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WEAK SATURATION NUMBERS FOR SPARSE GRAPHS

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Abstract

For a fixed graph F, a graph G is F*-saturated* if there is no copy of F in G, but for any edge $e \notin G$, there is a copy of F in $G + e$. The minimum number of edges in an F-saturated graph of order n will be denoted by $\mathbf{sat}(n, F)$. A graph G is weakly F-saturated if there is an ordering of the missing edges of G so that if they are added one at a time, each edge added creates a new copy of F. The minimum size of a weakly F-saturated graph G of order n will be denoted by $\mathbf{wsat}(n, F)$. The graphs of order n that are weakly F-saturated will be denoted by $\mathbf{wSAT}(n, F)$, and those graphs in $\mathbf{wSAT}(n, F)$ with $\mathbf{wsat}(n, F)$ edges will be denoted by $\underline{\mathbf{wSAT}}(n, F)$. The precise value of $\mathbf{wsat}(n, T)$ for many families of sparse graphs, and in particular for many trees, will be determined. More specifically, families of trees for which $\mathbf{wsat}(n, T) = |T| - 2$ will be determined. The maximum and minimum values of $\mathbf{wsat}(n, T)$ for the class of all trees will be given. Some properties of $\mathbf{wsat}(n, T)$ and $\mathbf{wSAT}(n, T)$ for trees will be discussed.

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1. Introduction

Only finite graphs without loops or multiple edges will be considered. Notation will be standard, and generally follow the notation of [3]. For a graph G the vertex set $V(G)$ and the edge set $E(G)$ will be represented by just G when it is clear from the context.

For a fixed graph F, a graph G is *F-saturated* if there is no copy of F in G, but for any edge $e \notin G$, there is a copy of F in $G + e$. The collection of F-saturated graphs of order n is denoted by $\mathbf{SAT}(n, F)$, and the *saturation number*, denoted $\mathbf{sat}(n, F)$, is the minimum number of edges in a graph in $\mathbf{SAT}(n, F)$. A graph G is *weakly F-saturated* if there is an ordering of the missing edges of G so that if they are added one at a time, they form a complete graph and each edge added creates a new copy of F. The minimum size of a weakly F-saturated graph G of order n will be denoted by $\mathbf{wsat}(n, F)$. The graphs of order n that are weakly F-saturated will be denoted by $\mathbf{wSAT}(n, F)$, and those graphs in $\mathbf{wSAT}(n, F)$ with $\mathbf{wsat}(n, F)$ edges will be denoted by $\underline{\mathbf{wSAT}}(n, F)$. Clearly $\mathbf{wsat}(n, F) \leq \mathbf{sat}(n, F)$ as any F-saturated graph is also weakly F-saturated.

There are several general results on saturated and weakly saturated graphs and hypergraphs in print. These include a paper by Tuza [14] on sparse saturated graphs, results by Sidorowicz [11] and Borowiecki and Sidorowicz [1] on weakly saturated graphs, and papers by Erdős, Füredi, and Tuza [4] on saturated r-uniform hypergraphs, and Pikhurko [10] on weakly saturated hypergraphs. A survey of such results can be found in a paper by J. Faudree, R. Faudree, and Schmitt [6].

The objective is to determine the exact value of $\mathbf{wsat}(n, F)$ for many families of sparse graphs F, and in particular when F is a tree. Also, general bounds on $\mathbf{wsat}(n, F)$ will be presented.

Families of graphs F for which $\mathbf{wsat}(n, F) = |E(F)| - 1$ will be exhibited, and in particular trees for which $\mathbf{wsat}(n, T) = |T| - 2$ will be presented. The maximum and minimum values of $\mathbf{wsat}(n, G)$ for graphs in general and in particular values of $\mathbf{wsat}(n, T)$ for the class of all trees will be given. Some properties of $\mathbf{wsat}(n, G)$ and $\mathbf{wSAT}(n, G)$ for sparse graphs and in particular for trees will be discussed.

2. Known Results

Clearly $\mathbf{wsat}(n, F) \leq \mathbf{sat}(n, F)$ as any F-saturated graph is also weakly F-saturated. Lovász [9] proved the following result, which was earlier conjectured by Bollobás and verified for $3 \leq p < 7$ in [2].

Theorem 1 [9]**.** *For integers n and p,* $\mathbf{wsat}(n, K_p) = \mathbf{sat}(n, K_p) = \binom{n}{2} - \binom{n-p+2}{2}$.

The graph $K_{p-2} + \overline{K}_{n-p+2} \in \underline{\mathbf{wSAT}}(n, K_p)$ and has a minimal number of edges. In the special case when $p = 3$, it is easily seen that any tree T_n on n vertices is in $\underline{\mathbf{wSAT}}(n, K_3)$. Thus, there is not a unique graph in $\underline{\mathbf{wSAT}}(n, K_3)$, and the same is true for $\underline{\mathbf{wSAT}}(n, K_p)$.

Borowiecki and Sidorowicz [1]) considered the weak saturation number of cycles, and proved the following results.

Theorem 2 [1]**.** *For $n \geq 2p + 1$, $\mathbf{wsat}(n, C_{2p+1}) = n - 1$.*

Theorem 3 [1]**.** *For $n \geq 2p$, $\mathbf{wsat}(n, C_{2p}) = n$.*

It is clear that $P_n \in \underline{\mathbf{wSAT}}(n, C_{2k+1})$, and it is easily verified that any tree T_n on n vertices with diameter at least $2k$ is in $\underline{\mathbf{wSAT}}(n, C_{2k+1})$. Also, it is easily verified that C_n for n odd, and the graph obtained from C_{n-1} by adding a pendant edge for n even are in $\underline{\mathbf{wSAT}}(n, C_{2k})$. These examples verify that it is not true in general that $\mathbf{sat}(n, F) = \mathbf{wsat}(n, F)$, since, for example, $\mathbf{sat}(n, C_4) = \lfloor \frac{3n-5}{2} \rfloor$ (see Ollmann [12]) and $\mathbf{wsat}(n, C_4) = n$.

The following natural question was first raised by Tuza in [13].

Question 1 [13]**.** Are there necessary and/or sufficient conditions for $\mathbf{wsat}(n, F)$ to equal $\mathbf{sat}(n, F)$?

3. General Elementary Preliminary Results

Consider a graph F of order p with q edges and with minimum degree $\delta = \delta(F)$. If $G \in \mathbf{wSAT}(n, F)$, then when the appropriate first edge is added, there must be a copy of F. Thus, there are at least $q-1$ edges in the p vertices that give a copy of F. Also, every vertex v of G must have degree at least $\delta - 1$, since when the first edge is added incident to v, this must result in a vertex of degree at least δ. Thus, $\mathbf{wsat}(n, G) \geq (2(q-1) + (n-p)(\delta-1))/2 = q - 1 + (\delta-1)(n-p)/2$. Also, observe that the graph $H(\delta)$ obtained from $K_{p-1} \cup \overline{K}_{n-p+1}$ by adding precisely $\delta - 1$ edges from each vertex in the $\overline{K}_{n-p+1}$ to K_{p-1} is in $\mathbf{wSAT}(n, F)$ and has $\binom{p-1}{2} + (\delta-1)(n-p+1)$ edges. This upper bound in the next result was observed by Sidorowicz (Theorem 2, [11]) in a more general setting, but an example which implies the upper bound was given here since it is easily described and completes the proof of the result. This gives the following theorem.

Theorem 4. *Let F be a graph with p vertices, q edges, and minimal degree δ. Then, $q - 1 + (\delta - 1)(n - p)/2 \leq \mathbf{wsat}(n, F) \leq (\delta - 1)n + (p - 1)(p - 2\delta)/2$ for any $n \geq p$.*

Both the lower bound and the upper bound in Theorem 4 occur for appropriate δ. For example, consider the graph $F_{p,\delta}$ of order p obtained from the complete graph

K_{p-1} by attaching a vertex of degree δ. If $\delta = 1$, then it is easily verified that $K_{p-1} \cup \overline{K}_{n-p+1}$ implies $\mathbf{wsat}(n, F_{p,1}) = \binom{p-1}{2}$, and if $\delta = p-1$, then Theorem 1 of Lovász gives the result $\mathbf{wsat}(n, F_{p,p-1}) = (\delta-1)n + (p-1)(p-2\delta)/2$.

For $\delta > 1$ a stronger lower bound can be proved. Consider a graph F with minimum degree $\delta = \delta(F)$, and assume $G \in \mathbf{wSAT}(n, F)$ for n sufficiently large. Partition the vertices of G into two sets A and B, with B being the vertices of degree $\delta - 1$ and A the remaining vertices. Let $|B| = k$, and so $|A| = n - k$. The vertices in B form an independent set, since the addition of a first edge to a vertex $v \in B$ must result in v and all of its neighbors having degree at least δ. Likewise, each vertex in A must have degree at least $\delta - 2$ relative to A to be able to add edges between B and A, since at most 2 vertices of B can be used. This gives the following inequality:

$$(\delta-1)k + (\delta-2)(n-k) \le \delta(n-k).$$

This implies $k \le 2n/(\delta+1)$, and so

$$\mathbf{wsat}(n, F) \ge (\delta-1)(\frac{2n}{\delta+1}) + \frac{\delta(n - \frac{2n}{\delta+1})}{2} = \frac{\delta n}{2} - \frac{n}{\delta+1}.$$

Theorem 5. *If F is a graph with p vertices and minimal degree δ, then,*

$$\frac{\delta n}{2} - \frac{n}{\delta+1} \le \mathbf{wsat}(n, F) \le (\delta-1)n + \frac{(p-1)(p-2\delta)}{2}$$

for any n sufficiently large.

This lower bound cannot be improved significantly. Consider the graph $F_{2\delta}$ that consists of two vertex disjoint complete graphs $K_\delta \cup K_\delta$ along with a perfect matching between the δ vertices in each complete graph. Thus, $F_{2\delta}$ is δ-regular graph with 2δ vertices and δ^2 edges. See Figure 1 for the graph $F_{2\delta}$ when $\delta = 4$. Consider the graph $H_{2\delta}$. This graph consists of two vertex disjoint copies of $K_\delta \cup K_\delta$ along with a matching between the first $\delta - 1$ vertices of each complete graph K_δ and also $\delta - 1$ edges from the last vertex of the second K_δ to the first $\delta - 1$ vertices of the first K_δ. Thus, $H_{2\delta}$ has 2δ vertices and $\delta^2 + \delta - 2$ edges. See Figure 1 for the graph $H_{2\delta}$ when $\delta = 4$. It is straightforward to check that $H_{2\delta} \in \mathbf{wSAT}(2\delta, F_{2\delta})$. First add the edge between the last two vertices in the K_δ's, then the remaining edges from the last vertex in the first K_δ to the vertices in the second K_δ, and then the remaining edges can be added in any order.

Let $H_{3\delta}$ be the graph obtained from $H_{2\delta}$ by adding a vertex disjoint K_δ and $\delta - 1$ matching edges between $H_{2\delta}$ and the new K_δ. Likewise, $H_{(i+1)\delta}$ can be formed from $H_{i\delta}$ in the same way. It is straightforward to confirm that $H_{3\delta} \in \mathbf{wSAT}(3\delta, F_{2\delta})$, and more generally $H_{i\delta} \in \mathbf{wSAT}(i\delta, F_{2\delta})$ for any $i \ge 2$. For $m \ge 3$ the graph $H_{m\delta}$ has $m\delta$ vertices and $m(\delta+2)(\delta-1)/2$ edges. Thus, if $n = m\delta$,

$F_{2(2)}$ $H_{2(2)}$

Figure 1

then $H_n \in \mathbf{wSAT}(n, F_{2\delta})$ and has $n(\delta+2)(\delta-1)/(2\delta) = (\delta/2 + 1/2 - 1/\delta)n$ edges.

Theorems 4 and 5 imply there are constants c_1 and c_2 dependent on F but independent of n such that $(\delta-1)n/2 + c_1 \le \mathbf{wsat}(n, F) \le (\delta-1)n + c_2$ and $(\frac{\delta^2+\delta-2}{2\delta+2})n + c_1 \le \mathbf{wsat}(n, F) \le (\delta-1)n + c_2$ respectively. Observe that if $\delta = 1$, such as what would be true for trees, then $c_1 \le \mathbf{wsat}(n, F) \le c_2$, or more specifically, $q - 1 \le \mathbf{wsat}(n, F) \le \binom{p-1}{2}$. On the other hand if $\delta(F) \ge 2$, then $\mathbf{wsat}(n, F) \ge (\frac{\delta^2+\delta-2}{2\delta+2})n$, and so is not independent of n.

If in the argument made for the lower bound for Theorem 4, the requirement that when the first edge is added incident to the vertex v, the degree must be at least $\delta - 1$ relative to the vertices that have already had an edge added, then this would imply that $\mathbf{wsat}(n, F) \ge q - 1 + (\delta-1)(n-p)$. Thus, the following question is a natural one.

Question 2. What properties will insure that a graph F with p vertices, q edges, and minimal degree δ will satisfy

$$q - 1 + (\delta-1)(n-p) \le \mathbf{wsat}(n, F) \le (p-1)(p-1)/2 + (\delta-1)(n-p+1)$$

for any $n \ge p$?

A slightly stronger result can be stated if more is known about the graph F. For example if $\mathbf{wsat}(p, F)$ is known, then the complete graph K_{p-1} in Theorem 4 can be replaced by any graph in $\underline{\mathbf{wSAT}}(p, F)$. This gives the following result.

Theorem 6. *Let F be a graph with p vertices, q edges, and minimal degree δ. Then,*

$$\mathbf{wsat}(n, F) \le \mathbf{wsat}(p, F) + (\delta-1)(n-p)$$

for any $n \ge p$.

Clearly, for any tree T_p of order p, $\mathbf{wsat}(n, T_p) \ge p - 2$, since the addition of any edge to a graph in $\underline{\mathbf{wSAT}}(n, T_p)$ results in a graph with at least $p-1$ edges. Also, the graph $K_{p-1} \cup \overline{K}_{n-p+1} \in \mathbf{wSAT}(n, T_p)$, since the addition of any edge incident to a vertex in K_{p-1} results in a copy of T_p containing the edge. Thus, the following is true.

Corollary 7. *For any tree T_p with p vertices,*

$$p-2 \leq \mathbf{wsat}(n, T_p) \leq \binom{p-1}{2},$$

and each of the bounds is sharp.

Each of the bounds in the previous corollary is attained. In particular, it will be shown later that the lower bound is true for P_p and the upper bound by $K_{1,p-1}$.

Before stating the next general result, several definitions are needed.

Definition. Given a graph G, a rooted tree T with root v is an *endtree* of G if there is a cutvertex v' of G such that some of the components of $G - v'$ along with v' induce a tree rooted at v' isomorphic to T.

More specifically, there is a special endtree which is a star.

Definition. Given a graph F, an *endstar* $S = K_{1,s}$ of F is an induced star of F such that the center of S has degree $s+1$ in F and s of the other vertices of S have degree 1 in F. The minimal *enddegree* of F, denoted by $\delta_e(F)$, is the degree of the smallest endstar.

For example, the broom B_{r_1,r_2} with $r_1 + r_2$ vertices containing a path with r_1 vertices in the handle and a star with r_2 vertices of degree 1 would have enddegree 1 if $r_1 \geq 3$. It would also have an endstar K_{1,r_2}. A double star with 2 centers of stars connected by an edge with $r_1 \leq r_2$ vertices of degree 1 respectively, would have enddegree r_1.

Definition. A rooted tree T with root v is *minimum weakly saturated*, if $\mathbf{wsat}(n, T) = |T| - 2$ for any $n \geq |T|$, and v is the root of each of the copies of T obtained when edges are added to obtain the complete graph.

Examples of minimum weakly saturated rooted trees appear in Figure 2. It is easy to verify that each of the rooted trees T in Figure 2 are minimum weakly saturated. For example, consider the tree $T_1^* = (v_1, v_2, v_3)$. Starting with the edge v_1v_2 in a graph G of order $n \geq 3$ containing only the edge v_1v_2, the edges in G can be added in the following order to get a complete graph such that each new edge is in a new copy of T_1^*: edges incident to v_2, edges incident to v_1, and the remaining edges of G. In the case of T_2^*, consider the tree T' obtained from T_2^* by deleting the vertex u_2. Starting with the tree T' in a graph G of order $n \geq |T_2^*|$ containing only the edges of the tree T', the edges in G can be added in the same order as for T_1^* (edges from v_1, then v_2, and then the remaining edges) to obtain a complete graph such that each new edge is in a new copy of T_2^*. Similar arguments can be made for the remaining trees T_i^* for $i \geq 3$.

The saturation number of a graph that has a rooted endtree that is minimum weakly saturated is easily determined as the following result verifies.

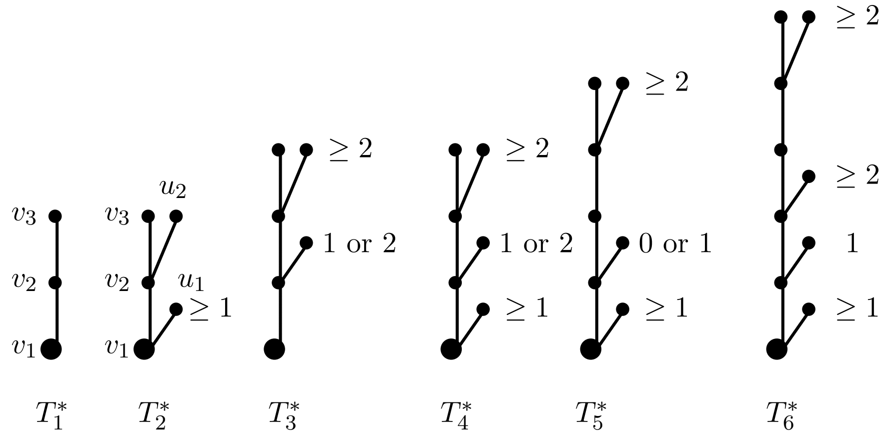

Figure 2. Some examples of minimum weakly saturated rooted trees.

Theorem 8. *Let F be a graph of order p with q edges that contains a rooted endtree $T_{p'}$ of order p' that is minimum weakly saturated. Then, for $n \geq 2p-p'-1$,*

$$\mathbf{wsat}(n, F) = q - 1.$$

Proof. Let G be a graph of order n containing the graph F' obtained from F by deleting an edge from the tree $T_{p'}$. Let S be the vertices of G not in F'. Since $T_{p'}$ is minimum weakly saturated, edges can be added generating new copies of F such that the graph spanned by $S \cup T_{p'}$ is complete. Since this complete graph has at least p vertices, G can be extended to a complete graph as well. This completes the proof of Theorem 8. ∎

4. Results for General Graphs

The existence of endstars in a graph F has a significant impact on the weak saturation number. The following results gives upper bounds on $\mathbf{wsat}(n, F)$.

Theorem 9. *Let F be a graph with p vertices and q edges with $\delta_e(F) = k \geq 1$. If $n \geq 2p - k$, then*

$$\mathbf{wsat}(n, F) \leq q - 1 + \binom{k}{2}.$$

Proof. Let v be the center of the endstar S with k edges, u the vertex of S adjacent to v not in the endstar with k edges, and F' the graph obtained from F by deleting one of the edges from the endstar. Consider the graph $H = F' \cup K_k \cup \overline{K}_{n-p-k+1}$. The claim is that $H \in \mathbf{wSAT}(n, G)$. New copies of F are obtained by adding edges in the following order: edges from v to $K_k \cup \overline{K}_{n-p-k+1}$, edges from u to K_k, edges from a vertex in K_k to a vertex in $\overline{K}_{n-p-k+1}$, edges from u to $\overline{K}_{n-p-k+1}$, and finally edges in $\overline{K}_{n-p-k+1}$. This results in a complete graph with at least p vertices, and thus all remaining edges of F' can be added. This completes the proof of Theorem 9. ■

Let $F = F(p,q,k,s)$ be a connected graph of order p with q edges such that $\delta_e(F) = k$, and F contains an induced subgraph isomorphic to the broom $B_{3,s}$ for $s \geq 1$. Let (u_0, u_1, u_2) be the path in $B_{3,s}$, and so $d_F(u_1) = 2$ and $d_F(u_2) = s+1$. Let $V = \{v_1, v_2, \ldots, v_s\}$ the vertices of degree 1 in $B_{3,s}$ not on the path (u_0, u_1, u_2).

Theorem 10. *If $n \geq 2p-k$, $k \geq 1$, and $F = F(p,q,k,s)$ contains an induced broom $B_{3,s}$ for $s \geq 1$, then*

$$\mathbf{wsat}(n, G) \leq q + ks.$$

Proof. Let F' be the graph obtained from F by deleting the edge u_0u_1, adding k vertices $U = \{u_{2,1}, u_{2,2}, \ldots, u_{2,k}\}$, adding all edges between U and V, and finally adding the edge $u_2u_{2,1}$. New copies of F are obtained by adding edges in the following order: the edges u_0u_1, $u_0u_{2,1}$, u_0u_2, the edges $u_{2,1}u_{2,i}$ for $i \geq 2$, the edges $u_0u_{2,i}$ for $i \geq 2$, and then the remaining edges of U. This results in a complete graph K_k that is disjoint from a copy of G. Then the proof of Theorem 9 implies that this graph can be completed to a complete graph, which completes the proof of Theorem 10. ■

The special case when $s = 1$ of Theorem 10 gives the following immediate corollary.

Corollary 11. *If $n \geq 2p-k$, $k \geq 1$, and F is a connected graph of order p with q edges that contains an induced path with at least 4 vertices and $\delta_e(F) = k$, then*

$$\mathbf{wsat}(n, F) \leq q + k.$$

The saturation numbers $\mathbf{wsat}(n, F)$ for all connected graphs of order at most 5 can be obtained from the previous results along with those that will be proved for trees in the next section, except for three graphs—namely $K_{2,3}$, $K_5 - K_2$, and $K_5 - 2K_2$. That is one rationale for raising the following two questions and proving the results for $K_{2,3}$ and $K_5 - K_2$.

The difference between upper bounds and the lower bounds on $\mathbf{wsat}(n, F)$ for graphs with $\delta(F) = 1$ have been shown to be independent of n. For $\delta_e(F) \geq 2$ this is not in general known. It would be of interest to know if there are are results for $\mathbf{wsat}(n, F_{p,\delta})$, where $F_{p,\delta}$ is the graph of order p obtained from a K_{p-1} by adding a vertex of degree δ. In particular, does $\mathbf{wsat}(n, F_{p,\delta})$ have the form of Theorem 1? More specifically, there is the following question, which is known to be true for $\delta = 1$ and $\delta = p - 1$.

Question 3. Is $\mathbf{wsat}(n, F_{p,\delta}) = \binom{p-1}{2} + (n - p + 1)(\delta - 1)$ for $2 \leq \delta < p - 1$?

A similar question could be asked for the family of graphs $K_p - sK_2$ for $s \leq p/2$. More specifically, there is the following question.

Question 4. Is $\mathbf{wsat}(n, K_p - sK_2) = \binom{p-1}{2} - s + (n - p + 1)(p - 3)$ for $1 \leq s < (p - 1)/2$?

In Question 3, the case when $p = 5$ and $\delta = 3$ ($F_{5,3}$) can be answered.

Theorem 12. *For $n \geq 5$, $\mathbf{wsat}(n, K_5 - K_2) = 2n - 2$.*

Proof. To prove the result it is sufficient to show that if $G \in \mathbf{wSAT}(n, K_5 - K_2)$, then G has at least $2n - 2$ edges. The addition of the first edge implies the existence of a subgraph H of G with 5 vertices and 8 edges. This completes to a complete graph H_1 with 5 vertices. If a vertex $v \in G$ has two adjacencies in H_1, then an additional edge from v to H_1 can be added to get a copy of $K_5 - K_2$, and this will yield a complete graph with 6 vertices. This can be continued. If this terminates in a K_n, then G would have at least $8 + 2(n - 5) = 2n - 2$ edges. If not, then it terminates in a complete graph H^* with $m < n$ vertices that contains at least $2m - 2$ edges of the original graph G. Also, each vertex of $G - H^*$ will have at most 1 adjacency in H^*. This process can be started again with a subgraph H' with 5 vertices and 8 edges that will end in a complete graph $H^{'*}$ with m' edges just as before. However, with no loss of generality one can assume that H^* and $H^{'*}$ have a vertex in common by selecting the correct starting subgraph H'. The two graphs H^* and $H^{'*}$ will contain $m + m' - 1$ vertices and at least $(2m - 2) + (2m' - 2) = 2(m + m' - 1) - 2$ edges. This process can be continued, until all n vertices of G are contained in one of the complete graphs implying that G has at least $2n - 2$ vertices. ■

The question concerning $\mathbf{wsat}(n, K_{2,3})$ can easily be answered, which is the next result.

Theorem 13. *For $n \geq 5$, $\mathbf{wsat}(n, K_{2,3}) = n + 1$.*

Proof. Consider the graph H_5 obtained from the cycle $C = (x_1, x_2, x_3, x_4, x_5, x_1)$ by adding the chord x_1x_3. The graph $H_5 \in \mathbf{wSAT}(5, K_{2,3})$, since the addition of the chords $x_2x_4, x_3x_5, x_4x_1, x_5x_2$ in that order will generate new copies of $K_{2,3}$. The only possible graph of order 5 and size 5 in $\mathbf{wSAT}(5, K_{2,3})$ is $K_{2,3} - e$, and it is easily checked that this graph is not weakly $K_{2,3}$-saturated. Thus, $\mathbf{wsat}(5, K_{2,3}) = 6$, and $H_5 \in \underline{\mathbf{wSAT}}(5, K_{2,3})$.

Consider the graph G_n obtained from H_5 by placing the center of an induced star with $n-5$ edges at one of the vertices of H_5. The graph $G_n \in \mathbf{wSAT}(n, K_{2,3})$, since edges can be added to H_5 until it is complete, edges from each vertex of $G_n - H_5$ to a vertex in H_5 can be added, and then edges can be added between any pair of vertices in $G_n - H_5$ obtaining a new copy of $K_{2,3}$ in each case. Thus, $\mathbf{wsat}(n, K_{2,3}) \leq n+1$.

Assume $G \in \mathbf{wSAT}(n, K_{2,3})$ with at most n edges. Since G must be connected there is a subgraph H' of order 5 that contains an induced $K_{2,3} - e$ and the graph induced by $G - E(H')$ is a forest. This follows from the fact that if $|E(H')| \geq 6$, then $|E(G)| \geq n+1$. The only edges that can be added to G that generate a new $K_{2,3}$ is one edge of H' and any edge from a vertex in $G - H'$ to one of the vertices of degree 3 in H', if the vertex is adjacent to the other vertices of degree 3 in H'. This, however implies that $\mathbf{wsat}(n, K_{2,3}) > n$, which gives a contradiction that completes the proof of Theorem 13. ■

5. Results for Trees

Consider the class $\mathcal{T}_p$ of labeled trees of order p. Since for many trees $\mathbf{wsat}(n, T_p) = p-2$, it is natural to question whether this is true for nearly all trees. The following result answers that question about the probability $P(\mathbf{wsat}(n, T_p) = p-2)$.

Theorem 14. *For the class of labeled trees $T_p \in \mathcal{T}_p$,*

$$\lim_{n\to\infty} P(\mathbf{wsat}(n, T_p) = p-2) \to 1.$$

Proof. The Cayley Tree Formula implies that there are p^{p-2} non-identical labeled trees on p vertices. For each pair of labels $i < j \in I_p = \{1, 2, \ldots p\}$, consider the family $\mathcal{F}_{i,j}$ of $(p-2)^{p-4}$ non-identical trees with labels $I - \{i, j\}$. For each tree $T \in \mathcal{F}_{i,j}$, find a longest path in T, say with endvertices with labels $k < \ell$. Form a tree T_1 with p vertices from T by adding a vertex with label i adjacent to the vertex with label k and adding a vertex with label j adjacent to the vertex with label ℓ. Repeat this construction to form a tree T_2 except that the vertex with label i is made adjacent to the vertex with label ℓ and the vertex with label j is made adjacent to the vertex with label k. This process generates $2\binom{p}{2}(p-2)^{p-4} = p(p-1)(p-2)^{p-4}$ trees of order p. It is easily seen that these trees are pairwise non-identical.

Each of these $p(p-1)(p-2)^{p-4}$ trees T' has a suspended endpath with 3 vertices, which implies by Theorem 8 that the tree is minimally weakly saturated, and so $\mathbf{wsat}(n, T') = p-2$. Since

$$\lim_{p\to\infty} \frac{p(p-1)(p-2)^{p-4}}{p^{p-2}} = 1,$$

this completes the proof of Theorem 14. ■

The existence of induced brooms and paths in trees impacts the magnitude of the weak saturation number. The following two corollaries are a result of Theorem 10 applied to trees.

Corollary 15. *If $n \geq 2p - k$ and T is a tree of order p that contains an induced broom $K_{3,s}$ for $s \geq 1$ and $\delta_e(T) = k$, then*

$$\mathbf{wsat}(n, T) \leq p - 1 + ks.$$

Corollary 16. *If $n \geq 2p - k$ and T is a tree of order p that contains an induced path with at least 4 vertices and $\delta_e(T) = k$, then*

$$\mathbf{wsat}(n, T) \leq p + k - 1.$$

The following result for trees follows immediately from Theorem 9.

Corollary 17. *Let T be a tree with p vertices with $\delta_e(G) = k \geq 1$. If $n \geq 2p - k$, then*

$$\mathbf{wsat}(n, T) \leq p - 2 + \binom{k}{2}.$$

Note that if T is a tree of order p with $\delta_e(T) = 1$, then $\mathbf{wsat}(n, T) = p - 2$, and if $\delta_e(T) = 2$, then $\mathbf{wsat}(n, T) \leq p - 1$. Thus, for all binary trees T, (and tree with $\Delta(T) \leq 3$),

$$p - 2 \leq \mathbf{wsat}(n, T) \leq p - 1.$$

Definition. For a tree T, the *internal tree* T_I is the subtree of T obtained by deleting the vertices of degree 1 of T. The maximum internal degree $\Delta_I(T)$ of T is $\Delta(T_I)$.

If T is a tree such that $\Delta_I(T) = \ell$, then the vertices of T of degree 1 can be partitioned into ℓ sets, which implies there is some endstar with at most $(p-1)/\ell$ vertices of degree 1. This gives the following result.

Corollary 18. *If $n \geq 2p - \frac{p-1}{\ell}$ and T is a tree of order p with $\Delta_I(T) = \ell$, then*

$$\mathbf{wsat}(n, T) \leq p - 2 + \binom{\frac{p-1}{\ell} - 1}{2}.$$

In the remainder of this section the weak saturation number for special classes of trees will be determined starting with the star, which has the largest weak saturation number. The following result was proved by Borowiecki and Sidorowicz (Theorem 13, [1]). The proof is given here, since it is short and the idea is used elsewhere.

Theorem 19 [1]. *For $k \geq 2$ and $n > k$,*

$$\mathbf{wsat}(n, K_{1,k}) = \binom{k}{2}.$$

Proof. Consider the graph $K_k \cup \overline{K}_{n-k}$. Adding an edge between a vertex not in K_k to a vertex in K_k generates a $K_{1,k}$. Addition of the rest of these edges results in a graph in which each vertex has degree at least k. It is easily seen that any additional edge added gives a $K_{1,k}$ containing the edge.

Let G be a graph of order n in $\mathbf{wSAT}(n, K_{1,k})$. When the first edge is added to G it must be incident to a vertex v_1 that has degree at least $k-1$ in G. All of the remaining edges that would be incident to v_1 can be added to form G_1. The next edge added to G_1 must be incident to a vertex v_2 that now has degree $k-1$ in G_1, and so v_2 must have degree at least $k-2$ in $G - v_1$. Thus argument can be continued to obtain vertices $v_3, \ldots, v_{k-1}$ such that v_j has degree at least $k-j$ in $G - \{v_1, v_2, \ldots, v_{j-1}\}$. Thus, G has at least $1 + 2 + \cdots + (k-1) = \binom{k}{2}$ edges. This completes the proof of Theorem 19. ■

For positive integers $k_1 \leq k_2$ and $s \geq 2$, let $B(k_1, k_2, s)$ denote the tree composed of 2 stars with k_1 and k_2 edges with their centers connected by a path with s vertices. This tree, sometimes called a double broom, has $p = k_1 + k_2 + s$ vertices. In the case when $s = 2$, the tree $B(k_1, k_2, s)$ is called a double star. The following result gives the weak saturation numbers for these trees.

Theorem 20. *For positive integers $k_1 \leq k_2$ and $s \geq 2$, let $T_p = B(k_1, k_2, s)$, where $p = k_1 + k_2 + s$. Then for $n \geq 2p$,*

(1) $\mathbf{wsat}(n, T_p) = p - 2$ *if $s \geq 4$ and is even,*

(2) $\mathbf{wsat}(n, T_p) = p - 1$ *if $s \geq 5$ and is odd,*

(3) $\mathbf{wsat}(n, T_p) = p - 2 + \binom{k_1}{2}$ *if $s = 3$, and*

(4) $\mathbf{wsat}(n, T_p) = p - 2 + \binom{k_1 - 1}{2}$ *if $s = 2$.*

Proof. Let $T = T_p$, and let $v_1, v_2, \ldots, v_s$ be the vertices of the path with s vertices with v_1 the center of the star with k_1 edges and v_s the center of the star with k_2 edges.

(1) Consider the tree T' obtained from T by deleting one of the edges from the star with k_2 edges. Start with a graph G with n vertices that contains just the edges in the tree T', and let S be the set of vertices of G not in T'. Adding edges in G in the following order will always result in a new copy of T: edges between vertices in S and v_s, $v_{s-2}, \ldots, v_2, v_1, v_3, \ldots, v_{s-1}$. Thus, all vertices of S are adjacent to all of the vertices $\{v_1, v_2, \ldots, v_s\}$ of the path. Then all edges between vertices in S can be added, and so this results in a complete graph with at least p vertices. Clearly, then the edges of G can be added until a complete

graph is reached. At least $p-2$ edges are needed for a tree of order p, so this completes the proof of this case.

(2) Consider the tree T' obtained from T by adding a vertex $v*$, edges v_1v^*, v^*v_2 and deleting the edge v_1v_2. Let $P = (v_1, v*, v_2, \ldots, v_s)$. Start with a graph G with n vertices that contains just the edges in the tree T', and let S be the set of vertices of G not in T'. Adding edges in G in the following order will always result in a new copy of T: chords of length 2 along the path P starting with v_1v_2, chords of length 3 along the path P starting with v_1v_3, edges between vertices in S and v_s, $v_{s-1}, \ldots, v_2, v*, v_1$. Then all edges between vertices in S can be added, and so this results in a complete graph with at least p vertices. Clearly, then the edges of G can be added until a complete graph is reached.

If the starting T-saturated graph G has just $p-2$ edges, then after the addition of the first edge there will be a copy of the tree T. It is easily checked that the only edges that can be added in the closure process are edges incident to the vertices $\{v_1, v_3, \ldots, v_s\}$. This will result in a complete bipartite graph with the vertices $\{v_1, v_3, \ldots, v_s\}$ adjacent to all of the remaining vertices of G. No additional edges can be added. Thus, $\mathbf{wsat}(n, T) > p-2$.

(3) and (4) Let T' be the tree obtained from T by deleting one of the edges in the star with k_2 edges. Let G be the graph of order n with edges from the tree T' and a disjoint K_{k_1} (K_{k_1-1} in case (4)). Let S be the vertices of G not in T' and S' the vertices in G not in T' or K_{k_1} (K_{k_1-1} in case (4)). Adding edges in G in the following order will always result in a new copy of T: each of the edges from v_s to S, each of the edges from v_1 to S, each of the edges from v_2 to the vertices of K_{k_1} in case (3), each of the edges from a vertex in K_{k_1} (K_{k_1-1} in case (4)) to S', and the edges between vertices in S'. This results in a complete graph with at least p vertices, and so G can clearly be extended to a complete graph.

Observe that in the case when $s=2$, (Case (4)), each edge in T is incident to a vertex of degree at least k_1. Thus, if $G \in \mathbf{wSAT}(n, T)$, then there must be a vertex, $u_1 \neq v_1, v_s$ that will have degree at least $k_1 - 2$ in $G - \{v_1, v_s\}$. Also, there must be a vertex, $u_2 \neq v_1, v_s, u_1$ that will have degree at least $k_1 - 3$ in $G - \{v_1, v_s, u_1\}$. This pattern continues, and so there must be at least $1+2+\cdots+k_1-2 = \binom{k_1-1}{2}$ edges in G not in T. Thus, in Case (4) $\mathbf{wsat}(n, T) = p-2+\binom{k_1-1}{2}$. The same argument holds in case (3) except that k_1 is replaced by $k_1 - 1$. This completes the proof of Theorem 20 ■

For nonnegative integers $d_1, d_2, \ldots, d_r$, with $r \geq 3$ and $d_1, d_r > 0$, the tree $T = C(d_1, d_2, \ldots, d_r)$ is the caterpillar with spine the path $(v_1, v_2, \ldots, v_r)$ of r vertices such that for each i $(1 \leq i \leq r)$ there is a star with d_i edges off of the path and centered a vertex v_i. Thus, T has $r + d_1 + d_2 + \cdots + d_r \geq 5$ vertices. The following gives the weak saturation numbers for caterpillars.

Theorem 21. *Let $T = C(d_1, d_2, \ldots, d_r)$ with $p = r + d_1 + d_2 + \cdots + d_r$ vertices.*

(1) *If* $d_i = k > 0$ *for some* i, *and both* $d_{i-1}, d_{i+1} > 0$, *then* $\mathbf{wsat}(n,T) \le p - 2 + \binom{k-1}{2}$.
(2) *If* $d_i = k > 0$ *for some* i, *and precisely one of* d_{i-1}, $d_{i+1} > 0$, *then* $\mathbf{wsat}(n,T) \le p - 2 + \binom{k}{2}$.
(3) *If* $d_i = k > 0$ *for some* i, *and both* $d_{i-1}, d_{i+1} = 0$, *then* $\mathbf{wsat}(n,T) \le p - 2 + \binom{k+1}{2}$.
(4) *If* $d_1 = k$ *and* $d_2 > 0$ *(with the same property for* d_s*), then* $\mathbf{wsat}(n,T) \le p - 2 + \binom{k-1}{2}$.
(5) *If* $d_1 = k$ *and* $d_2 = 0$ *(with the same property for* d_s*), then* $\mathbf{wsat}(n,T) \le p - 2 + \binom{k}{2}$.
(6) *If there is precisely an even positive number of consecutive* $d_i = 0$, *then* $\mathbf{wsat}(n,T) = p - 2$.
(7) *If there is precisely an odd number at least* 3 *of consecutive* $d_i = 0$, *then* $\mathbf{wsat}(n,T) \le p - 1$.

Proof. Let $G \in \mathbf{wSAT}(n,T)$. (1) Let T' be the tree obtained from T by deleting one of the endvertices of one of the stars. Let G be a graph of order n containing the induced tree T' and a vertex disjoint K_{k-1}. Let S be the vertices of $G - T'$ and label the vertices in K_{k-1} by $\{w_1, w_2, \dots w_{k-1}\}$. Additional copies of T can be obtained by adding the edges to G in the following order: the edge deleted from T to obtain T', edges from v_i to S if $d_i > 0$. At this point all of the vertices of $v_i \in T$ with $d_i > 0$ will be adjacent to all of the vertices of S. Adding an edge, say u_1w_1 with $u_1 \in S - K_{k-1}$ and $w_1 \in K_{k-1}$ will allow the vertex v_i of the path in T to be replaced by w_1 with w_1 being the center of a star with k edges in the new copy of T. Consecutively adding the edges $u_1w_2, \dots, u_1w_{k-1}$ will result in new copies of T. This results in a complete graph K_k, and this can be continued. Thus, all of the edges in S can be added, which results in a complete graph with at least p vertices. Hence, edges can be added to G to make it complete.

(2) This proof of is identical to that of (1) except that the complete graph disjoint from T' is a K_k, and the edges u_1w_j are part of path from v_{i-1} to v_{i+2} instead of being part of a star with k edges.

(3) This proof is identical to that of (1) except that the complete graph disjoint from T' is a K_{k+1}, and the edges u_1w_j are part of path from v_{i-2} to v_{i+1} (along with an edge of K_{k+1}) instead of being part of a star with k edges.

(4) and (5) The proofs of (4) and (5) are analogues of the proofs of (1) and (2).

(6) Let T' be the tree obtained from T by deleting one of the endvertices of one of the stars, and let G of a graph of order n containing an induced copy of T'. Let S be the vertices of G not in T'. Let $w_1, w_2, \dots, w_{2r}$ be the vertices of a path such that $d(w_i) = 0$ for $1 < i < 2r$ and such that $d(w_1), d(w_{2r}) > 0$.

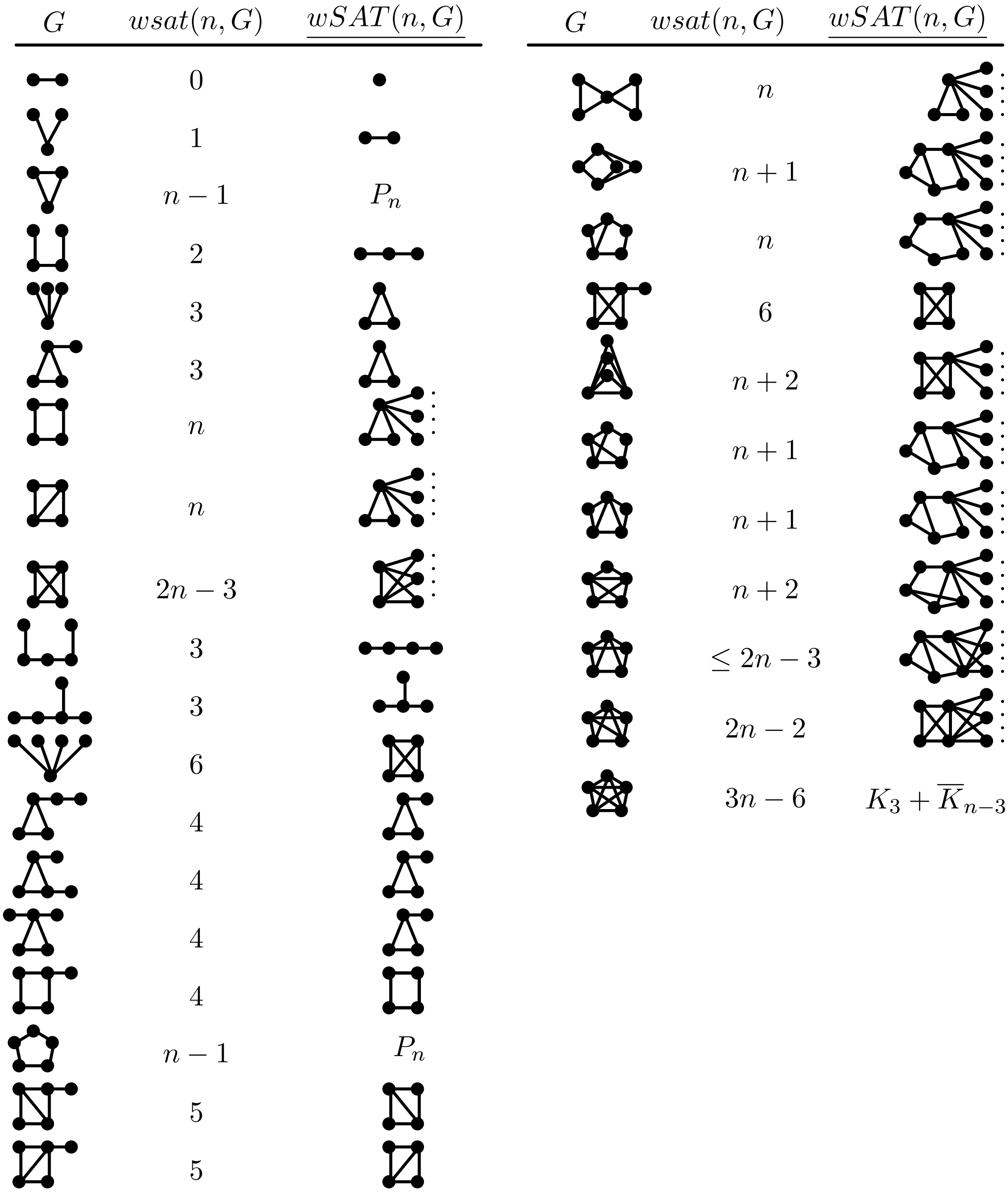

Figure 3. Weak saturation numbers for small graphs.

Additional copies of T can be obtained by adding the edges of G in the following order: the edge deleted from T to obtain T', edges from v_i to S if $d_i > 0$, the edges from vertices in S to $w_3, \ldots, w_{2r-1}$, and the edges from vertices in S to $w_{2r-2}, \ldots, w_2$. At this point all vertices in S are adjacent to all of the vertices of $\{w_1, w_2, \ldots, w_{2r}\}$, and so any edge between vertices of S can be added and it will be on a path that can replace the path $(w_1, w_2, \ldots, w_{2r})$. This gives a complete graph with $|S|$ vertices, which can be extended to a complete graph for G.

(7) Let T' be the tree obtained from T by deleting one of the endvertices of one of the stars, and let G of a graph of order n containing the an induced copy of T'. Let S be the vertices of G not in T', and add an edge e in S. Let $w_1, w_2, \ldots, w_{2r+1}$ be the vertices of a path such that $d(w_i) = 0$ for $1 < i < 2r$ and such that $d(w_1), d(w_{2r}) > 0$. Additional copies of T can be obtained by adding the edges of G in the following order: the edge deleted from T to obtain T', edges from v_i to S if $d_i > 0$, and the edges from vertices in S to $w_3, \ldots, w_{2r-1}$. At this point all vertices in S are adjacent to all of the vertices of $\{w_1, w_3, \ldots, w_{2r+1}\}$, and so any edge between a vertex of S and a vertex of e can be added and it will be on a path that can replace the path $(w_1, w_2, \ldots, w_{2r})$. A repetition of this will result in a complete graph with $|S|$ vertices, which can be extended to a complete graph for G. This completes the proof of (7) and of Theorem 21. ■

Using the results of this section, $\mathbf{wsat}(n, T)$ can easily be determined for all small order trees, and in particular all trees with at most 10 vertices. For trees with at most $p \leq 6$ vertices the weak saturation number is $p-2$ except for stars. For trees with at most $p \leq 10$ vertices the weak saturation number is also $p-2$ except for stars, double stars in which each endstar has at least 3 edges, double brooms with connecting paths with a odd number of vertices, one caterpillar $C(2, 0, 2, 0, 2)$, and the tree of order 10 obtained from a $K_{1,3}$ by adding endstars with 2 edges to each vertex of the $K_{1,3}$. Thus, all but 22 of the 202 trees T_p of order at most 10 have $\mathbf{wsat}(n, T_p) = p-2$.

In Figure 3 is a list of all connected graphs F with at most 5 vertices along with their weak saturation number $\mathbf{wsat}(n, F)$ and a graph in $\underline{\mathbf{wSAT}}(n, F)$.

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