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RAINBOW CONNECTION NUMBER OF DENSE GRAPHS

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Abstract

An edge-colored graph G is rainbow connected, if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection number of a connected graph G, denoted $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. In this paper we show that $rc(G) \leq 3$ if $|E(G)| \geq \binom{n-2}{2} + 2$, and $rc(G) \leq 4$ if $|E(G)| \geq \binom{n-3}{2} + 3$. These bounds are sharp.

Keywords: edge-colored graph, rainbow coloring, rainbow connection number.

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1. Introduction

We use [1] for terminology and notation not defined here and consider finite and simple graphs only.

An edge-colored graph G is called *rainbow connected*, if any two vertices are connected by a path whose edges have different colors. The concept of rainbow connection in graphs was introduced by Chartrand *et al.* in [2]. The *rainbow connection number* of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. The rainbow connection number has been studied for several graph classes. These results are presented in a recent survey [5]. Rainbow connection has an interesting application for the secure transfer of classified information between agencies, see [3].

In [4] the following problem was suggested:

Problem 1. For every $k, 1 \leq k \leq n-1$, compute and minimize the function $f(n,k)$ with the following property: If $|E(G)| \geq f(n,k)$, then $rc(G) \leq k$.

The authors of [4] got the following results:

Proposition 2. $f(n,k) \geq \binom{n-k+1}{2} + (k-1)$.

For convenience we repeat the proof given in [4].

Proof. We construct a graph G_k as follows: Take a $K_{n-k+1} - e$ and denote the two vertices of degree $n-k-1$ with u_1 and u_2. Now take a path P_k with vertices labeled $w_1, w_2, \ldots, w_k$ and identify the vertices u_2 and w_1. The resulting graph G_k has order n and size $|E(G_k)| = \binom{n-k+1}{2} + (k-2)$. For its diameter we obtain $d(u_1, w_k) = diam(G) = k+1$. Hence $f(n,k) \geq \binom{n-k+1}{2} + (k-1)$. ■

Moreover, in [4] $f(n,k)$ has been determined for $k = 1, 2, n-1, n-2$.

Proposition 3. $f(n,1) = \binom{n}{2}, f(n,n-1) = n-1, f(n,n-2) = n$.

Theorem 4. *Let G be a connected graph of order $n \geq 3$. If $\binom{n-1}{2} + 1 \leq |E(G)| \leq \binom{n}{2} - 1$, then $rc(G) = 2$.*

Hence $f(n,2) = \binom{n-1}{2} + 1$. In this paper we will show that $f(n,3) = \binom{n-2}{2} + 2$ and $f(n,4) = \binom{n-3}{2} + 3$. One might think that equality in Proposition 1.2 always holds for any $1 \leq k \leq n$. But, as one will see, our proof technique cannot be easily used for general $k \geq 5$.

2. MAIN RESULTS

At first, we give some notation which will be used in the sequel.

Definition. Let G be a connected graph. The *distance* between two vertices u and v in G, denoted by $d(u,v)$, is the length of a shortest path between them in G. The *distance between a vertex* v *and a set* $S \subset V(G)$ is defined as $d(v,S) = \min_{x\in S} d(v,x)$. The k*-step open neighborhood of a set* $S \subset V(G)$ is defined as $N^k(S) = \{x \in V(G) \mid d(x,S) = k\}, k \in \{0,1,2,\ldots\}$. When $k = 1$, we may omit the qualifier "1-step" in the above name and the superscript 1 in the notation. The *neighborhood of a vertex* v in $\overline{G}$, denoted by $\overline{N}(v)$, is defined as $\overline{N}(v) = \{x \mid xv \notin E(G)\}$.

We will first present a new and shorter proof for Theorem 4, which we restate for convenience.

Theorem 5. *Let G be a connected graph of order $n \geq 3$. If $\binom{n-1}{2}+1 \leq |E(G)| \leq \binom{n}{2} - 1$, then $rc(G) = 2$.*

Proof. Our proof will be by induction on n. For $n = 3$, we have $f(n,n-1) = n-1 = 2 = \binom{3-1}{2} + 1$. For $n = 4$, we have $f(n,n-2) = n = 4 = \binom{4-1}{2} + 1$. So we may assume $n \geq 5$.

Since $|E(G)| \leq \binom{n}{2} - 1$, we have $1 \leq \delta(G) \leq n-2$. Choose a vertex $w \in V(G)$ with $d(w) = \delta(G)$ and set $d(w) = n-2-t$ with $0 \leq t \leq n-3$. Let $H = G - w$. Then $|E(H)| \geq \binom{n-1}{2}+1-d(w) = \binom{n-2}{2}+n-2+1-(n-2-t) = \binom{n-2}{2}+1+t = \binom{(n-1)-1}{2}+1+t \geq \binom{n-2}{2}+1$. Hence, H is connected; otherwise $E(H) < \binom{n-2}{2}+1$.

Now let $\overline{N}(w) = \{v_1, v_2, \ldots, v_t, v_{t+1}\}$.

Claim. *$N(v_i) \cap N(w) \neq \emptyset$ for $1 \leq i \leq t+1$.*

Proof of the Claim. Suppose $N(v_i) \cap N(w) = \emptyset$ for some i with $1 \leq i \leq t+1$. Then $d(v_i) \leq (t+1) - 1 = t$, thus $E(\overline{G}) \geq |\overline{N}_H(v_i)| + |\overline{N}_G(w)| \geq (n-t-2) + (t+1) = n-1 > n-2$, a contradiction, since $E(\overline{G}) \leq n-2$. □

Hence for every vertex v_i, there is a vertex $u_i \in N(w)$ such that $u_iv_i \in E(G)$ for $1 \leq i \leq t+1$. Let H' be a subgraph of H with $V(H') = V(H)$ and $E(H') = E(H) - \{v_1u_1, \ldots, v_tu_t\}$. Then $|E(H')| \geq \binom{n-2}{2} + 1 + t - t = \binom{n-2}{2} + 1 = \binom{(n-1)-1}{2} + 1$. So H' is connected, and by induction we have $rc(H') \leq 2$. Now take a 2-rainbow coloring of H'. Let $c(v_{t+1}u_{t+1}) = 1$. Then, set $c(v_iu_i) = 1$ for $1 \leq i \leq t$ and $c(e) = 2$ for all edges e which are incident with w. It is easy to check that G is 2-rainbow connected. ■

In the following we give the new results of this paper.

Theorem 6. *Let G be a connected graph of order $n \geq 4$. If $|E(G)| \geq \binom{n-2}{2} + 2$, then $rc(G) \leq 3$.*

Proof. Our proof will be by induction on n. For $n = 4$, we have $f(n, n-1) = n - 1 = 3 = \binom{4-2}{2} + 2$. For $n = 5$, we have $f(n, n-2) = n = 5 = \binom{5-2}{2} + 2$. So we may assume $n \geq 6$.

By Theorem 5, we have $rc(G) \leq 2$ for $|E(G)| \geq \binom{n-1}{2} + 1$. Hence we may assume $|E(G)| \leq \binom{n-1}{2}$. This implies $\delta(G) \leq \frac{(n-1)(n-2)}{n} = n - 3 + \frac{2}{n} < n - 2$.

Claim 1. *$diam(G) \leq 3$.*

Proof of Claim 1. Suppose $diam(G) \geq 4$ and consider a diameter path $v_1, v_2, \ldots, v_{D+1}$ with $D \geq 4$. Then $d(v_1) + d(v_4) \leq n - 2$ and $d(v_2) + d(v_5) \leq n - 2$, implying $|E(G)| \leq \binom{n}{2} - 2(2n-3-(n-2)) = \binom{n}{2} - 2(n-1) = \binom{n-2}{2} - 1 < \binom{n-2}{2} + 2$, a contradiction. □

Claim 2. *If $\delta(G) = 1$, then $rc(G) \leq 3$.*

Proof of Claim 2. Let w be a vertex with $d(w) = \delta(G) = 1$, and let $H = G - w$. Then $|E(H)| \geq \binom{n-2}{2} + 2 - 1 = \binom{n-2}{2} + 1 = \binom{(n-1)-1}{2} + 1$. Hence $rc(H) \leq 2$ by Theorem 5. Take a 2-rainbow coloring for H, and set $c(e) = 3$ for the edge incident with w. Then $rc(G) \leq 3$. □

Hence we may assume $\delta(G) \geq 2$. Let $w_1, w_2 \in V(G)$ with $w_1w_2 \notin E(G)$. Suppose $N(w_1) \cap N(w_2) = \emptyset$. Let $H = G - \{w_1, w_2\}$. Then $|E(H)| \geq \binom{n-2}{2} + 2 - (n-2) = \binom{n-3}{2} + 1 = \binom{(n-2)-1}{2} + 1$. Thus H is connected. Hence $rc(H) \leq 2$ by Theorem 5. Consider a 2-rainbow coloring of H with colors 1 and 2. Since $diam(G) \leq 3$, there is a w_1w_2-path $w_1u_1u_2w_2$. Let $c(u_1u_2) = 1$, and then set $c(w_1u_1) = 2, c(w_2u_2) = 3$ and $c(e) = 3$ for all other edges incident with w_1 or w_2. Then G is 3-rainbow connected.

Hence we may assume $N(w_1) \cap N(w_2) \neq \emptyset$ if $w_1, w_2 \in V(G)$ and $w_1w_2 \notin E(G)$. Choose a vertex w with $d(w) = \delta(G)$ and set $d(w) = n - 2 - t$ with $1 \leq t \leq n - 4$. As in the proof of Theorem 5, there exist vertices $u_i \in N(w)$ such that $u_iv_i \in E(G)$ for $1 \leq i \leq t + 1$, where $\overline{N}(w) = \{v_1, v_2, \ldots, v_t, v_{t+1}\}$. Let $H = G - w$, and let H' be a subgraph of H with $V(H') = V(H)$ and $E(H') = E(H) - \{u_1v_1, \ldots, u_{t-1}v_{t-1}\}$. Then $|E(H')| \geq \binom{n-2}{2} + 2 - (n-2-t) - (t-1) = \binom{n-2}{2} - n + 5 = \binom{(n-1)-2}{2} + 2$.

Hence, if H' is connected, then by induction, H' is 3-rainbow connected. Now take a 3-rainbow coloring of H'. Let $c(u_iv_i) \in \{1, 2\}$ for $i = t, t+1$, and then set $c(u_iv_i) = 1$ for $1 \leq i \leq t - 1$ and $c(e) = 3$ for all edges e incident with w. Then G is 3-rainbow connected.

Claim 3. *If H' is disconnected, then H' has at most 2 components and one of them is a single vertex.*

***Proof of Claim* 3.** Suppose, on the contrary, that H' has $k \geq 3$ components. Let n_i be the number of vertices of the ith component. Thus $n_1+\cdots+n_k = n-1$, and then

$$\begin{aligned} |E(H')| &\leq \binom{n_1}{2} + \cdots + \binom{n_k}{2} = \sum\nolimits_{i=1}^{k} \frac{n_i^2 - n_i}{2} \\ &= \frac{1}{2}\left(\sum\nolimits_{i=1}^{k} n_i^2 - (n-1)\right) \leq \frac{1}{2}\left[1 + 1 + (n-1-2)^2 - n + 1\right] \\ &= \frac{1}{2}\left(n^2 - 7n + 12\right) < \binom{n-3}{2} + 2, \end{aligned}$$

a contradiction. So H' has two components, that is, $n_1 + n_2 = n - 1$. If $n_1 \geq 2$, then

$$\begin{aligned} |E(H')| &\leq \binom{n_1}{2} + \binom{n_2}{2} = \frac{n_1^2 + n_2^2 - (n-1)}{2} \\ &\leq \frac{1}{2}\left[2^2 + (n-3)^2 - n + 1\right] \\ &= \frac{1}{2}\left(n^2 - 7n + 14\right) < \binom{n-3}{2} + 2, \end{aligned}$$

thus completing the proof. □

Let $H_1 = \{v\}, H_2$ be two components of H'. We know $v \in N(w)$ (otherwise, $\delta(G) = 1$). Let $N(v) = \{w, v_1, \ldots, v_{d(v)-1}\}$. Obviously, $d(v) \leq t$, and all edges $v_i v, 1 \leq i \leq d(v) - 1$ are deleted edges. Since $|E(H_2)| = |E(H')| \geq \binom{(n-2)-1}{2} + 2$, H_2 is 2-rainbow connected by Theorem 5. Consider a 2-rainbow coloring of H_2 with colors $1, 2$. Set $c(vv_i) = 3, 1 \leq i \leq d(v) - 1, c(wv) = 1, c(e) = 3$ for all other edges incident with $w, c(e) = 2$ for all other deleted edges. Then for every $x \in V(G)\backslash w$, there is a rainbow path between w and x, and for every $x \in N(w)$, there is a rainbow path vwx. For every $x \in N^2(w)\backslash N(v)$, we know $xv \notin E(G)$, and then $N(x) \cap N(v) \neq \emptyset$, which means that there exist some $v_i, 1 \leq i \leq d(v) - 1$ with $v_i \in N(x) \cap N(v)$, i.e., there is a rainbow path between v and x. So G is 3-rainbow connected. ■

Theorem 7. *Let G be a connected graph of order $n \geq 5$. If $|E(G)| \geq \binom{n-3}{2} + 3$, then $rc(G) \leq 4$.*

Proof. We apply the proof idea from the proof of Theorem 6.

Our proof will be by induction on n. For $n = 5$, we have $f(n, n-1) = n-1 = 4 = \binom{5-3}{2} + 3$, and for $n = 6$, we have $f(n, n-2) = n = 6 = \binom{6-3}{2} + 3$. So we may assume $n \geq 7$.

By Theorem 6, we have $rc(G) \leq 3$ for $|E(G)| \geq \binom{n-2}{2} + 2$. Hence we may assume $|E(G)| \leq \binom{n-2}{2} + 1$. This implies $\delta(G) \leq \frac{(n-2)(n-3)+2}{n} = n - 5 + \frac{8}{n} < n - 3$.

Claim 4. $diam(G) \leq 4$.

***Proof of Claim* 4.** Suppose $diam(G) \geq 5$ and consider a diameter path $v_1, v_2, \ldots, v_{D+1}$ with $D \geq 5$. Then $d(v_i) + d(v_{i+3}) \leq n-2$ for $i = 1, 2, 3$, implying $|E(G)| \leq \binom{n}{2} - 3(2n-3-(n-2)) = \binom{n}{2} - 3(n-1) = \binom{n-3}{2} - 3 < \binom{n-3}{2} + 3$, a contradiction. □

Claim 5. *If* $\delta(G) = 1$*, then* $rc(G) \leq 4$.

***Proof of Claim* 5.** Let w be a vertex with $d(w) = \delta(G) = 1$, and let $H = G - w$. Then $|E(H)| \geq \binom{n-3}{2} + 3 - 1 = \binom{n-3}{2} + 2 = \binom{(n-1)-2}{2} + 2$. Hence $rc(H) \leq 3$ by Theorem 6. Take a 3-rainbow coloring for H, and set $c(e) = 4$ for the edge incident with w. Then $rc(G) \leq 4$. □

Hence we may assume $\delta(G) \geq 2$.

Case 1. There are $w_1, w_2 \in V(G), w_1w_2 \notin E(G)$, with $N(w_1) \cap N(w_2) = \emptyset$ and $d(w_1) + d(w_2) \leq n-3$.

Let $H = G - \{w_1, w_2\}$. Then $|E(H)| \geq \binom{n-3}{2} + 3 - (n-3) = \binom{n-4}{2} + 2 = \binom{(n-2)-2}{2} + 2$. We claim that H is connected. Otherwise, by the proof of Theorem 6, we know that H has at most 2 components and one of them is a single vertex. Thus $\delta(G) = 1$, a contradiction. Then $rc(H) \leq 3$ by Theorem 6. Consider a 3-rainbow coloring of H with colors $1, 2, 3$. If there is a rainbow path $P = xyz$ of length 2 between $N(w_1)$ and $N(w_2)$, where $x \in N(w_1), z \in N(w_2)$, then let $c(xy) = 1, c(yz) = 2$ and set $c(w_1x) = 3, c(w_2z) = 4$ and $c(e) = 4$ for all other edges incident with w_1 or w_2. Then G is 4-rainbow connected. If all paths of length 2 between $N(w_1)$ and $N(w_2)$ are not rainbow, then we choose a path $P = xyz$, where $x \in N(w_1), z \in N(w_2)$. Let $c(xy) = c(yz) = 1$, and then keep the colors of all the edges in $E(H)$ except for yz. Then set $c(yz) = 4, c(w_1x) = 2, c(w_2z) = 3$ and $c(e) = 4$ for all other edges incident with w_1 or w_2. It is only need to check that G is 4-rainbow connected. Since $\delta \geq 2$, then there exists a v such that $c(w_1v) = 4$. For every $w \in V(G) \setminus N(w_1)$, there is a rainbow path P from w_1 to w not containing yz. Otherwise, there is a rainbow path of length 2 between $N(w_1)$ and $N(w_2)$, and so w_1vPw is a rainbow path. For w_2, the proof is similar.

Case 2. For all $w_1, w_2 \in V(G), w_1w_2 \notin E(G)$, we have $N(w_1) \cap N(w_2) \neq \emptyset$ or $d(w_1) + d(w_2) \geq n-2$.

We know that in this case $diam(G) \leq 3$. Choose a vertex w with $d(w) = \delta(G)$, and set $d(w) = n-2-t$ with $2 \leq t \leq n-4$.

Subcase 2.1. $N^3(w) = \emptyset$. As in the proof of Theorem 5, there exist vertices $u_i \in N(w)$ such that $u_iv_i \in E(G)$ for $1 \leq i \leq t+1$, where $\overline{N}(w) = \{v_1, v_2, \ldots, v_t, v_{t+1}\}$. Let $H = G - w$, and let H' be a subgraph of H with

$V(H') = V(H)$ and $E(H') = E(H) - \{u_1v_1, \ldots, u_{t-2}v_{t-2}\}$. Then $|E(H')| \geq \binom{n-3}{2} + 3 - (n-2-t) - (t-2) = \binom{n-3}{2} - n + 7 = \binom{(n-1)-3}{2} + 3$.

If H' is connected, then by induction, H' is 4-rainbow connected. Now take a 4-rainbow coloring of H'. Let $c(u_iv_i) \in \{1, 2, 3\}$ for $i = t-1, t, t+1$. Then set $c(u_iv_i) = 1$ for $1 \leq i \leq t-2$ and $c(e) = 4$ for all edges e incident with w. Then G is 4-rainbow connected.

If H' is disconnected, we claim that H' has at most 3 components. Otherwise, $|E(H')| < \binom{n-4}{2} + 3$. If H' has exactly 3 components H_1, H_2, H_3, we may assume that $|H_3| \geq |H_2| \geq |H_1| \geq 1, |H_1| + |H_2| + |H_3| = n-1$. If $|H_2| \geq 2$, then $|E(H')| \leq \binom{|H_1|}{2} + \binom{|H_2|}{2} + \binom{|H_3|}{2} \leq 1 + \binom{n-4}{2} < \binom{n-4}{2} + 3$. So $|H_1| = |H_2| = 1$, and let $V(H_1) = \{u_1\}, V(H_2) = \{u_2\}$ and $u_1, u_2 \in N(w)$. Then $|E(H_3)| \geq \binom{n-4}{2} + 3 \geq \binom{(n-3)-1}{2} + 3$. Hence, by Theorem 5, H_3 is 2-rainbow connected. Now consider a 2-rainbow coloring of H' with colors $1, 2$. Set $c(wu_1) = 1, c(wu_2) = 2, c(e) = 4$ for all the edges e incident with w, and set $c(f) = 3$ for all edges f incident with u_1 or u_2 except for wu_1, wu_2, as well as $c(g) = 1$ for all other deleted edges g. Then G is 4-rainbow connected.

If H' has exactly 2 components H_1, H_2, we may assume that $|H_2| \geq |H_1| \geq 1$. First, $|H_1| = 1$, and let $V(H_1) = \{u_1\}$ and $u_1 \in N(w)$. Then $|E(H_2)| \geq \binom{n-4}{2} + 3 \geq \binom{(n-2)-2}{2} + 3$. Hence, by Theorem 6, H_2 is 3-rainbow connected. Now consider a 3-rainbow coloring of H' with colors $1, 2, 3$. Set $c(wu_1) = 1, c(e) = 4$ for all edges e incident with w or u_1 except for wu_1 and set $c(g) = 2$ for all other deleted edges g. Then G is 4-rainbow connected. Second, $|H_1| \geq 2$. Since $n \geq 7$, then $|H_2| \geq 3$. Thus if $|H_1| \geq 3$, we have

$$\begin{aligned}
|E(H_1)| &\geq \binom{n-4}{2} + 3 - \binom{|H_2|}{2} \\
&= \frac{1}{2}\left[|H_1|^2 - 3|H_1| + 4\right] + |H_1||H_2| - 3|H_2| - 2|H_1| + 7 \\
&\geq \binom{|H_1|-1}{2} + 1 + 3(n-4) - 3(n-1) + |H_1| + 7 \\
&\geq \binom{|H_1|-1}{2} + 1.
\end{aligned}$$

Similarly, $|E(H_2)| \geq \binom{|H_2|-1}{2} + 1$. Obviously if $|H_1| = 2$, H_1, H_2 are 2-rainbow connected. Hence when $|H_1| \geq 2$, both H_1, H_2 are 2-rainbow connected. Consider a 2-rainbow coloring of H' with colors $1, 2$. Set $c(wv) = 4$ for all $v \in V(H_1)$, $c(wv) = 3$ for all $v \in V(H_2)$, $c(uv) = 4$ for all $u \in V(H_2) \cap N(w), v \in V(H_1) \cap N^2(w)$, $c(uv) = 3$ for all $u \in V(H_1) \cap N(w), v \in V(H_2) \cap N^2(w)$, $c(e) = 1$ for all other edges e. Then G is 4-rainbow connected.

Subcase 2.2. $N^3(w) \neq \emptyset$. For every $u \in N^3(w)$, $wu \notin E(G)$ and $N(w) \cap N(u) = \emptyset$, then $d(w) + d(u) = n-2$, that is, $N(u) = N^2(w) \cup N^3(w) \backslash \{u\}$. Let

$N(w) = \{u_1, \ldots, u_{n-t-2}\}, N^2(w) = \{v_1, \ldots, v_p\}, p \geq 1, N^3(w) = \{v_{p+1}, \ldots, v_{t+1}\}$.

If $p = 1$, v_1 is a cut vertex and $G[N^2(w) \cup N^3(w)]$ is a complete graph. Let H_1, H_2 be two blocks of $G - v_1$, we may assume that H_2 is a complete graph. Let $N_{H_1}(v_1) = \{u_1, \ldots, u_s\}, 1 \leq s \leq n - t - 2$. Then $K_{2,s}$ is a spanning subgraph of $G[w, v_1, u_1, \ldots, u_s]$. If $s \geq 2$, then $K_{2,s}$ is 4-rainbow connected. Now we give a 4-coloring of $K_{2,s}$ as follows:

$$c(e) = \begin{cases} j+1, & \text{if } e = u_i w,\ i \in \{3j+1, 3j+2, 3j+3\} \text{ for } 0 \leq j \leq 2, \\ 4, & \text{if } e = u_i w \text{ for } i > 9, \\ i \bmod 3, & \text{if } e = v_1 u_i \text{ for } i \leq 9, \\ 3, & \text{if } e = v_1 u_i \text{ for } i > 9. \end{cases}$$

For every $u_k (s < k \leq n - t - 2), u_k v_j \notin E(G)$ and $N(u_k) \cap N(v_j) = \emptyset$ for $2 \leq j \leq t + 1$, then $N(u_k) = N(w) \cup \{w\} \backslash \{u_k\}$. Set $c(u_k u_j) = c(w u_j)$ for $1 \leq j \leq s$, $c(e) = 1$ for all other edges e in $E(H_1)$, $c(e) = 4$ for $e \in E(H_2)$. Then G is 4-rainbow connected. If $s = 1$, then G is 3-rainbow connected.

If $p = 2$, let $H_1 = G[w \cup N(w) \cup N^2(w)], H_2 = G[N^2(w) \cup N^3(w)]$, then $|H_1| + |H_2| = n + 2, |H_1| \geq 5, |H_2| \geq 3$. If $|H_2| = 3$, then $d(v_3) = 2 = d(w)$, thus $n = 6$. If $|H_2| = 4, n = 7$, set $c(wu_1) = 4, c(wu_2) = 3, c(u_1 v_1) = 2, c(u_2 v_2) = 1, c(e) = 1$ for all $e \in E(H_2)$, then G is 4-rainbow connected. If $|H_2| = 4, n \geq 8$, $|E(H_1)| \geq \binom{n-3}{2} + 3 - 5 = \binom{(n-2)-2}{2} + n - 6$, then H_1 is 3-rainbow connected. Consider a 3-rainbow coloring of H_1 with $2, 3, 4$. Set $c(e) = 1$ for all $e \in E(H_2)$, then G is 4-rainbow connected. If $|H_2| \geq 5, |H_1| \geq 6$, then

$$\begin{aligned} |E(H_1)| &\geq \binom{n-3}{2} + 3 - \binom{|H_2|}{2} \\ &= \frac{1}{2}\left[|H_1|^2 - 5|H_1| + 6\right] + 2 + |H_1||H_2| - 3|H_1| - 5|H_2| + 13 \\ &\geq \binom{|H_1| - 2}{2} + 2 + 5(n + 2 - 5) - 5(n + 2) + 2|H_1| + 13 \\ &\geq \binom{|H_1| - 2}{2} + 2. \end{aligned}$$

Hence H_1 is 3-rainbow connected. Consider a 3-rainbow coloring of H_1 with colors $2, 3, 4$. Set $c(e) = 1$ for all $e \in E(H_2)$, then G is 4-rainbow connected. When $|H_2| \geq 5, |H_1| = 5$, set $c(wu_1) = 4, c(wu_2) = 3, c(u_1 v_1) = 2, c(u_2 v_2) = 1, c(e) = 1$ for all $e \in E(H_2)$, then G is 4-rainbow connected.

Now we may assume that $p \geq 3$. For every $v_i \in N^2(w)$, there is a vertex $u_i \in N(w)$ such that $u_i v_i \in E(G)$. Let H be the graph be deleting w and edges $u_i v_i$ for $v_i \in N^2(w) \backslash \{v_1, v_2, v_3\}$ and edges $v_1 v_i$ for $p+1 \leq i \leq t+1$, then $|E(H)| \geq \binom{n-4}{2} + 3$. If H is connected, then by induction, H is 4-rainbow connected. Consider a 4-rainbow coloring of H with colors $1, 2, 3, 4$. Let $c(u_i v_i) \in \{1, 2, 3\}$ for $i = 1, 2, 3$.

We may assume that $c(u_1v_1) = 1$. Set $c(e) = 4$ for all edges e incident with w, $c(v_1v_i) = 2, p+1 \leq i \leq t+1$, $c(v_iu_i) = 3$ for all other edges between $N(w)$ and $N^2(w)$. Then G is 4-rainbow connected.

If H is disconnected, then similarly as in the proof of Subcase 2.1, H has at most 3 components. If H has exactly 3 components with two single vertices in $N(w)$, denoted by u, u', then H_3 is 2-rainbow connected. Consider a 2-rainbow coloring of H_3 with colors $1, 2$. Let $c(u_1v_1) = 1$, set $c(wu) = c(wu') = 3, c(e) = 4$ for all edges e incident with w, $c(v_1v_i) = 2, p+1 \leq i \leq t+1$, $c(f) = 3$ for all edges f incident with u except for wu, $c(g) = 4$ for all edges g incident with u' except for wu', $c(h) = 1$ for all the remaining edges h. Then G is 4-rainbow connected.

Assume that H has exactly two components H_1, H_2. First, $|H_1| = 1$, let $V(H_1) = \{u\} \subseteq N(w)$, then H_2 is 3-rainbow connected by the proof of Subcase 2.1. Now consider a 3-rainbow coloring of H_2 with colors $1, 2, 3$. Set $c(wu) = 1, c(e) = 4$ for all edges e incident with w or u except for wu, $c(f) = 1$ for all the remaining edges f. Then G is 4-rainbow connected. Second, $|H_1| \geq 2$, then both H_1, H_2 are 2-rainbow connected by the proof of Subcase 2.1. We may assume that H_2 contains $N^3(w)$. Now consider a 2-rainbow coloring of H_1, H_2 with colors $1, 2$. Set $c(wv) = 3$ for all $v \in V(H_1)$, $c(wv) = 4$ for all $v \in V(H_2)$ $c(v_1v_i) = 4, p+1 \leq i \leq t+1, c(e) = 3$ for all the remaining edges e. Then G is 4-rainbow connected. ■

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