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CHARACTERIZATIONS OF THE FAMILY OF ALL GENERALIZED LINE GRAPHS—FINITE AND INFINITE—AND CLASSIFICATION OF THE FAMILY OF ALL GRAPHS WHOSE LEAST EIGENVALUES ≥ -2

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Abstract

The infimum of the least eigenvalues of all finite induced subgraphs of an infinite graph is defined to be its least eigenvalue. In [P.J. Cameron, J.M. Goethals, J.J. Seidel and E.E. Shult, *Line graphs, root systems, and elliptic geometry*, J. Algebra **43** (1976) 305–327], the class of all finite graphs whose least eigenvalues $\geqslant -2$ has been classified: (1) *If a* (*finite*) *graph is connected and its least eigenvalue is at least* -2, *then either it is a generalized line graph or it is represented by the root system* E_8. In [A. Torgašev, *A note on infinite generalized line graphs*, in: Proceedings of the Fourth Yugoslav Seminar on Graph Theory, Novi Sad, 1983 (Univ. Novi Sad, 1984) 291–297], it has been found that (2) *any countably infinite connected graph with least eigenvalue* $\geqslant -2$ *is a generalized line graph*. In this article, the family of all generalized line graphs—countable and uncountable—is described algebraically and characterized structurally and an extension of (1) which subsumes (2) is derived.

Keywords: generalized line graph, enhanced line graph, representation of a graph, extended line graph, least eigenvalue of a graph.

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For unexplained graph theoretic terms and notation, the reader is referred to [15]. For information on Hilbert spaces, we rely on [6]. Each graph considered in this article is simple; its order need not be finite. Let G be a graph; let a, b be two vertices of G; we write $\langle\langle a, b\rangle\rangle = 1$ to mean that a, b are adjacent whereas $\langle\langle a, b\rangle\rangle = 0$ implies that they are not adjacent. The set $N(a) \cup \{a\}$, known as the closed neighbourhood of a, is denoted by $N[a]$ or $N_G[a]$. The *order* of G is denoted by $|G|$. Any graph obtained from G by deleting some vertices—known as an *induced subgraph* of G—is called in this article an *innergraph* of G. To denote that a graph H is an innergraph of G, we write $H \preccurlyeq G$. If $X \subseteq V(G)$, then the innergraph of G with vertex set X is denoted by $G[X]$; any innergraph with a finite vertex set $\{x_1, x_2, \ldots, x_n\}$ is denoted by $G[x_1, x_2, \ldots, x_n]$ also. Let H be a finite subgraph of G; if there is a vertex p in $V(G) \setminus V(H)$ such that $|N(p) \cap V(H)|$ is odd, then H is called *odd* in G; otherwise it is *even* in G. Let e be an edge of G; when the degree of one of its endpoints is 1, it is called a *pendant edge*; in that case, the other endpoint is said to *support* e. As in [11], the infimum of the least eigenvalues of all finite innergraphs of G—note that when G is finite, this infimum coincides with the least eigenvalue of G—is defined to be the *least eigenvalue* of G and denoted by $\lambda(G)$. The *line graph* $L(G)$ is defined as follows: its vertex set is $E(G)$; two vertices of $L(G)$ are adjacent if and only if they have a common vertex in G. A family $\mathcal{F}$ of subgraphs of G is called a *decomposition* of G, if every edge of G appears in exactly one member of $\mathcal{F}$. The following results have been obtained in [8] and [12], respectively:

Theorem 1. *A graph G is a line graph if and only if it decomposes into complete subgraphs such that each vertex of G appears in at most two of these subgraphs.*

Theorem 2. *A graph G is a line graph if and only if no innergraph is $K_{1,3}$ and whenever an innergraph is $K_{1,1,2}$, one of its triangles is even in G.*

The above results are usually confined to finite graphs. However, since their proofs involve neither counting arguments nor induction, they work for infinite graphs also. (See [15, Pages 280–82]; for proving the second result for infinite graphs, we need the following: If H is a complete subgraph of a graph G, then there exists a maximal complete subgraph of G containing H; this fact can be proved by using Zorn's lemma.)

Let $\mathfrak{L}$ denote the family of all graphs with least eigenvalues $\geqslant -2$. A notable property of finite line graphs is that they belong to $\mathfrak{L}$. (For information in this regard and for a proof in particular, see [2]. If G is an infinite line graph, then for any finite innergraph H of G, $\lambda(H) \geqslant -2$ because H also is a line graph whence $\lambda(G) = \inf\{\lambda(H) : H \preccurlyeq G \text{ and } |H| < \infty\} \geqslant -2$. Therefore infinite line graphs also belong to $\mathfrak{L}$.) This fact has prompted many authors to study intensively the set of all finite graphs in $\mathfrak{L}$, denoted by $\mathfrak{L}_f$ in this article. Hoffman has found an important subfamily of $\mathfrak{L}_f$, whose members are called generalized line graphs. (See [7].) For our purpose, we extend Hoffman's definition of these graphs, as done in [11].

Definition 3. Any graph in which every vertex is adjacent to all other vertices except one—i.e., any graph obtained from a complete graph by removing a perfect matching—is called a *cocktail party graph*. For any nonnegative integer n, the cocktail party graph with $2n$ vertices is denoted by $\mathrm{CP}(n)$. Let G be a graph and

$\{H_\alpha : \alpha \in V(G)\}$ be a collection of cocktail party graphs such that for each $\alpha \in V(G)$, $V(H_\alpha) \cap E(G) = \emptyset$ and for all distinct $\alpha, \beta \in V(G)$, $V(H_\alpha) \cap V(H_\beta) = \emptyset$. Then the *generalized line graph* $L[G;\ H_\alpha, \alpha \in V(G)]$ is obtained from the union of $L(G)$ and the graphs H_α, $\alpha \in V(G)$ by forming additional edges: a vertex e in $L(G)$ is adjacent to all vertices in H_α whenever α is an endpoint of e in G. When $V(G)$ is finite, say $\{v_1, v_2, \ldots, v_n\}$ and for each $i \leqslant n$, the cocktail party graph associated with v_i is a finite graph, say $\mathrm{CP}(\sigma_i)$, then the generalized line graph is denoted by $L(G; \sigma_1, \sigma_2, \ldots, \sigma_n)$ also. Let $\mathcal{G}$ denote the family of all generalized line graphs.

Remark 4. (1) The generalized line graph described above coincides with $L(G)$ when all cocktail party graphs are null; it is a cocktail party graph when $G = K_1$. Thus line graphs and cocktail party graphs belong to $\mathcal{G}$.

(2) Let Γ be an innergraph of a generalized line graph $L[G;\ H_\alpha, \alpha \in V(G)]$. For each $\alpha \in V(G)$, let $A_\alpha = \{v \in V(\Gamma) \cap V(H_\alpha) :$ for some $x \in V(\Gamma) \cap V(H_\alpha) \setminus \{v\},\ \langle\langle x, v\rangle\rangle = 0\}$ and $B_\alpha = (V(\Gamma) \cap V(H_\alpha)) \setminus A_\alpha$. (Note that for each $\alpha \in V(G)$, $\Gamma[A_\alpha]$ is a cocktail party graph.) Let G' be the graph defined according to the following three conditions: $V(G') = V(G) \cup [\cup_{\alpha \in V(G)} B_\alpha]$; $V(\Gamma) \cap E(G) \subset E(G')$; if $\alpha \in V(G)$, then for each $x \in B_\alpha$, $\alpha x \in E(G')$. Now, for each $\alpha \in V(G')$, let H'_α be the cocktail party graph defined as follows: if $\alpha \in V(G)$, then $H'_\alpha = \Gamma[A_\alpha]$; if $\alpha \in (V(G') \setminus V(G))$, then $H'_\alpha = \mathrm{CP}(0)$. It can be verified that Γ can be taken as the generalized line graph $L[G';\ H'_\alpha, \alpha \in V(G')]$. Thus it follows that any innergraph of a generalized line graph also is a generalized line graph.

(3) Note that $L(K_2; 1, 1) = K_{1,4}$ whence $K_{1,3}$ also belongs to $\mathcal{G}$ and $L(K_3; 1, 0, 0) = K_{1,1,3}$.

(4) Since any finite generalized line graph belongs to $\mathfrak{L}_f$—see [7] or [2] for a proof—it follows that $\mathcal{G} \subset \mathfrak{L}$.

Let $\mathcal{G}_f$ be the family of all finite generalized line graphs; a characterization of this family, analogous to that of the family of all finite line graphs given by Theorem 1 has been obtained in [4]. The following definition conceptualizes the characterization given by [4].

Definition 5. A graph Ω is called an *extended line graph*, if there exists a decomposition $\mathfrak{F} = \{F_j : j \in J\}$ of Ω such that the following hold.

(1) For any $j \in J$, every vertex of F_j is adjacent to all other vertices of F_j except at most one vertex, i.e., F_j can be obtained from a complete graph by removing a matching.

(2) For all distinct $j, k \in J$, F_j and F_k have at most one common vertex.

(3) Every vertex lies in at most two members of $\mathfrak{F}$.

(4) If a vertex v lies in two distinct members F_j, F_k of $\mathfrak{F}$, then v is adjacent to all vertices in $[V(F_j) \cup V(F_k)] \setminus \{v\}$.

Let $\mathcal{X}$ be the family of all extended line graphs; note that by Theorem 1, line graphs belong to this family.

Remark 6. Let $\Omega = L[G;\ H_\alpha, \alpha \in V(G)]$ be a generalized line graph; for each $\alpha \in V(G)$, let $F_\alpha = \Omega[V(H_\alpha) \cup E_\alpha]$ where E_α is the set of all edges in G which are incident with α. Taking $J = V(G)$, it is easy to check that (1), (2), (3) and (4) of Definition 5 hold, i.e., Ω is an extended line graph. Therefore, $\mathcal{G} \subseteq \mathcal{X}$.

Let W be a subset of a Hilbert space such that the norm of each vector in W is $\sqrt{2}$. (Every Hilbert space $\mathbb{H}$ considered in this article is real; the inner product of any two vectors $\alpha, \beta \in \mathbb{H}$ is denoted by $\langle \alpha, \beta \rangle$.) If ψ is a map from the vertex set of a graph G to W such that for all distinct $x, y \in V(G)$, $\langle \psi(x), \psi(y) \rangle = \langle\!\langle x, y \rangle\!\rangle$, then ψ is called a *representation* of G in W. The family of all representable graphs is denoted by $\mathcal{R}$. If a graph has a representation in some set $\{\pm\mu \pm \nu : \mu, \nu \in \mathcal{O} \text{ and } \mu \neq \nu\}$ where $\mathcal{O}$ is an orthonormal set in a Hilbert space, then it is called *amicable*. The family of all amicable graphs is denoted by $\mathcal{A}$.

Let $\{e_1, e_2, \ldots, e_8\}$ be an orthonormal basis of $\mathbb{R}^8$. The set E_8—known as an exceptional root system in the literature—is defined to be

$$\{\pm e_i \pm e_j : 1 \leqslant i < j \leqslant 8\} \cup \left\{ \tfrac{1}{2} \textstyle\sum_{i=1}^{8} \varepsilon_i e_i : \text{for all } i \leqslant 8,\ \varepsilon_i = \pm 1 \text{ and } \prod_{i=1}^{8} \varepsilon_i = 1 \right\}.$$

A representation of a graph in E_8 is shown in Figure 1. Note that this graph appears in Figure 2. Later, we will prove that no graph in this figure is amicable. Therefore, $\mathcal{A} \subsetneq \mathcal{R}$.

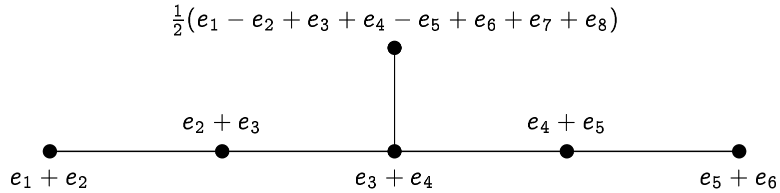

Figure 1. A representation of a graph in E_8.

Lemma 7. *Every extended line graph is amicable.*

Proof. Let Ω be an extended line graph; then there exists a decomposition $\{F_j : j \in J\}$ of Ω such that the conditions (1), (2), (3) and (4) of Definition 5 hold. For each $j \in J$, let A_j be a subset of $V(F_j)$ such that for each $v \in V(F_j)$, there is exactly one vertex $x \in A_j$ with $\langle\!\langle v, x \rangle\!\rangle = 0$ (x and v may be same). We can assume that $\mathcal{O} := J \cup V(\Omega)$ is an orthonormal set in a Hilbert space and Ω does not have isolated vertices. Let $\xi : V(\Omega) \to \{\pm\mu \pm \nu : \mu, \nu \in \mathcal{O} \text{ and } \mu \neq \nu\}$ be the map defined as follows: Let $u \in V(\Omega)$. If u is a vertex of two members of $\mathfrak{F}$, say F_α, F_β, then let $\xi(u) = \alpha + \beta$. Suppose that u belongs to exactly one member, say F_α. If $u \in A_\alpha$, let $\xi(u) = \alpha + u$; otherwise let $\xi(u) = \alpha - v$ where v is the vertex in A_α such that $\langle\langle u, v\rangle\rangle = 0$. It is easy to verify that ξ is a representation of Ω. ∎

It has been proved that $\{G \in \mathfrak{L}_f \setminus \mathcal{G}_f : G \text{ is connected}\}$ is finite—see [2, Theorem 4.10]. This fact and the natural relationship between line graphs and generalized line graphs have motivated various authors to study $\mathcal{G}_f$ comprehensively. Let $\mathcal{M}_f$ be the class of all minimal forbidden graphs for $\mathcal{G}_f$. (Let $\mathfrak{G}$ be a hereditary family of graphs; if a graph G does not belong to $\mathfrak{G}$, whereas every proper innergraph of G belongs to $\mathfrak{G}$, then G is called a *minimal forbidden graph* for $\mathfrak{G}$.) Note that $\mathcal{M}_f$ determines $\mathcal{G}_f$: A finite graph G is a generalized line graph if and only if no innergraph of G belongs to $\mathcal{M}_f$. Various algebraic properties of $\mathcal{G}_f$ have been found in [4, 5] and five different methods of computing $\mathcal{M}_f$ have been found in [4, 9, 13, 5, 14]. Countably infinite graphs in $\mathfrak{L}$ also have been studied: in [10], it has been shown that any countably infinite connected graph with least eigenvalue $\geqslant -2$ is a generalized line graph and in [11], all countably infinite connected graphs with least eigenvalues > -2 have been determined. The current article generalizes many of the results on $\mathfrak{L}_f$. We describe $\mathcal{G}$ by using vectorial representability of its members, characterize it structurally and find the set of all minimal forbidden graphs for $\mathcal{G}$, denoted in this article by $\mathcal{M}$. In this process we obtain a classification of $\mathfrak{L}$, and prove that $\mathcal{M} = \mathcal{M}_f$ and that $\mathcal{G}$ is determined by $\mathcal{M}$. The main tool for proving our results is a notion introduced in [14]:

Definition 8. A graph G is called an *enhanced line graph* when the following conditions hold for every innergraph H of G.

(E1) If $H = C_4$, then it is even in G.

(E2) If $H = K_{1,3}$, then two vertices of H have same neighbourhood in G.

(E3) If $H = K_{1,1,2}$, then either its nonadjacent vertices have same neighbourhood in G or one of its triangles is even in G.

(E4) If $H = K_{1,1,3}$, then for some $x, y \in V(H)$, $N_G(x) = N_G(y)$ and $H - \{x, y\}$ is an even triangle in G.

(E5) If H is connected and $V(H)$ has three distinct vertices such that their neighbourhoods in H are same, then H is either $K_{1,3}$ or $K_{1,4}$ or $K_{1,1,3}$.

Let $\mathcal{E}$ denote the family of all enhanced line graphs. By using Theorem 2, it can be easily verified that every line graph belongs to this family; we can even prove a generalization of this fact: $\mathcal{A}$ is a subfamily of $\mathcal{E}$; the latter fact itself is subsumed by Proposition 9, to derive which, we need the following: Let ψ be a representation of a graph G in a set $\{\pm\mu\pm\nu : \mu,\nu \in \mathcal{O}$ and $\mu \neq \nu\}$ where $\mathcal{O}$ is an orthonormal set in a Hilbert space. If a,b are two distinct vertices such that $\psi(a)$ and $\psi(b)$ are linear combinations of the same vectors in $\mathcal{O}$—i.e., $\psi(a)$ and $\psi(b)$ are of the forms $\pm\alpha\pm\beta$ and $\pm\alpha\mp\beta$—then a and b are called *associates with respect to* ψ. Note that a vertex cannot be an associate of two different vertices. If two vertices a,b are associates, then $N(a) = N(b)$ because for any $x \in V(G)\setminus\{a,b\}$, $\langle\psi(a),\psi(x)\rangle = \langle\psi(b),\psi(x)\rangle$ and $\langle\psi(a),\psi(b)\rangle = 0$.

Proposition 9. *Let G be a graph such that every finite innergraph of G is amicable. Then G is an enhanced line graph.*

Proof. Let $\mathcal{O}$ be a countably infinite orthonormal set in a Hilbert space. Let $\mathcal{O}^* = \{-\nu,\nu : \nu \in \mathcal{O}\}$ and $W = \{\pm\mu\pm\nu : \mu,\nu \in \mathcal{O}$ and $\mu \neq \nu\}$; note that by the hypothesis, every finite innergraph of G has a representation in W. First, suppose that a,b,c,d are vertices of G such that $H := G[a,b,c,d] = C_4$ and $\langle\!\langle a,c\rangle\!\rangle = 0$. Let $p \in V(G)\setminus\{a,b,c,d\}$ and ψ be a representation of $G[a,b,c,d,p]$ in W. It is easy to verify that $\|\psi(a)-\psi(b)+\psi(c)-\psi(d)\|^2 = 0$ whence $\psi(a)+\psi(c) = \psi(b)+\psi(d)$; therefore $|N(p)\cap V(H)| = \sum_{x\in V(H)}\langle\!\langle p,x\rangle\!\rangle = 2\langle\psi(p),\psi(a)+\psi(c)\rangle$; so, it follows that H is even, i.e., the conclusion of (E1) holds.

Now, suppose that (E2) does not hold. Then, there are vertices p,a,b,c such that $H := G[p,a,b,c] = K_{1,3}$, $\deg_H p = 3$ and a,b,c have different neighbourhoods in G. Therefore, there are three vertices x,y,z such that in $K := G[p,a,b,c,x,y,z]$, a,b,c have different neighbourhoods. Let ψ be a representation of K in W. It is easy to verify that two of $\{a,b,c\}$ are associates with respect to ψ; these two have same neighbourhood in K—a contradiction.

Now suppose that (E3) does not hold. Then there are vertices a,b,p,q such that $G[a,b,p,q] = K_{1,1,2}$, $\langle\!\langle p,q\rangle\!\rangle = 0$, $N(p) \neq N(q)$ and both $G[a,b,p]$ and $G[a,b,q]$ are odd. Therefore, there are vertices x,y,z such that in $K := G[a,b,p,q,x,y,z]$, p,q have different neighbourhoods and both $G[p,a,b]$ and $G[q,a,b]$ are odd. Let ψ be a representation of K in W. Then for some $\alpha,\beta,\gamma \in \mathcal{O}^*$, $\{\psi(a),\psi(b)\} = \{\alpha+\beta,\alpha+\gamma\}$. If $\langle\alpha,\psi(p)\rangle \neq 0 \neq \langle\alpha,\psi(q)\rangle$, then $\psi(p),\psi(q)$ are associates whence $N_K(p) = N_K(q)$—a contradiction. Therefore, we can assume that $\langle\alpha,\psi(p)\rangle = 0$. Then $\psi(p) = \beta+\gamma$ whence $G[a,b,p]$ is even in K—again, a contradiction.

Next, suppose that (E4) does not hold; i.e., there are vertices p,q,a,b,c such that $H := G[p,q,a,b,c] = K_{1,1,3}$, $\deg_H p = \deg_H q = 4$ and the conclusion of (E4) does not hold. Then, there are vertices $x,y,z \in [V(G)\setminus V(H)]$ such that the following statements hold:

- either $\langle\!\langle x,b\rangle\!\rangle \neq \langle\!\langle x,c\rangle\!\rangle$ or $|N(x)\cap\{p,q,a\}|$ is odd;
- either $\langle\!\langle y,c\rangle\!\rangle \neq \langle\!\langle y,a\rangle\!\rangle$ or $|N(y)\cap\{p,q,b\}|$ is odd;
- either $\langle\!\langle z,a\rangle\!\rangle \neq \langle\!\langle z,b\rangle\!\rangle$ or $|N(z)\cap\{p,q,c\}|$ is odd.

Thus the conclusion of (E4) does not hold when G is replaced by $K := G[p, q, a, b, c, x, y, z]$. Let ψ be a representation of K in W. Then for some $\alpha, \beta, \gamma, \delta \in \mathcal{O}^*$, $\{\psi(p), \psi(a), \psi(b), \psi(c)\} = \{\alpha + \beta, \beta + \gamma, \alpha + \delta, \alpha - \delta\}$; from this we find that $\psi(q) = \alpha + \gamma$ whence the vertices which are associates have same neighbourhood in K and the rest of $V(H)$ form an even triangle in K—a contradiction.

Now, suppose that a, b, c are distinct vertices of a connected innergraph H such that $N_H(a) = N_H(b) = N_H(c)$. Let $p \in N_H(a)$; then $G[p, a, b, c] = K_{1,3}$. Let K be a maximal connected innergraph of H such that $p, a, b, c \in V(K)$ and $|K| \leqslant 6$. Let ψ be a representation of K in W. Then for some $\alpha, \beta, \gamma, \delta \in \mathcal{O}^*$, $\{\psi(p), \psi(a), \psi(b), \psi(c)\} = \{\alpha + \beta, \beta + \gamma, \alpha + \delta, \alpha - \delta\}$. If $|K| = 4$, then $H = K$ whence the conclusion of (E5) holds. So, let q be a vertex in $V(K) \setminus \{p, a, b, c\}$ such that $G[p, q, a, b, c]$ is connected; note that $\psi(q)$ is either $\alpha + \gamma$ or $\beta - \gamma$; in each case, the connectivity of K and the property of a, b, c ensure that $V(K)$ does not have any other vertex of G, whence $K = K_{1,4}$ or $K_{1,1,3}$. Now, by the choice of K, it follows that $H = K$. Thus the conclusion of (E5) holds.

Summarizing, we find that all the conditions of Definition 8 hold; therefore, G is an enhanced line graph. ■

Using Theorem 2 effectively, the set of all minimal forbidden graphs for the family of all line graphs has been found in [1]. In this article, the next result—its finite version has been derived in [14]—is similarly used for determining the set of all minimal forbidden graphs for the family of all generalized line graphs.

Theorem 10. *Any enhanced line graph G—possibly infinite—is a generalized line graph.*

Proof. Since every component of G is an enhanced line graph and the conclusion holds for G when it does for every component, we can assume that G is connected; let its vertex set be V. If three distinct vertices of G have same neighbourhood, then by connectivity of G and by (E5), $G \in \{K_{1,3}, K_{1,4}, K_{1,1,3}\}$ whence by Remark 4, $G \in \mathcal{G}$. Therefore, we assume that for each $v \in V$, $|\{x \in V : N(x) = N(v)\}| \leqslant 2$. Let U be a subset of V such that for each $x \in V$, there is exactly one vertex $u \in U$ with $N(u) = N(x)$. It is easy to verify that $F := G[U]$ is connected. We can assume that $F \not\leqslant K_3$ for otherwise being an innergraph of CP(3), $G \in \mathcal{G}$ by Remark 4. Now from (E2), (E3) and Theorem 2 it follows that F is a line graph, i.e., there is a graph I with $L(I) = F$.

Let q be any vertex in $V \setminus U$. Then there is a (unique) vertex $p \in U$ with $N(p) = N(q)$. If a, b are two distinct nonadjacent vertices in $N_F(p)$, then by the choice of U, there is a vertex x with $\langle\langle x, a\rangle\rangle \neq \langle\langle x, b\rangle\rangle$ whence $G[p, a, q, b]$ is odd—a contradiction to (E1). Therefore, $N_F[p]$ is a clique. Now we claim that p is a pendant edge in the graph I. Otherwise, there are edges a, b in $E(I) \setminus \{p\}$ such that they are incident with different ends of p. Since $G[p, a, b] = K_3$, in I, p, a, b form a triangle. Therefore, $G[p, a, b]$ is even in F. Since $N_F[p]$ is a clique, it follows that $\deg_F p = 2$. Since $F \neq K_3$, there is a vertex $r \in U \setminus \{p, a, b\}$ such that $G[p, a, b, r]$ is connected. Obviously $\langle\langle a, r\rangle\rangle = \langle\langle b, r\rangle\rangle = 1$ and $\langle\langle r, p\rangle\rangle = 0$. Since $N(r) \neq N(p)$, there must be a vertex $x \in U \setminus \{a, b\}$ with $\langle\langle r, x\rangle\rangle = 1$. Since $G[p, a, b]$ is even, $\langle\langle a, x\rangle\rangle = \langle\langle b, x\rangle\rangle$. Now we find that (E4) is violated. So, our claim holds.

For each $\alpha \in V(I)$, let $X_\alpha = \{u \in U :$ for some $v \in (V \setminus U)$, $N(u) = N(v)$ and α supports u in $I\}$ and $Y_\alpha = \{v \in (V \setminus U) :$ for some $u \in X_\alpha$, $N(v) = N(u)\}$. Let K be the spanning subgraph of I whose edge set is $U \setminus \cup_{\alpha \in V(I)} X_\alpha$. Now we observe the following: $L(K) = G[U \setminus \cup_{\alpha \in V(K)} X_\alpha]$; $\{E(K), X_\alpha, Y_\alpha : \alpha \in V(K)\}$ is a partition of V; for each $\alpha \in V(K)$, $H_\alpha := G[X_\alpha \cup Y_\alpha]$ is a cocktail party graph; if $\alpha \in V(K)$, $u \in E(K)$ and $v \in V(H_\alpha)$, then $\langle\langle u, v\rangle\rangle = 1$ if and only if u is incident with α in K; for all distinct $\alpha, \beta \in V(K)$, there is no edge from $X_\alpha \cup Y_\alpha$ to $X_\beta \cup Y_\beta$. From the preceding five facts, it follows that G is $L[K;\ H_\alpha, \alpha \in V(K)]$. ∎

The next result yields a set of different descriptions of $\mathcal{G}$, including a characterization using which $\mathcal{M}$ is computed by Theorem 13.

Theorem 11. *For any graph G—possibly infinite—the following are equivalent.*

(1) *G is a generalized line graph.*

(2) *G is an extended line graph.*

(3) *G is an amicable graph.*

(4) *G is an enhanced line graph.*

Proof. Remark 6 is (1) $\Rightarrow$ (2); Lemma 7 is (2) $\Rightarrow$ (3); by Proposition 9, (3) $\Rightarrow$ (4); Theorem 10 is (4) $\Rightarrow$ (1). ∎

Theorem 12 generalizes the result that *every countably infinite connected graph with least eigenvalue $\geqslant -2$ is a generalized line graph* [10] and yields an objective of this article, viz., the classification of $\mathfrak{L}$.

Theorem 12. *For a connected graph G—possibly infinite—the following are equivalent.*

(1) *The least eigenvalue of G is at least -2.*

(2) *G is representable.*

(3) *Either G is a generalized line graph or it has a representation in E_8.*

Proof. It is a well known fact that the result holds when G is finite. (For a proof, see [2]; for additional details, see [3].) So, let us assume that G is infinite. First, suppose that (1) holds. Let X be a finite subset of $V(G)$. We can choose a finite subset Y of $V(G)$ such that $X \subset Y$, $G[Y]$ is connected and it cannot be represented by E_8. (Note that E_8 is finite.) Since $\lambda(G[Y]) \geqslant -2$, by the fact mentioned in the beginning of the proof, $G[Y]$ is a generalized line graph; therefore, $G[X]$ also is a generalized line graph whence by Theorem 11, we find that $G[X] \in \mathcal{A}$; now by Proposition 9, $G \in \mathcal{E}$ whence by Theorem 10, $G \in \mathcal{G}$; therefore, (3) holds.

Now, suppose that (3) holds. Since G cannot have a representation in E_8, $G \in \mathcal{G}$; therefore, by Theorem 11, $G \in \mathcal{A}$; since $\mathcal{A} \subset \mathcal{R}$, (2) holds. Next, suppose that (2) holds. Let X be any finite subset of $V(G)$. Then $G[X] \in \mathcal{R}$ whence by the fact mentioned in the beginning of this proof, $\lambda(G[X]) \geqslant -2$. Therefore, $\lambda(G) \geqslant -2$, i.e., (1) holds. ■

It is easy to verify that $\mathcal{M}_f \subseteq \mathcal{M}$; let $G \in \mathcal{M}$; then for each $v \in V(G)$, $G - v \in \mathcal{G}$; therefore, G is finite for otherwise by Proposition 9 and Theorem 11, G would belong to $\mathcal{G}$. Therefore, $G \in \mathcal{M}_f$. Thus it follows that $\mathcal{M}_f$ and $\mathcal{M}$ are same. In [14], $\mathcal{M}_f$ has been computed by using the finite version of Theorem 10 but not directly. (See [4, 9, 5] for different methods.) This has been done in [13] and [3] by using results for finite graphs which are similar to Theorem 10. For the sake of completeness, we give a method of computing $\mathcal{M}$ which is similar to the method of [13] but shorter.

Theorem 13. *A graph G—possibly infinite—is a generalized line graph if and only if no innergraph of G belongs to $\mathfrak{M} := \{G_i : 1 \leqslant i \leqslant 31\}$.* (See Figure 2.)

Proof. Routinely but easily, the following can be verified.

- For each $i \in \{10, 11\}$, (E1) does not hold in G_i.
- For each $i \in \{18, 23, 24, 25, 26, 27, 28, 29, 30, 31\}$, (E2) does not hold in G_i.
- For each $i \in \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 25, 26, 28, 29, 30\}$, (E3) does not hold in G_i.
- For each $i \in \{4, 7, 8, 9\}$, (E4) does not hold in G_i.
- For each $i \in \{1, 2, 3, 4, 5, 6, 7\}$, (E5) does not hold in G_i.

From this information, it follows that for each $i \leqslant 31$, $G_i \notin \mathcal{E}$; therefore, if $G \in \mathcal{G}$, then by Theorem 11, no innergraph of G belongs to $\mathfrak{M}$. Conversely, proving the following is enough: if a graph is not an enhanced line graph, then one of its innergraphs belongs to $\mathfrak{M}$. Let $G \notin \mathcal{E}$. Then for an innergraph H of G, one of the conditions of Definition 8 fails.

Case E1. H is isomorphic to C_4 and odd in G. Then, there is a vertex $p \in V(G) \setminus V(H)$ such that $|N(p) \cap V(H)|$ is odd whence $G[\{p\} \cup V(H)]$ is either G_{10} or G_{11}.

Case E2. $H = K_{1,3}$ and all vertices of H have different neighbourhoods. Let $V(H) = \{p, a, b, c\}$ and assume that $\deg_H p = 3$. Since $N(a) \neq N(b)$, there is a vertex $x \in V(G)$ with $\langle\langle x, a\rangle\rangle \neq \langle\langle x, b\rangle\rangle$. Let $S = G[p, a, b, c, x]$. Leaving the

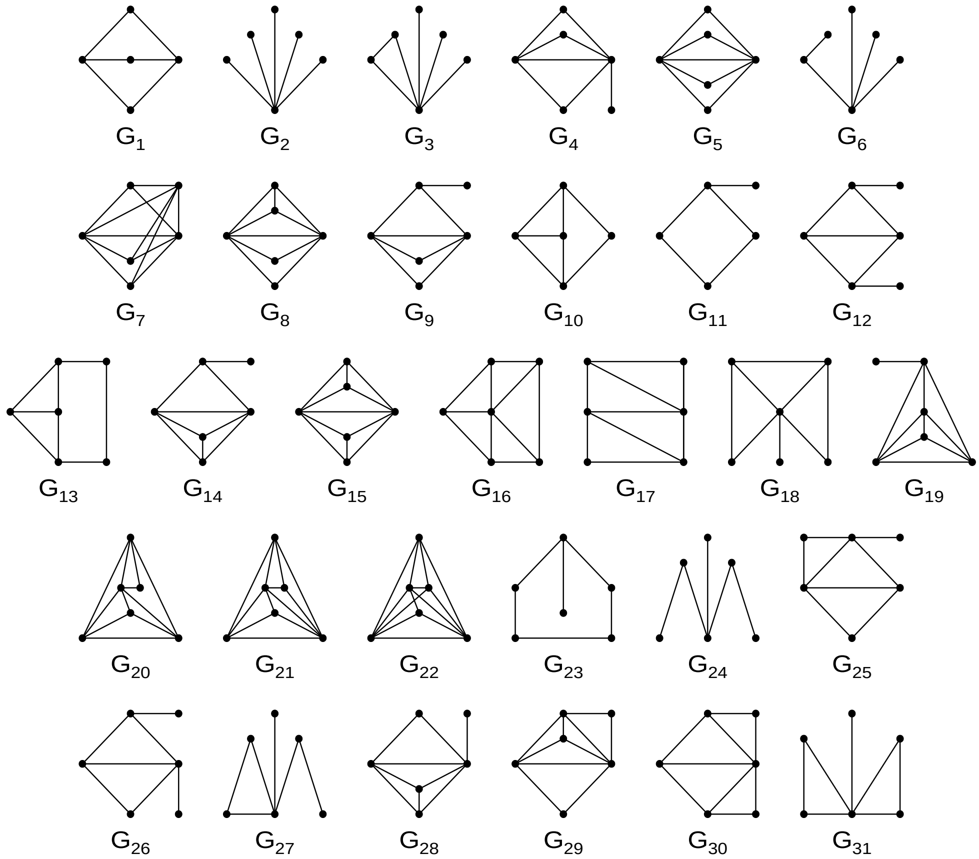

Figure 2. The set of all minimal forbidden graphs for $\mathcal{E}$.

possibility that $x \in (N(c) \setminus N(p))$, for then $S = G_{11}$, we find that S is one of the graphs H_1, H_2, H_3 in Figure 3. Note that $N_S(c) \neq N_S(a)$ or $N_S(b)$. Therefore,

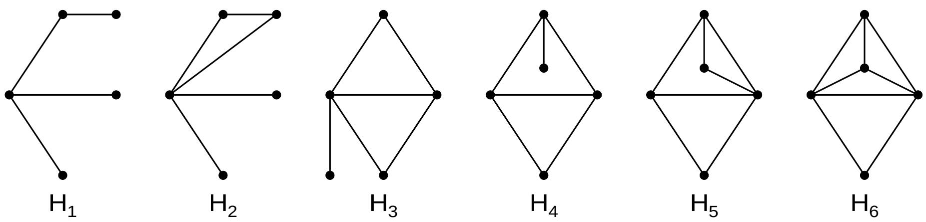

Figure 3. Some auxiliary graphs in $\mathcal{E}$.

by the hypothesis there is a vertex y such that $N_T(b) \neq N_T(c) \neq N_T(a)$ where $T = G[p, a, b, c, x, y]$. It can be verified that either $T \in \{G_{18}\} \cup \{G_i : 23 \leqslant i \leqslant 31\}$ or $G_{11} \preccurlyeq T$.

Case E3. There are vertices a, b, p, q such that $G[a, b, p, q] = K_{1,1,2}$, $\langle\langle p, q\rangle\rangle = 0$, $N(p) \neq N(q)$ and both $G[a, b, p]$ and $G[a, b, q]$ are odd. Then there is a vertex $x \in V(G)$ with $\langle\langle x, p\rangle\rangle \neq \langle\langle x, q\rangle\rangle$. Note that $I := G[a, b, p, q, x]$ is one of the graphs H_4, H_5, H_6 in Figure 3 and either $G[a, b, p]$ or $G[a, b, q]$ is odd in I. By the hypothesis, there is a vertex y such that both $G[a, b, p]$ and $G[a, b, q]$ are odd in $K := G[a, b, p, q, x, y]$. It can be verified that either $G_{10} \preccurlyeq K$ or $G_{11} \preccurlyeq K$ or $K \in \{G_i : 12 \leqslant i \leqslant 22\} \cup \{G_{25}, G_{26}, G_{28}, G_{29}, G_{30}\}$.

Case E4. There are vertices p, q, a, b, c such that $H := G[p, q, a, b, c] = K_{1,1,3}$, $\deg_H p = \deg_H q = 4$ and the conclusion of (E4) does not hold. If a, b, c have different neighbourhoods, then Case E2 holds. So, let us assume that $N(a) = N(b)$. Then by the hypothesis, $G[p, q, c]$ is odd whence for some $x \in V(G)$, $|N(x) \cap \{p, q, c\}|$ is odd. It is easy to verify that for some $i \in \{4, 7, 8, 9, 10, 11\}$, $G_i \preccurlyeq G[p, q, a, b, c, x]$.

Case E5. There are three distinct vertices a, b, c in a connected innergraph H of G such that $N_H(a) = N_H(b) = N_H(c)$ and $H \notin \{K_{1,3}, K_{1,4}, K_{1,1,3}\}$. By connectivity of H, there is a vertex $p \in N_H(a)$; then $G[p, a, b, c] = K_{1,3}$. Since $H \neq K_{1,3}$, there is a vertex $x \in V(H) \setminus \{p, a, b, c\}$ such that $S := G[p, a, b, c, x]$ is connected. If $x \notin N(p)$, then $S = G_1$; so let $\langle\langle x, p\rangle\rangle = 1$; then S is $K_{1,4}$ or $K_{1,1,3}$. Then there has to be a vertex $y \in V(G) \setminus \{p, a, b, c, x\}$ such that $T = G[p, a, b, c, x, y]$ is connected whence we find that either $G_1 \preccurlyeq T$ or $T \in \{G_i : 2 \leqslant i \leqslant 7\}$. ∎

Let $G \in \mathfrak{M}$; then by Theorem 13, $G \notin \mathcal{G}$ and for each $v \in V(G)$, it is easy to verify that $G - v \in \mathcal{A}$ whence by Theorem 11, $G - v \in \mathcal{G}$; therefore $G \in \mathcal{M}$. Conversely, let $G \in \mathcal{M}$; then by Theorem 13, an innergraph H of G belongs to $\mathfrak{M}$; since $H \notin \mathcal{G}$, we find that $G = H$, i.e., $G \in \mathfrak{M}$. Thus, it follows that $\mathcal{M} = \mathfrak{M}$.

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