# BOUNDS ON THE SIGNED 2-INDEPENDENCE NUMBER IN GRAPHS 

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#### Abstract

Let $G$ be a finite and simple graph with vertex set $V(G)$, and let $f$ : $V(G) \rightarrow\{-1,1\}$ be a two-valued function. If $\sum_{x \in N[v]} f(x) \leq 1$ for each $v \in V(G)$, where $N[v]$ is the closed neighborhood of $v$, then $f$ is a signed 2independence function on $G$. The weight of a signed 2 -independence function $f$ is $w(f)=\sum_{v \in V(G)} f(v)$. The maximum of weights $w(f)$, taken over all signed 2-independence functions $f$ on $G$, is the signed 2-independence number $\alpha_{s}^{2}(G)$ of $G$.

In this work, we mainly present upper bounds on $\alpha_{s}^{2}(G)$, as for example $\alpha_{s}^{2}(G) \leq n-2\lceil\Delta(G) / 2\rceil$, and we prove the Nordhaus-Gaddum type inequality $\alpha_{s}^{2}(G)+\alpha_{s}^{2}(\bar{G}) \leq n+1$, where $n$ is the order and $\Delta(G)$ is the maximum degree of the graph $G$. Some of our theorems improve well-known results on the signed 2-independence number.


Keywords: bounds, signed 2-independence function, signed 2-independence number, Nordhaus-Gaddum type result.
2010 Mathematics Subject Classification: 05C69.

## 1. Introduction

Domination and independence in graphs are well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi and Slater $[2,3]$.

All graphs considered are undirected, simple and finite. The vertex set and edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$. The order $n=n(G)$ and size $q=q(G)$ of a graph $G$ is the number of vertices and edges, respectively. The open neighborhood of $v \in V(G)$ is $N_{G}(v)=\{u \in V(G) \mid u v \in E(G)\}$ and the closed neighborhood of $v$ is $N_{G}[v]=N_{G}(v) \cup\{v\}$. The degree of $v$ in $G$,
denoted by $d_{G}(v)$, is the cardinality of $N_{G}(v)$. We write $\Delta(G)$ and $\delta(G)$ for the maximum and minimum degree of $G$. If the graph $G$ is clear from context, we simply use $N(v), N[v], d(v), \Delta$ and $\delta$ instead of $N_{G}(v), N_{G}[v], d_{G}(v), \Delta(G)$ and $\delta(G)$, respectively. For two disjoint subsets $A$ and $B$ of $V(G)$, let $e(A, B)$ denote the number of edges between $A$ and $B$. A graph $G$ is $r$-partite with vertex classes $V_{1}, V_{2}, \ldots, V_{r}$ if $V(G)=V_{1} \cup V_{2} \cup \cdots \cup V_{r}, V_{i} \cap V_{j}=\emptyset$ whenever $1 \leq i<j \leq r$, and no edge joins two vertices in the same class. The subgraph of $G$ induced by $A$ is denoted by $G[A]$. The complete graph of order $n$ is denoted by $K_{n}$. A graph is $K_{r+1}$-free if it does not contain the complete graph $K_{r+1}$ as a subgraph. The complement of a graph $G$ is denoted by $\bar{G}$.

For a two-valued function $f: V(G) \rightarrow\{-1,1\}$, the weight of $f$ is $w(f)=$ $\sum_{v \in V(G)} f(v)$. For a subset $A \subseteq V(G)$, we define $f(A)=\sum_{v \in A} f(v)$ and so $w(f)=f(V(G))$. For a vertex $v$ in $V(G)$, we denote $f(N[v])$ by $f[v]$ for notational convenience. The function $f$ is defined in [1] to be a signed dominating function of $G$ if $f[v]=f(N[v]) \geq 1$ for every $v \in V(G)$. The signed domination number of $G$ is the minimum weight of a signed dominating function on $G$.

The function $f: V(G) \rightarrow\{-1,1\}$ is defined in [7] to be a signed 2-independence function on $G$ if $f[v]=f(N[v]) \leq 1$ for every $v \in V(G)$. The signed 2independence number $\alpha_{s}^{2}(G)$ of $G$ is the maximum weight of a signed 2-independence function on $G$. Hence the signed 2-independence number is a certain dual to the signed domination number of a graph. Results on the signed 2-independence number can be found in $[4,5,7]$.

In this paper we continue the investigations of the signed 2-independence number. We mainly present upper bounds on $\alpha_{s}^{2}(G)$ for general graphs and $K_{r+1}$-free graphs. In addition, we prove the Nordhaus-Gaddum type inequality $\alpha_{s}^{2}(G)+\alpha_{s}^{2}(\bar{G}) \leq n+1$. Some of our results improve known bounds on the signed 2-independence numbers of graphs given by Henning [4] in 2002 and Shan, Sohn and Kang [5] in 2003.

Zelinka [7] determined the signed 2-independence number of complete graphs, and he established a sharp upper bound on $\alpha_{s}^{2}(G)$ for regular graphs $G$.
Theorem 1 [7]. If $G$ is isomorphic to the complete graph $K_{n}$, then $\alpha_{s}^{2}(G)=0$ when $n$ is even and $\alpha_{s}^{2}(G)=1$ when $n$ is odd.

Theorem 2 [7]. If $G$ is an r-regular graph of order $n$, then $\alpha_{s}^{2}(G) \leq n /(r+1)$ when $r$ is even and $\alpha_{s}^{2}(G) \leq 0$ when $r$ is odd.

## 2. Main Results

Theorem 3. If $G$ is a graph of order $n$, then

$$
2-n \leq \alpha_{s}^{2}(G) \leq n-2\left\lceil\frac{\Delta}{2}\right\rceil
$$

Proof. Let $w \in V(G)$ be a vertex of maximum degree $d(w)=\Delta$, and let $f$ be a signed 2-independence function on $G$ for which $f(V(G))=\alpha_{s}^{2}(G)$. We define the two sets $P=\{v \in V(G) \mid f(v)=1\}$ and $M=\{v \in V(G) \mid f(v)=-1\}$. If $|P|=p$ and $|M|=m$, then $n=p+m$ and $\alpha_{s}^{2}(G)=p-m=n-2 m$.

Assume first that $f(w)=1$ and therefore $w \in P$. The condition $f[w] \leq 1$ leads to the inequality $|N(w) \cap P|-|N(w) \cap M| \leq 0$, and since $w$ is a vertex of maximum degree, we have $|N(w) \cap P|+|N(w) \cap M|=\Delta$. Combining the last two inequalities, we deduce that $m \geq|N(w) \cap M| \geq\lceil\Delta / 2\rceil$, and this yields to

$$
\alpha_{s}^{2}(G)=n-2 m \leq n-2\left\lceil\frac{\Delta}{2}\right\rceil .
$$

Assume second that $f(w)=-1$ and so $w \in M$. As $f[w] \leq 1$ and $d(w)=\Delta$, we obtain $|N(w) \cap P|-|N(w) \cap M| \leq 2$ and $|N(w) \cap P|+|N(w) \cap M|=\Delta$. Combining these two inequalities, we conclude that

$$
m \geq|N(w) \cap M|+1=\frac{2|N(w) \cap M|+2}{2} \geq \frac{\Delta}{2}
$$

and thus $m \geq\lceil\Delta / 2\rceil$. This implies $\alpha_{s}^{2}(G)=n-2 m \leq n-2\lceil\Delta / 2\rceil$ as above, and the upper bound on $\alpha_{s}^{2}(G)$ is proved.

For the first inequality define $f: V(G) \rightarrow\{-1,1\}$ by $f(v)=1$ for an arbitrary vertex $v \in V(G)$ and $f(x)=-1$ for each vertex $x \in V(G)-\{v\}$. Obviously, $f$ is a signed 2-independence function on $G$ of weight $2-n$ and thus $\alpha_{s}^{2}(G) \geq 2-n$.

If $G$ is isomorphic to the star $K_{1, \Delta}$, then

$$
\alpha_{s}^{2}(G)=n-2\left\lceil\frac{\Delta}{2}\right\rceil \text {, }
$$

and therefore the upper bound on $\alpha_{s}^{2}(G)$ in Theorem 3 is sharp.
Corollary 4. If $G$ is a graph of order $n$, then $\alpha_{s}^{2}(G)=n$ if and only if $G$ is the empty graph.

Proof. If $G$ is the empty graph, then $f: V(G) \rightarrow\{-1,1\}$ with $f(v)=1$ for each vertex $v \in V(G)$ is a signed 2-independence function on $G$ of weight $n$ and thus $\alpha_{s}^{2}(G)=n$.

Conversely, assume that $\alpha_{s}^{2}(G)=n$. If we suppose that $G$ is not the empty graph, then $\Delta \geq 1$, and Theorem 3 leads to the contradiction $n=\alpha_{s}^{2}(G) \leq n-2$. Therefore $G$ is the empty graph, and the proof is complete.

Obviously, $\alpha_{s}^{2}\left(K_{2}\right)=0=n-2$, and therefore equality in the left inequality of Theorem 3 is achieved. However, if $G$ is a graph of order $n \geq 3$, then the next result improves the lower bound in Theorem 3.

Theorem 5. If $G$ is a graph of order $n \geq 3$, then $\alpha_{s}^{2}(G) \geq 4-n$.
Proof. If $G$ has two non-adjacent vertices $u$ and $v$, then $f: V(G) \rightarrow\{-1,1\}$ with $f(u)=f(v)=1$ and $f(x)=-1$ for each $x \in V(G)-\{u, v\}$ is a signed 2independence function on $G$ of weight $4-n$ and thus $\alpha_{s}^{2}(G) \geq 4-n$. Otherwise, $G$ is the complete graph. If $n \geq 4$, then it follows from Theorem 1 that $\alpha_{s}^{2}(G) \geq$ $0 \geq 4-n$, and if $n=3$, then Theorem 1 implies that $\alpha_{s}^{2}(G)=1=4-n$.

As an application of Theorems 1, 2, 3 and Corollary 4, we will prove the following Nordhaus-Gaddum type result.

Theorem 6. If $G$ is a graph of order n, then

$$
\alpha_{s}^{2}(G)+\alpha_{s}^{2}(\bar{G}) \leq n+1
$$

with equality if and only if $n$ is odd and $G=K_{n}$ or $\bar{G}=K_{n}$.
Proof. Theorem 3 implies that

$$
\begin{align*}
\alpha_{s}^{2}(G)+\alpha_{s}^{2}(\bar{G}) & \leq n-\Delta(G)+n-\Delta(\bar{G}) \\
& =n-\Delta(G)+n-(n-\delta(G)-1)  \tag{1}\\
& =n+1-\Delta(G)+\delta(G)
\end{align*}
$$

and the desired bound follows, since $\delta(G)-\Delta(G) \leq 0$. If $n$ is odd and $G=K_{n}$ or $\bar{G}=K_{n}$, then we deduce from Theorem 1 and Corollary 4 that $\alpha_{s}^{2}(G)+\alpha_{s}^{2}(\bar{G})=$ $n+1$.

If $\Delta(G)-\delta(G) \geq 1$, then the inequality chain (1) leads to $\alpha_{s}^{2}(G)+\alpha_{s}^{2}(\bar{G}) \leq n$.
Assume now that $\Delta(G)=\delta(G)=\delta$, implying that $G$ is $\delta$-regular.
Assume first that $n$ is even. If $\delta$ is even, then $\delta(\bar{G})=n-\delta-1$ is odd, and Theorem 2 implies that $\alpha_{s}^{2}(\bar{G}) \leq 0$ and thus $\alpha_{s}^{2}(G)+\alpha_{s}^{2}(\bar{G}) \leq n$. If $\delta$ is odd, then Theorem 2 implies that $\alpha_{s}^{2}(G) \leq 0$ and thus $\alpha_{s}^{2}(G)+\alpha_{s}^{2}(\bar{G}) \leq n$.

Finally assume that $n$ is odd. The handshaking lemma implies that $\delta$ and $\delta(\bar{G})$ are even. If $\delta=0$ or $\delta(\bar{G})=0$, then $\bar{G}=K_{n}$ or $G=K_{n}$ and thus $\alpha_{s}^{2}(G)+\alpha_{s}^{2}(\bar{G})=n+1$. In the remaining case that $\delta \geq 2$ and $\delta(\bar{G}) \geq 2$, Theorem 2 shows that

$$
\alpha_{s}^{2}(G)+\alpha_{s}^{2}(\bar{G}) \leq \frac{n}{\delta+1}+\frac{n}{\delta(\bar{G})+1} \leq \frac{2 n}{3}<n
$$

and the proof of Theorem 6 is complete.
The following upper bound on $\alpha_{s}^{2}(G)$ was obtained by Henning [4] in 2002.
Theorem 7 [4]. If $G$ is a connected graph of order $n \geq 2$ and size $q$, then

$$
\alpha_{s}^{2}(G) \leq \frac{4 q-n}{5}
$$

We now improve the bound in Theorem 7.
Theorem 8. If $G$ is a connected graph of order $n \geq 2$ and size $q$, then

$$
\alpha_{s}^{2}(G) \leq \frac{4 q+(2-\delta-2\lceil\delta / 2\rceil) n}{2+\delta+2\lceil\delta / 2\rceil}
$$

Proof. Let $f$ be a signed 2-independence function on $G$ for which $f(V(G))=$ $\alpha_{s}^{2}(G)$, and let $P, M, p$ and $m$ be defined as in the proof of Theorem 3 . Then $n=p+m$ and $\alpha_{s}^{2}(G)=p-m=2 p-n$. The condition $f[v] \leq 1$ implies that $|N(v) \cap P| \leq|N(v) \cap M|$ for $v \in P$ and $|N(v) \cap P| \leq|N(v) \cap M|+2$ for $v \in M$. Thus we obtain

$$
\delta \leq d(v)=|N(v) \cap P|+|N(v) \cap M| \leq 2|N(v) \cap M|
$$

and so $|N(v) \cap M| \geq\left\lceil\frac{\delta}{2}\right\rceil$ for each $v \in P$. Hence we deduce that

$$
\begin{equation*}
e(P, M)=\sum_{v \in P}|N(v) \cap M| \geq p\left\lceil\frac{\delta}{2}\right\rceil=(n-m)\left\lceil\frac{\delta}{2}\right\rceil \tag{2}
\end{equation*}
$$

In addition, we have

$$
\begin{align*}
e(P, M) & =\sum_{v \in M}|N(v) \cap P| \leq \sum_{v \in M}(|N(v) \cap M|+2) \\
& =2|E(G[M])|+2 m \tag{3}
\end{align*}
$$

Combining (2) and (3), we find that

$$
\begin{equation*}
2|E(G[M])| \geq p\left\lceil\frac{\delta}{2}\right\rceil-2 m \tag{4}
\end{equation*}
$$

Furthermore, we deduce from (2) that

$$
\begin{align*}
e(P, M)+|E(G[P])| & =\sum_{v \in P}|N(v) \cap M|+\frac{1}{2} \sum_{v \in P}|N(v) \cap P| \\
& =\frac{1}{2} \sum_{v \in P}|N(v) \cap M|+\frac{1}{2} \sum_{v \in P}|N(v) \cap M| \\
& +\frac{1}{2} \sum_{v \in P}|N(v) \cap P|  \tag{5}\\
& =\frac{1}{2} \sum_{v \in P}|N(v)|+\frac{1}{2} \sum_{v \in P}|N(v) \cap M| \\
& \geq \frac{1}{2} p \delta+\frac{1}{2} p\left[\frac{\delta}{2}\right\rceil .
\end{align*}
$$

According to (4) and (5), we have

$$
\begin{aligned}
2 q & =2 e(P, M)+2|E(G[P])|+2|E(G[M])| \\
& \geq p \delta+2 p\left\lceil\frac{\delta}{2}\right\rceil-2 m=p\left(2+\delta+2\left\lceil\frac{\delta}{2}\right\rceil\right)-2 n
\end{aligned}
$$

Hence

$$
p \leq \frac{2 q+2 n}{2+\delta+2\lceil\delta / 2\rceil}
$$

and so we obtain the desired bound as follows

$$
\alpha_{s}^{2}(G)=2 p-n \leq \frac{4 q+(2-\delta-2\lceil\delta / 2\rceil) n}{2+\delta+2\lceil\delta / 2\rceil}
$$

Note that

$$
\frac{4 q+(2-\delta-2\lceil\delta / 2\rceil) n}{2+\delta+2\lceil\delta / 2\rceil} \leq \frac{4 q-n}{5}
$$

for $\delta \geq 1$, and therefore Theorem 8 is an improvement of Theorem 7 .
For the next result, we use the famous theorem of Turán [6].
Theorem 9 [6]. Let $r \geq 1$ be an integer. If $G$ is a $K_{r+1}$-free graph of order $n$, then

$$
|E(G)| \leq \frac{r-1}{2 r} n^{2}
$$

Theorem 10. If $G$ is a $K_{r+1}$-free graph of order $n$ with $r \geq 2$ and minimum degree $\delta \geq 1$, then

$$
\alpha_{s}^{2}(G) \leq n+\frac{r(2+\lceil\delta / 2\rceil)}{r-1}-\sqrt{\left(\frac{r(2+\lceil\delta / 2\rceil)}{r-1}\right)^{2}+\frac{4 r n\lceil\delta / 2\rceil}{r-1}}
$$

Proof. Let $f, P, M, p$ and $m$ be defined as in the proof of Theorem 3. By (2) and (3), we obtain

$$
\begin{equation*}
(n-m)\lceil\delta / 2\rceil \leq e(P, M) \leq 2|E(G[M])|+2 m \tag{6}
\end{equation*}
$$

Since $G$ is $K_{r+1}$-free, the induced subgraph $G[M]$ is also $K_{r+1}$-free, and hence it follows from Theorem 9 that $|E(G[M])| \leq(r-1) m^{2} / 2 r$. Using (6), we obtain

$$
(n-m)\lceil\delta / 2\rceil \leq e(P, M) \leq \frac{r-1}{r} m^{2}+2 m
$$

and so

$$
m^{2}+\frac{r}{r-1}(2+\lceil\delta / 2\rceil) m-\frac{r}{r-1} n\lceil\delta / 2\rceil \geq 0
$$

This yields

$$
m \geq-\frac{r}{2(r-1)}(2+\lceil\delta / 2\rceil)+\sqrt{\left(\frac{r}{2(r-1)}(2+\lceil\delta / 2\rceil)\right)^{2}+\frac{r}{r-1} n\lceil\delta / 2\rceil},
$$

and we obtain the desired bound as follows

$$
\alpha_{s}^{2}(G)=n-2 m \leq n+\frac{r(2+\lceil\delta / 2\rceil)}{r-1}-\sqrt{\left(\frac{r(2+\lceil\delta / 2\rceil)}{r-1}\right)^{2}+\frac{4 r n\lceil\delta / 2\rceil}{r-1}} .
$$

Since

$$
\frac{r(2+\lceil\delta / 2\rceil)}{r-1}-\sqrt{\left(\frac{r(2+\lceil\delta / 2\rceil)}{r-1}\right)^{2}+\frac{4 r n\lceil\delta / 2\rceil}{r-1}} \leq \frac{3 r}{r-1}-\sqrt{\left(\frac{3 r}{r-1}\right)^{2}+\frac{4 r n}{r-1}}
$$

for $\delta \geq 1$, the next known result is an immediate consequence of Theorem 10 .
Corollary 11 [5]. If $G$ is an $r$-partite graph of order $n$ with $r \geq 2$, then

$$
\alpha_{s}^{2}(G) \leq n+\frac{3 r}{r-1}-\sqrt{\left(\frac{3 r}{r-1}\right)^{2}+\frac{4 r n}{r-1}} .
$$

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Received 21 February 2012
Revised 3 September 2012
Accepted 3 September 2012

