BROKEN CIRCUITS IN MATROIDS—DOHMEN'S INDUCTIVE PROOF

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Abstract

Dohmen [4] gives a simple inductive proof of Whitney's famous broken circuits theorem. We generalise his inductive proof to the case of matroids.

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1. Introduction

The main aim of this note is to generalise Dohmen's inductive proof of Whitney's broken circuits theorem to matroids. The presented version of this proof is much more general. This is feasible because Dohmen's proof uses the notion of vertices in graphs only in one essential point, where an equivalence relation $\sim_e$ on the set of vertices is defined. We omit that definition and avoid the notion of vertices by introducing the "reduced matroid" and slightly modifying the rest of Dohmen's original proof.

Let M denote the matroid on the finite set E with rank function ρ and a family of cycles $\mathcal{C}$. Recall the following equivalent definitions of matroids (e.g., Oxley [6]).

Definition. A *matroid* M is a pair $M = (E, \mathcal{C})$, where $\mathcal{C}$ is a family of subsets of a finite set E such that

(c_1) if $C \in \mathcal{C}$ and $C' \subset C$, then $C' \notin \mathcal{C}$,

(c_2) if $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$, $e \in C_1 \cap C_2$, then there exists $C \in \mathcal{C}$ such that $C \subseteq (C_1 \cup C_2) \setminus \{e\}$.

A *loop* is a one-element cycle.

Definition. A *matroid* M is a pair $M = (E, \rho)$, where E is a finite set and ρ is a function $\rho : E \to \{0, 1, 2, \dots\}$ such that for any $A, B \subseteq E$,

(r_1) $0 \leq \rho(A) \leq |A|$,

(r_2) if $A \subseteq B \subseteq E$, then $\rho(A) \leq \rho(B)$,

(r_3) $\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$.

A set I is *independent* if it includes no cycles or, equivalently, if $\rho(I) = |I|$. A cycle is any minimal set which is not independent (i.e. it is dependent). $\rho(A)$ is the number of elements of any maximal independent set $I \subseteq A$.

The *characteristic polynomial* of a matroid M is usually defined (see, e.g. [3]) by

$$(1) \qquad \chi(M; \lambda) = \sum_{S \subseteq E} (-1)^{|S|} \lambda^{\rho(E) - \rho(S)},$$

then $\chi(M; \lambda)$ is a polynomial in λ with alternating signs:

$$(2) \qquad \chi(M; \lambda) = \sum_{k=0}^{\rho(E)} (-1)^k a_k(M) \lambda^{\rho(E) - k}.$$

Note that $\chi(M; \lambda) = 0$ if and only if M contains a loop. From now on we assume that M is the loop-free matroid.

Let us assume that E is linearly ordered. If C is a cycle in M and e is a maximal element of C then $C \setminus \{e\}$ is called a *broken circuit* of M. We will use the traditional name "broken circuit" instead of the more suitable name "broken cycle".

For the characteristic polynomial defined by (1) we have the following version of Whitney's broken circuit theorem (see e.g., Dohmen [3], Corollary 3.2):

Theorem 1. *For $k = 0, \dots, \rho(E)$ the coefficients $a_k(M)$ in (2) are the numbers of k-element subsets of E which do not include a broken circuit of M as a subset.*

Let $e \in E$. The *restriction $M - e$* of a matroid M to $E \setminus \{e\}$ is defined as the matroid on $E \setminus \{e\}$ whose family of independent sets is given by

$$(3) \qquad \mathcal{I}(M \setminus \{e\}) = \{I : I \subseteq E \setminus \{e\}, I \in \mathcal{I}(M)\}.$$

The *contraction $M|_e$* of a matroid M to $E \setminus \{e\}$ is defined as the matroid $M|_e = (E \setminus \{e\}, \rho|_e)$ where

$$(4) \qquad \rho|_e(A) = \rho(A \cup \{e\}) - \rho(\{e\})$$

for any $A \subseteq E \setminus \{e\}$. It is well known (see [5] or e.g., [2]) that

$$(5) \qquad \chi(M; \lambda) = \chi(M - e; \lambda) - \chi(M|_e; \lambda).$$

2.   Inductive Proof

The main idea of Dohmen's inductive proof is to show that if the statement of Whitney theorem holds both for contraction $G|_e$ and for deletion $G - e$ then it also holds for G. To achieve this, Dohmen [4] introduced the equivalence relations $\sim_e$ on the set of edges and the set of vertices of G. In the case of matroids we have no set of vertices, so we have to define the linear ordering relation for deletion and contraction in a different way. We call elements $e, f \in E$ *parallel* if $\{e, f\} \in \mathcal{C}$.

Let P be some maximal set of parallel elements in E, and let e_P be the greatest element in such P. The *reduced matroid M°* is obtained from M by deleting all elements $e < e_P$ from each maximal set P of parallel elements.

The following lemma is obvious

Lemma 2. *If $C \in \mathcal{C}$, $e \in \mathcal{C}$, $f \notin \mathcal{C}$ and e and f are parallel, then $C \setminus \{e\} \cup \{f\} \in \mathcal{C}$.*

Lemma 3. *If e and f are parallel, then*

$$\chi(M; \lambda) = \chi(M - e; \lambda) = \chi(M - f; \lambda).$$

Proof. If e and f are parallel in M, then f is a loop in $M|_e$ and $\chi(M|_e; \lambda) = 0$. From (5) we obtain the assertion. ∎

From Lemma 3 we immediately obtain the following result.

Lemma 4. *If M° is reduced matroid of M, then*

$$(6) \qquad \chi(M; \lambda) = \chi(M^\circ; \lambda).$$

Replacing contraction $G|_e$ in Lemmas 2 – 5 and in the proof of the theorem in [4] by $(M|_e)^\circ$ and writing M instead of G, the generalisation of Dohmen's inductive proof is obtained.

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