 2\} \\ &\cup \{\max S(v_j, \alpha) + 1\} = [\min S(v_j, \alpha), \max S(v_j, \alpha) + 2]. \end{aligned}$$

Case 2. $2 \leq i \leq n$, $1 \leq j \leq p$. By the definition of γ and taking into account that $\max S(v_j, \alpha) - \min S(v_j, \alpha) = r - 1$ for $1 \leq j \leq p$, we have

$$\begin{aligned} S(v_j^{(i)}, \gamma) &= \{\min S(v_j, \alpha) + (i-1)(r+1) + 1, \dots, \max S(v_j, \alpha) \\ &+ (i-1)(r+1) + 1\} \cup \{\max S(v_j, \alpha) + (i-2)(r+1) + 2\} \\ &\cup \{\max S(v_j, \alpha) + (i-1)(r+1) + 2\} \\ &= [\min S(v_j, \alpha) + (i-1)(r+1), \max S(v_j, \alpha) + (i-1)(r+1) + 2]. \end{aligned}$$

Case 3. $i = n+1$, $1 \leq j \leq p$. By the definition of γ and taking into account that $\max S(v_j, \alpha) - \min S(v_j, \alpha) = r - 1$ for $1 \leq j \leq p$, we have

$$\begin{aligned}
S(v_j^{(n+1)}, \gamma) &= \{\min S(v_j, \alpha) + n(r+1) + 1, \dots, \max S(v_j, \alpha) + n(r+1) + 1\} \\
&\cup \{\max S(v_j, \alpha) + (n-1)(r+1) + 2\} \\
&\cup \{\max S(v_j, \alpha) + (n-1)(r+1) + 1\} \\
&= [\min S(v_j, \alpha) + n(r+1) - 1, \max S(v_j, \alpha) + n(r+1) + 1].
\end{aligned}$$

Case 4. $n+2 \leq i \leq 2n$, $1 \leq j \leq p$. By the definition of γ and taking into account that $\max S(v_j, \alpha) - \min S(v_j, \alpha) = r-1$ for $1 \leq j \leq p$, we have

$$\begin{aligned}
S(v_j^{(i)}, \gamma) &= \{\min S(v_j, \alpha) + (2n+1-i)(r+1), \dots, \max S(v_j, \alpha) \\
&+ (2n+1-i)(r+1)\} \cup \{\max S(v_j, \alpha) + (2n+1-i)(r+1) + 1\} \\
&\cup \{\max S(v_j, \alpha) + (2n-i)(r+1) + 1\} = [\min S(v_j, \alpha) \\
&+ (2n-i+1)(r+1) - 1, \max S(v_j, \alpha) + (2n-i+1)(r+1) + 1].
\end{aligned}$$

Next we show that in the coloring γ all colors are used. Clearly, there exists an edge $v_{j_0}^{(1)} v_{k_0}^{(1)} \in E(G^1)$ such that $\gamma(v_{j_0}^{(1)} v_{k_0}^{(1)}) = 1$, since in the coloring α there exists an edge $v_{j_0} v_{k_0}$ with $\alpha(v_{j_0} v_{k_0}) = 1$ and $\gamma(v_{j_0}^{(1)} v_{k_0}^{(1)}) = \beta(v_{j_0}^{(1)} v_{k_0}^{(1)}) = \alpha(v_{j_0} v_{k_0})$. Similarly, there exists an edge $v_{j_1}^{(n+1)} v_{k_1}^{(n+1)} \in E(G^{n+1})$ such that $\gamma(v_{j_1}^{(n+1)} v_{k_1}^{(n+1)}) = W(G) + n(r+1) + 1 = W(G) + W(C_{2n}) + nr$, since in the coloring α there exists an edge $v_{j_1} v_{k_1}$ with $\alpha(v_{j_1} v_{k_1}) = W(G)$ and $\gamma(v_{j_1}^{(n+1)} v_{k_1}^{(n+1)}) = \beta(v_{j_1}^{(n+1)} v_{k_1}^{(n+1)}) = \alpha(v_{j_1} v_{k_1}) + n(r+1) + 1$.

Now, by Lemma 7, we have that γ is an interval $(W(G) + W(C_{2n}) + nr)$ -coloring of the graph $G \square C_{2n}$ for $n \geq 2$. \blacksquare

From Theorems 5 and 13, we have:

Corollary 14. *If $n = p2^q$, where p is odd and q is nonnegative, then*

$$W(K_{2n} \square C_{2n}) \geq 2n^2 + 4n - 1 - p - q.$$

Note that the lower bound in Corollary 14 is close to the upper bound for $W(K_{2n} \square C_{2n})$, since $\Delta(K_{2n} \square C_{2n}) = 2n+1$ and $\text{diam}(K_{2n} \square C_{2n}) = n+1$, by Theorem 2, we have $W(K_{2n} \square C_{2n}) \leq 2n^2 + 4n + 1$.

4. GRIDS, CYLINDERS AND TORI

Interval edge-colorings of grids, cylinders and tori were first considered by Giaro and Kubale in [7], where they proved the following:

Theorem 15. *If $G = G(n_1, \dots, n_k)$ or $G = C(m, 2n)$, $m \in \mathbb{N}, n \geq 2$, or $G = T(2m, 2n)$, $m, n \geq 2$, then $G \in \mathfrak{N}$ and $w(G) = \Delta(G)$.*

For the greatest possible number of colors in interval colorings of grid graphs, the first author and Karapetyan [20] proved the following theorems:

Theorem 16. *For any $m \in \mathbb{N}, n \geq 2$, we have $W(C(m, 2n)) \geq 3m + n - 2$.*

Theorem 17. *For any $m, n \geq 2$, we have $W(T(2m, 2n)) \geq \max\{3m + n, 3n + m\}$.*

First we consider grids. It is easy to see that $W(G(2, n)) = 2n - 1$ for any $n \in \mathbb{N}$. Now we provide a lower bound for $W(G(m, n))$ when $m, n \geq 2$.

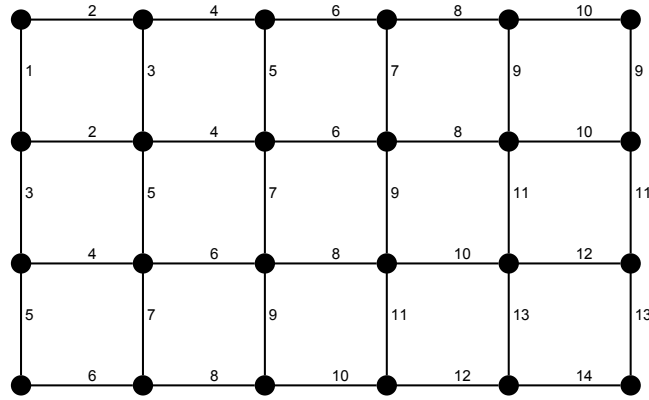


Figure 1. Interval 14-coloring of the graph $G(4, 6)$.

Theorem 18. *For any $m, n \geq 2$, we have $W(G(m, n)) \geq 2(m + n - 3)$.*

Proof. For the proof, we are going to construct an edge-coloring of the graph $G(m, n)$ that satisfies the specified conditions.

Define an edge-coloring α of $G(m, n)$ as follows:

(1) for $i = 1, \dots, m - 1, j = 1, \dots, n - 1$, let

$$\alpha\left(v_j^{(i)} v_j^{(i+1)}\right) = 2(i + j) - 3;$$

(2) for $i = 1, \dots, m - 1$, let

$$\alpha\left(v_n^{(i)} v_n^{(i+1)}\right) = 2(n + i) - 5;$$

(3) for $j = 1, \dots, n - 1$, let

$$\alpha \left(v_j^{(1)} v_{j+1}^{(1)} \right) = 2j;$$

(4) for $i = 2, \dots, m$, $j = 1, \dots, n - 1$, let

$$\alpha \left(v_j^{(i)} v_{j+1}^{(i)} \right) = 2(i + j) - 4.$$

It is easy to see that α is an interval $(2(m + n - 3))$ -coloring of $G(m, n)$ when $m, n \geq 2$. ■

Figure 1 shows the interval 14-coloring α of the graph $G(4, 6)$ described in the proof of Theorem 18.

Note that the lower bound in Theorem 18 is not far from the upper bound for $W(G(m, n))$, since $G(m, n)$ is bipartite, $2 \leq \Delta(G(m, n)) \leq 4$ and $\text{diam}(G(m, n)) = m + n - 2$, by Theorem 3, we have $W(G(m, n)) \leq 3(m + n - 2) + 1$.

From Theorems 10 and 18, we have:

Corollary 19. *If $n_1 \geq \dots \geq n_{2k} \geq 2$ ($k \in \mathbb{N}$), then*

$$W(G(n_1, \dots, n_{2k})) \geq 2 \sum_{i=1}^{2k} n_i - 6k,$$

and if $n_1 \geq \dots \geq n_{2k+1} \geq 2$ ($k \in \mathbb{N}$), then

$$W(G(n_1, \dots, n_{2k+1})) \geq 2 \sum_{i=1}^{2k} n_i + n_{2k+1} - 6k - 1.$$

Next we consider cylinders. In [18], Khchoyan proved the following:

Theorem 20. *For any $n \geq 3$, we have*

- (1) $C(2, n) \in \mathfrak{N}$,
- (2) $w(C(2, n)) = 3$,
- (3) $W(C(2, n)) = n + 2$,
- (4) *if $w(C(2, n)) \leq t \leq W(C(2, n))$, then $C(2, n)$ has an interval t -coloring.*

Now we prove some general results on cylinders.

Theorem 21. *For any $m \geq 3, n \in \mathbb{N}$, we have $C(m, 2n + 1) \in \mathfrak{N}$ and*

$$w(C(m, 2n + 1)) = \begin{cases} 4, & \text{if } m \text{ is even,} \\ 6, & \text{if } m \text{ is odd.} \end{cases}$$

Proof. First we show that if m is even, then $C(m, 2n + 1)$ has an interval 4-coloring. For $1 \leq i \leq \frac{m}{2}$, define a subgraph C^i of the graph $C(m, 2n + 1)$ as follows:

$$C^i = \left(V^{2i-1} \cup V^{2i}, E^{2i-1} \cup E^{2i} \cup \left\{ v_j^{(2i-1)} v_j^{(2i)} : 1 \leq j \leq 2n + 1 \right\} \right).$$

Clearly, C^i is isomorphic to $C(2, 2n+1)$ for $1 \leq i \leq \frac{m}{2}$. By Theorem 20, $C(2, 2n+1) \in \mathfrak{N}$ and there exists an interval 3-coloring α of $C(2, 2n+1)$. Now we define an edge-coloring β of $C(m, 2n+1)$. First we color the edges of C^i according to α for $1 \leq i \leq \frac{m}{2}$. Then we color the edges $v_j^{(2i)}v_j^{(2i+1)} \in E_j$ with color 4 for $1 \leq i \leq \frac{m}{2} - 1, 1 \leq j \leq 2n+1$. It is easy to see that β is an interval 4-coloring of $C(m, 2n+1)$. This shows that $C(m, 2n+1) \in \mathfrak{N}$ and $w(C(m, 2n+1)) \leq 4$. On the other hand, $w(C(m, 2n+1)) \geq \Delta(C(m, 2n+1)) = 4$; thus $w(C(m, 2n+1)) = 4$ for even m .

Now assume that m is odd. First we show that $C(3, 2n+1)$ has an interval 6-coloring. Define an edge-coloring γ of $C(3, 2n+1)$ as follows:

$$(1) \quad \gamma(v_1^{(1)}v_1^{(2)}) = 6 \text{ and for } j = 2, \dots, 2\lfloor \frac{n+1}{2} \rfloor, \text{ let } \gamma(v_j^{(1)}v_j^{(2)}) = 4;$$

$$(2) \quad \gamma(v_{2\lfloor \frac{n+1}{2} \rfloor+1}^{(1)}v_{2\lfloor \frac{n+1}{2} \rfloor+1}^{(2)}) = 2 \text{ and for } j = 2\lfloor \frac{n+1}{2} \rfloor + 2, \dots, 2n+1, \text{ let}$$

$$\gamma(v_j^{(1)}v_j^{(2)}) = 3;$$

$$(3) \quad \gamma(v_1^{(2)}v_1^{(3)}) = 3 \text{ and for } j = 2, \dots, 2\lfloor \frac{n+1}{2} \rfloor, \text{ let } \gamma(v_j^{(2)}v_j^{(3)}) = 2;$$

$$(4) \quad \text{for } j = 2\lfloor \frac{n+1}{2} \rfloor + 1, \dots, 2n+1, \text{ let } \gamma(v_j^{(2)}v_j^{(3)}) = 1;$$

$$(5) \quad j = 1, \dots, \lfloor \frac{n+1}{2} \rfloor, \text{ let}$$

$$\gamma(v_{2j-1}^{(1)}v_{2j}^{(1)}) = \gamma(v_{2j-1}^{(2)}v_{2j}^{(2)}) = 5 \text{ and } \gamma(v_{2j}^{(1)}v_{2j+1}^{(1)}) = \gamma(v_{2j}^{(2)}v_{2j+1}^{(2)}) = 3;$$

$$(6) \quad \text{for } j = \lfloor \frac{n+1}{2} \rfloor + 1, \dots, n, \text{ let}$$

$$\gamma(v_{2j-1}^{(1)}v_{2j}^{(1)}) = \gamma(v_{2j-1}^{(2)}v_{2j}^{(2)}) = 4 \text{ and } \gamma(v_1^{(1)}v_{2n+1}^{(1)}) = \gamma(v_1^{(2)}v_{2n+1}^{(2)}) = 4;$$

$$(7) \quad \text{for } j = \lfloor \frac{n+1}{2} \rfloor + 1, \dots, n, \text{ let } \gamma(v_{2j}^{(1)}v_{2j+1}^{(1)}) = \gamma(v_{2j}^{(2)}v_{2j+1}^{(2)}) = 2;$$

$$(8) \quad \text{for } j = 1, \dots, \lfloor \frac{n+1}{2} \rfloor, \text{ let } \gamma(v_{2j-1}^{(3)}v_{2j}^{(3)}) = 1 \text{ and } \gamma(v_{2j}^{(3)}v_{2j+1}^{(3)}) = 3;$$

$$(9) \quad \text{for } j = \lfloor \frac{n+1}{2} \rfloor + 1, \dots, n, \text{ let } \gamma(v_{2j-1}^{(3)}v_{2j}^{(3)}) = 2 \text{ and } \gamma(v_1^{(3)}v_{2n+1}^{(3)}) = 2;$$

$$(10) \quad \text{for } j = \lfloor \frac{n+1}{2} \rfloor + 1, \dots, n, \text{ let } \gamma(v_{2j}^{(3)}v_{2j+1}^{(3)}) = 3.$$

It is not difficult to see that γ is an interval 6-coloring of $C(3, 2n+1)$ for which $S(v_j^{(3)}, \gamma) = [1, 3]$ when $1 \leq j \leq 2n+1$.

Next we define an edge-coloring ϕ of $C(m, 2n+1)$ as follows: first we color the edges of the subgraph $C(3, 2n+1)$ of $C(m, 2n+1)$ according to γ . Secondly, we color the edges of the remaining subgraph $C(m-3, 2n+1)$ of $C(m, 2n+1)$ according to β , and finally, we color the edges $v_j^{(3)}v_j^{(4)} \in E_j$ with color 4 for $1 \leq j \leq 2n+1$. It is easy to see that ϕ is an interval 6-coloring of $C(m, 2n+1)$. This shows that $C(m, 2n+1) \in \mathfrak{N}$ and $w(C(m, 2n+1)) \leq 6$.

Now we prove that $w(C(m, 2n+1)) \geq 6$ for odd m . Let ψ be an interval $w(C(m, 2n+1))$ -coloring of $C(m, 2n+1)$ and $w(C(m, 2n+1)) \leq 5$. Consider the set $S(v_j^{(i)}, \psi)$ for $1 \leq i \leq m, 1 \leq j \leq 2n+1$. It is easy to see that if $d(v_j^{(i)}) = 3$, then $1 \leq \min S(v_j^{(i)}, \psi) \leq 3$, and if $d(v_j^{(i)}) = 4$, then $1 \leq \min S(v_j^{(i)}, \psi) \leq 2$. Hence, $3 \in S(v_j^{(i)}, \psi)$ for $1 \leq i \leq m, 1 \leq j \leq 2n+1$, but this implies that the edges with color 3 form a perfect matching in $C(m, 2n+1)$, which contradicts the fact that $C(m, 2n+1)$ does not have one. Thus $w(C(m, 2n+1)) = 6$ for odd m . ■

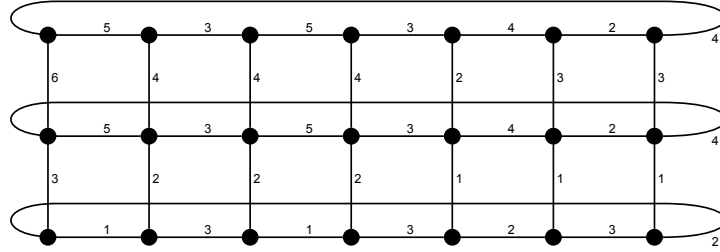


Figure 2. Interval 6-coloring of the graph $C(3, 7)$.

Figure 2 shows the interval 6-coloring γ of the graph $C(3, 7)$ described in the proof of Theorem 21.

Before we derive lower bounds for $W(C(2m, 2n))$ and $W(C(2m, 2n+1))$, let us note that Lemma 8, Theorems 15 and 21 imply the following:

Corollary 22. *If $G \square H$ is planar and both factors have at least 3 vertices, then $G \square H \in \mathfrak{N}$ and $w(G \square H) \leq 6$.*

Theorem 23. *If $m \in \mathbb{N}, n \geq 2$, then $W(C(2m, 2n)) \geq 4m + 2n - 2$, and if $m, n \in \mathbb{N}$, then $W(C(2m, 2n+1)) \geq 4m + 2n - 1$.*

Proof. For the proof of the theorem, it suffices to construct edge-colorings that satisfies the specified conditions. First we construct an interval $(4m + 2n - 2)$ -coloring of $C(2m, 2n)$ when $m \in \mathbb{N}, n \geq 2$.

Define an edge-coloring α of $C(2m, 2n)$ as follows:

(1) for $i = 1, \dots, m, j = 1, \dots, n$, let

$$\alpha \left(v_j^{(2i-1)} v_{j+1}^{(2i-1)} \right) = \alpha \left(v_j^{(2i)} v_{j+1}^{(2i)} \right) = 4i + 2j - 4;$$

(2) for $i = 1, \dots, m, j = n + 1, \dots, 2n - 1$, let

$$\alpha \left(v_j^{(2i-1)} v_{j+1}^{(2i-1)} \right) = \alpha \left(v_j^{(2i)} v_{j+1}^{(2i)} \right) = 4i - 2j + 4n - 1;$$

(3) for $i = 1, \dots, m$, let

$$\alpha \left(v_1^{(2i-1)} v_{2n}^{(2i-1)} \right) = \alpha \left(v_1^{(2i)} v_{2n}^{(2i)} \right) = 4i - 1;$$

(4) for $i = 1, \dots, m, j = 1, \dots, n$, let

$$\alpha \left(v_j^{(2i-1)} v_j^{(2i)} \right) = 4i + 2j - 5;$$

(5) for $i = 1, \dots, m, j = n + 1, \dots, 2n$, let

$$\alpha \left(v_j^{(2i-1)} v_j^{(2i)} \right) = 4i - 2j + 4n;$$

(6) for $i = 1, \dots, m - 1, j = 2, \dots, n + 1$, let

$$\alpha \left(v_j^{(2i)} v_j^{(2i+1)} \right) = 4i + 2j - 3;$$

(7) for $i = 1, \dots, m - 1, j = n + 2, \dots, 2n$, let

$$\alpha \left(v_j^{(2i)} v_j^{(2i+1)} \right) = 4i - 2j + 4n + 2;$$

(8) for $i = 1, \dots, m - 1$, let

$$\alpha \left(v_1^{(2i)} v_1^{(2i+1)} \right) = 4i.$$

Next we construct an interval $(4m + 2n - 1)$ -coloring of $C(2m, 2n + 1)$ when $m, n \in \mathbb{N}$. Define an edge-coloring β of $C(2m, 2n + 1)$ as follows:

(1) for $i = 1, \dots, m, j = 1, \dots, n + 1$, let

$$\beta \left(u_j^{(2i-1)} u_{j+1}^{(2i-1)} \right) = \beta \left(u_j^{(2i)} u_{j+1}^{(2i)} \right) = 4i + 2j - 4;$$

(2) for $i = 1, \dots, m, j = n + 2, \dots, 2n$, let

$$\beta \left(u_j^{(2i-1)} u_{j+1}^{(2i-1)} \right) = \beta \left(u_j^{(2i)} u_{j+1}^{(2i)} \right) = 4i - 2j + 4n + 1;$$

(3) for $i = 1, \dots, m$, let

$$\beta \left(u_1^{(2i-1)} u_{2n+1}^{(2i-1)} \right) = \beta \left(u_1^{(2i)} u_{2n+1}^{(2i)} \right) = 4i - 1;$$

(4) for $i = 1, \dots, m$, $j = 1, \dots, n + 2$, let

$$\beta \left(u_j^{(2i-1)} u_j^{(2i)} \right) = 4i + 2j - 5;$$

(5) for $i = 1, \dots, m$, $j = n + 3, \dots, 2n + 1$, let

$$\beta \left(u_j^{(2i-1)} u_j^{(2i)} \right) = 4i - 2j + 4n + 2;$$

(6) for $i = 1, \dots, m - 1$, $j = 2, \dots, n + 1$, let

$$\beta \left(u_j^{(2i)} u_j^{(2i+1)} \right) = 4i + 2j - 3;$$

(7) for $i = 1, \dots, m - 1$, $j = n + 2, \dots, 2n + 1$, let

$$\beta \left(u_j^{(2i)} u_j^{(2i+1)} \right) = 4i - 2j + 4n + 4;$$

(8) for $i = 1, \dots, m - 1$, let

$$\beta \left(u_1^{(2i)} u_1^{(2i+1)} \right) = 4i.$$

It is straightforward to check that α is an interval $(4m + 2n - 2)$ -coloring of $C(2m, 2n)$ when $m \in \mathbb{N}, n \geq 2$, and β is an interval $(4m + 2n - 1)$ -coloring of $C(2m, 2n + 1)$ when $m, n \in \mathbb{N}$. ■

Note that the lower bound in Theorem 23 is not so far from the upper bound for $W(C(m, n))$. Indeed, since $C(2m, 2n)$ is bipartite, $3 \leq \Delta(C(2m, 2n)) \leq 4$ and $\text{diam}(C(2m, 2n)) = 2m + n - 1$, by Theorem 3, we have $W(C(2m, 2n)) \leq 3(2m + n - 1) + 1$. Similarly, since $3 \leq \Delta(C(2m, 2n + 1)) \leq 4$ and $\text{diam}(C(2m, 2n + 1)) = 2m + n - 1$, by Theorem 2, we have $W(C(2m, 2n + 1)) \leq 3(2m + n) + 1$. Next we would like to compare obtained lower bounds for $W(C(m, n))$. If m is even and $m < n$, then the lower bound in Theorem 23 is better than in Theorem 16, if m is even and $m > n$, then the lower bound in Theorem 16 is better than in Theorem 23, and if m is even and $m = n$, then we obtain the same lower bound in both theorems.

In the following we consider tori. In [22], the first author proved that the torus $T(m, n) \in \mathfrak{N}$ if and only if mn is even. Since $T(m, n)$ is 4-regular, by Theorem 1, we obtain that $w(T(m, n)) = 4$ when mn is even. Now we derive a new lower bound for $W(T(m, n))$ when mn is even.

Theorem 24. For any $m, n \geq 2$, we have $W(T(2m, 2n)) \geq \max\{3m + n + 2, 3n + m + 2\}$, and for any $m \geq 2, n \in \mathbb{N}$, we have

$$W(T(2m, 2n + 1)) \geq \begin{cases} 2m + 2n + 2, & \text{if } m \text{ is odd,} \\ 2m + 2n + 3, & \text{if } m \text{ is even.} \end{cases}$$

Proof. First note that the lower bound for $W(T(2m, 2n))$ ($m, n \geq 2$) follows from Theorem 13. For the proof of a second part of the theorem, it suffices to construct an edge-coloring of $T(2m, 2n + 1)$ that satisfies the specified conditions.

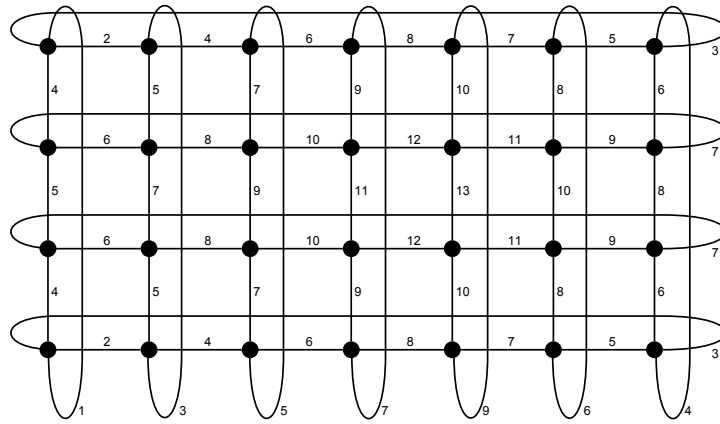


Figure 3. Interval 13-coloring of the graph $T(4, 7)$.

Define an edge-coloring α of $T(2m, 2n + 1)$ as follows:

(1) for $j = 1, \dots, n + 1$, let

$$\alpha(v_j^{(1)}v_{j+1}^{(1)}) = \alpha(v_j^{(2m)}v_{j+1}^{(2m)}) = 2j;$$

(2) for $j = n + 2, \dots, 2n$, let

$$\alpha(v_j^{(1)}v_{j+1}^{(1)}) = \alpha(v_j^{(2m)}v_{j+1}^{(2m)}) = 2(2n + 1 - j) + 3$$

and

$$\alpha(v_1^{(1)}v_{2n+1}^{(1)}) = \alpha(v_1^{(2m)}v_{2n+1}^{(2m)}) = 3;$$

(3) for $j = 1, \dots, n + 2$, let

$$\alpha(v_j^{(1)}v_j^{(2m)}) = 2j - 1;$$

(4) for $j = n + 3, \dots, 2n + 1$, let

$$\alpha \left(v_j^{(1)} v_j^{(2m)} \right) = 2(2n + 3 - j);$$

(5) for $i = 1, \dots, \lfloor \frac{m}{2} \rfloor$, $j = 1, \dots, n + 1$, let

$$\begin{aligned} \alpha \left(v_j^{(2i)} v_{j+1}^{(2i)} \right) &= \alpha \left(v_j^{(2i+1)} v_{j+1}^{(2i+1)} \right) \\ &= \alpha \left(v_j^{(2m-2i)} v_{j+1}^{(2m-2i)} \right) = \alpha \left(v_j^{(2m-2i+1)} v_{j+1}^{(2m-2i+1)} \right) = 4i + 2j; \end{aligned}$$

(6) for $i = 1, \dots, \lfloor \frac{m}{2} \rfloor$, $j = n + 2, \dots, 2n$, let

$$\begin{aligned} \alpha \left(v_j^{(2i)} v_{j+1}^{(2i)} \right) &= \alpha \left(v_j^{(2i+1)} v_{j+1}^{(2i+1)} \right) = \alpha \left(v_j^{(2m-2i)} v_{j+1}^{(2m-2i)} \right) \\ &= \alpha \left(v_j^{(2m-2i+1)} v_{j+1}^{(2m-2i+1)} \right) = 4i + 2(2n + 1 - j) + 3 \end{aligned}$$

and

$$\begin{aligned} \alpha \left(v_1^{(2i)} v_{2n+1}^{(2i)} \right) &= \alpha \left(v_1^{(2i+1)} v_{2n+1}^{(2i+1)} \right) \\ &= \alpha \left(v_1^{(2m-2i)} v_{2n+1}^{(2m-2i)} \right) = \alpha \left(v_1^{(2m-2i+1)} v_{2n+1}^{(2m-2i+1)} \right) = 4i + 3; \end{aligned}$$

(7) for $i = 1, \dots, \lceil \frac{m}{2} \rceil$, $j = 2, \dots, n + 1$, let

$$\alpha \left(v_j^{(2i-1)} v_j^{(2i)} \right) = \alpha \left(v_j^{(2m-2i+1)} v_j^{(2m-2i+2)} \right) = 4i + 2j - 3;$$

(8) for $i = 1, \dots, \lceil \frac{m}{2} \rceil$, $j = n + 2, \dots, 2n + 1$, let

$$\alpha \left(v_j^{(2i-1)} v_j^{(2i)} \right) = \alpha \left(v_j^{(2m-2i+1)} v_j^{(2m-2i+2)} \right) = 4(n + 1 + i) - 2j;$$

(9) for $i = 1, \dots, \lceil \frac{m}{2} \rceil$, let

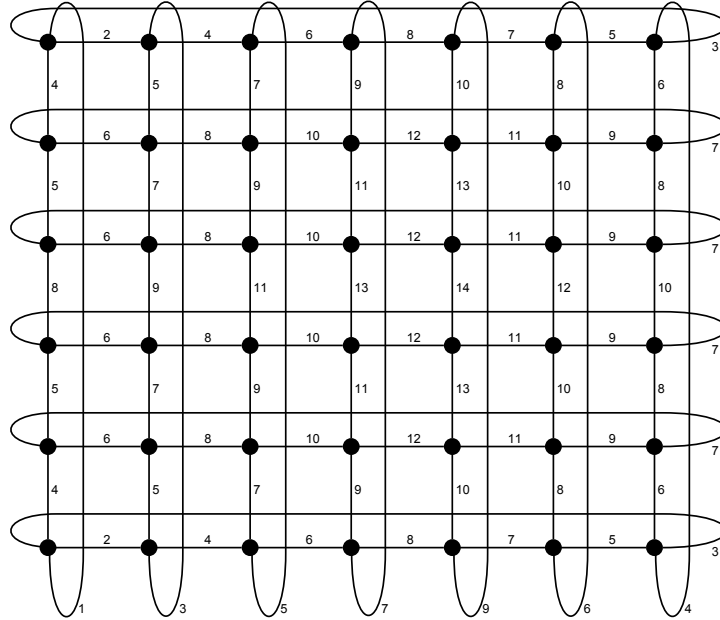
$$\alpha \left(v_1^{(2i-1)} v_1^{(2i)} \right) = \alpha \left(v_1^{(2m-2i+1)} v_1^{(2m-2i+2)} \right) = 4i;$$

(10) for $i = 1, \dots, \lfloor \frac{m}{2} \rfloor$, $j = 1, \dots, n + 2$, let

$$\alpha \left(v_j^{(2i)} v_j^{(2i+1)} \right) = \alpha \left(v_j^{(2m-2i)} v_j^{(2m-2i+1)} \right) = 4i + 2j - 1;$$

(11) for $i = 1, \dots, \lfloor \frac{m}{2} \rfloor$, $j = n + 3, \dots, 2n + 1$, let

$$\alpha \left(v_j^{(2i)} v_j^{(2i+1)} \right) = \alpha \left(v_j^{(2m-2i)} v_j^{(2m-2i+1)} \right) = 4i + 2(2n + 3 - j).$$

Figure 4. Interval 14-coloring of the graph $T(6, 7)$.

Let us show that the edges incident to any vertex of $T(2m, 2n + 1)$ are colored by four consecutive colors. For example, let $2 \leq i \leq \lfloor \frac{m}{2} \rfloor$ and $2 \leq j \leq n + 1$. By the points (5), (7) and (10) of the definition of α , for $2 \leq i \leq \lfloor \frac{m}{2} \rfloor$, $2 \leq j \leq n + 1$, we have

$$\begin{aligned} S(v_j^{(2i)}, \alpha) &= S(v_j^{(2m-2i)}, \alpha) = \{4i + 2j - 2, 4i + 2j\} \\ &\cup \{4i + 2j - 3\} \cup \{4i + 2j - 1\} = [4i + 2j - 3, 4i + 2j]. \end{aligned}$$

Similarly, it can be verified that the edges incident to other vertices of $T(2m, 2n + 1)$ are also colored by four consecutive colors. It is easy to see that $\alpha(v_1^{(1)}v_1^{(2m)}) = 1$. Now if m is odd, then $\alpha(v_{n+2}^{(m)}v_{n+2}^{(m+1)}) = 2m + 2n + 2$ and, by Lemma 7, α is an interval $(2m + 2n + 2)$ -coloring of $T(2m, 2n + 1)$ when m is odd. If m is even, then $\alpha(v_{n+2}^{(m)}v_{n+2}^{(m+1)}) = 2m + 2n + 3$ and, by Lemma 7, α is an interval $(2m + 2n + 3)$ -coloring of $T(2m, 2n + 1)$ when m is even. ■

Figure 3 and 4 show the interval colorings of the graphs $T(4, 7)$ and $T(6, 7)$ described in the proof of Theorem 24.

From Theorems 1, 15 and 24, we have:

Corollary 25. *If $G = T(2m, 2n)$ ($m, n \geq 2$) and $4 \leq t \leq \max\{3m + n + 2, 3n + m + 2\}$, then G has an interval t -coloring. Also, if $H = T(2m, 2n + 1)$ ($m \geq$*

$2, n \in \mathbb{N}$), m is odd and $4 \leq t \leq 2m + 2n + 2$, then H has an interval t -coloring, and if $H = T(2m, 2n + 1)$ ($m \geq 2, n \in \mathbb{N}$), m is even and $4 \leq t \leq 2m + 2n + 3$, then H has an interval t -coloring.

Let us note that the lower bound in Theorem 24 is not so far from the upper bound for $W(T(m, n))$. Indeed, since $T(2m, 2n)$ is bipartite, $\Delta(T(2m, 2n)) = 4$ and $\text{diam}(C(2m, 2n)) = m + n$, by Theorem 3, we have $W(T(2m, 2n)) \leq 3(m + n) + 1$. Similarly, since $\Delta(T(2m, 2n + 1)) = 4$ and $\text{diam}(T(2m, 2n + 1)) = m + n$, by Theorem 2, we have $W(T(2m, 2n + 1)) \leq 3(m + n + 1) + 1$.

5. n -DIMENSIONAL CUBES

It is well-known that the n -dimensional cube Q_n is the Cartesian product of n copies of K_2 . In [21], the first author investigated interval colorings of n -dimensional cubes and proved that $w(Q_n) = n$ and $W(Q_n) \geq \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$. In the same paper he also conjectured that $W(Q_n) = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$. Here, we prove this conjecture.

Let $e, e' \in E(Q_n)$ and $e = u_1u_2$, $e' = v_1v_2$. The distance between two edges e and e' in Q_n , we define as follows:

$$d(e, e') = \min_{1 \leq i \leq 2, 1 \leq j \leq 2} \{d(u_i, v_j)\}.$$

Let α be an interval t -coloring of Q_n . Define an edge span $\text{sp}_\alpha(e, e')$ of edges e and e' ($e, e' \in E(Q_n)$) in coloring α as follows:

$$\text{sp}_\alpha(e, e') = |\alpha(e) - \alpha(e')|.$$

For any $k, 0 \leq k \leq n - 1$, define an edge span at distance k $\text{sp}_{\alpha, k}$ in coloring α as follows:

$$\text{sp}_{\alpha, k} = \max \{ \text{sp}_\alpha(e, e') : e, e' \in E(Q_n) \text{ and } d(e, e') = k \}.$$

Clearly, $\text{sp}_{\alpha, 0} = n - 1$.

Theorem 26. *If $n \in \mathbb{N}$, then $W(Q_n) \leq \frac{n(n+1)}{2}$.*

Proof. Let α be an interval $W(Q_n)$ -coloring of Q_n . First we show that if $1 \leq k \leq n - 1$, then $\text{sp}_{\alpha, k} \leq \text{sp}_{\alpha, k-1} + n - k$.

Let $e, e' \in E(Q_n)$ be any two edges of Q_n with $d(e, e') = k$. Without loss of generality, we may assume that $\alpha(e) \geq \alpha(e')$. Since $d(e, e') = k$, there exist u and v vertices such that $u \in e$ and $v \in e'$ and $d(u, v) = k$. There are v_1, v_2, \dots, v_k ($v_i \neq v_j$ when $i \neq j$) vertices such that $d(u, v_i) = k - 1$ and $vv_i \in E(Q_n)$ for $i = 1, \dots, k$. Since Q_n is n -regular, we have

$$(*) \quad \min_{1 \leq i \leq k} \{ \alpha(v_i v) \} \leq \alpha(e') + n - k.$$

Let $\alpha(e'') = \min_{1 \leq i \leq k} \{ \alpha(v_i v) \}$. By (*), we obtain

$$\alpha(e') \geq \alpha(e'') - (n - k) \text{ and } d(e, e'') = k - 1.$$

Thus,

$$\begin{aligned} \text{sp}_\alpha(e, e') &= |\alpha(e) - \alpha(e')| \leq |\alpha(e) - \alpha(e'') + n - k| \leq |\alpha(e) - \alpha(e'')| + n - k \\ &\leq \text{sp}_{\alpha, k-1} + n - k. \end{aligned}$$

Since e and e' were arbitrary edges with $d(e, e') = k$, we obtain $\text{sp}_{\alpha, k} \leq \text{sp}_{\alpha, k-1} + n - k$. Now by induction on k with $\text{sp}_{\alpha, 0} = n - 1$, we obtain $\text{sp}_{\alpha, n-1} \leq \frac{n(n+1)}{2} - 1$. From this and taking into account that $d(e, e') \leq n - 1$ for all $e, e' \in E(Q_n)$, we get $W(Q_n) \leq \frac{n(n+1)}{2}$. ■

By Theorems 6 and 26, we obtain $W(Q_n) = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$. Moreover, by Theorem 1, we have that Q_n has an interval t -coloring if and only if $n \leq t \leq \frac{n(n+1)}{2}$.

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