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# BOUNDS ON THE SIGNED 2-INDEPENDENCE NUMBER IN GRAPHS

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**Abstract**

Let $G$ be a finite and simple graph with vertex set $V(G)$, and let $f : V(G) \to \{-1, 1\}$ be a two-valued function. If $\sum_{x \in N[v]} f(x) \leq 1$ for each $v \in V(G)$, where $N[v]$ is the closed neighborhood of $v$, then $f$ is a signed 2-independence function on $G$. The weight of a signed 2-independence function $f$ is $w(f) = \sum_{v \in V(G)} f(v)$. The maximum of weights $w(f)$, taken over all signed 2-independence functions $f$ on $G$, is the signed 2-independence number $\alpha_s^2(G)$ of $G$.

In this work, we mainly present upper bounds on $\alpha_s^2(G)$, as for example $\alpha_s^2(G) \leq n - 2\lceil \Delta(G)/2 \rceil$, and we prove the Nordhaus-Gaddum type inequality $\alpha_s^2(G) + \alpha_s^2(\overline{G}) \leq n + 1$, where $n$ is the order and $\Delta(G)$ is the maximum degree of the graph $G$. Some of our theorems improve well-known results on the signed 2-independence number.

**Keywords:** bounds, signed 2-independence function, signed 2-independence number, Nordhaus-Gaddum type result.

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## 1. Introduction

Domination and independence in graphs are well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi and Slater [2, 3].

All graphs considered are undirected, simple and finite. The vertex set and edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$. The *order* $n = n(G)$ and *size* $q = q(G)$ of a graph $G$ is the number of vertices and edges, respectively. The *open neighborhood* of $v \in V(G)$ is $N_G(v) = \{u \in V(G) |\ uv \in E(G)\}$ and the *closed neighborhood* of $v$ is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of $v$ in $G$, denoted by $d_G(v)$, is the cardinality of $N_G(v)$. We write $\Delta(G)$ and $\delta(G)$ for the *maximum* and *minimum degree* of $G$. If the graph $G$ is clear from context, we simply use $N(v), N[v]$, $d(v)$, $\Delta$ and $\delta$ instead of $N_G(v), N_G[v]$, $d_G(v)$, $\Delta(G)$ and $\delta(G)$, respectively. For two disjoint subsets $A$ and $B$ of $V(G)$, let $e(A,B)$ denote the number of edges between $A$ and $B$. A graph $G$ is *$r$-partite* with vertex classes $V_1, V_2, \ldots, V_r$ if $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$, $V_i \cap V_j = \emptyset$ whenever $1 \leq i < j \leq r$, and no edge joins two vertices in the same class. The subgraph of $G$ induced by $A$ is denoted by $G[A]$. The complete graph of order $n$ is denoted by $K_n$. A graph is *$K_{r+1}$-free* if it does not contain the complete graph $K_{r+1}$ as a subgraph. The *complement* of a graph $G$ is denoted by $\overline{G}$.

For a two-valued function $f : V(G) \to \{-1, 1\}$, the *weight* of $f$ is $w(f) = \sum_{v \in V(G)} f(v)$. For a subset $A \subseteq V(G)$, we define $f(A) = \sum_{v \in A} f(v)$ and so $w(f) = f(V(G))$. For a vertex $v$ in $V(G)$, we denote $f(N[v])$ by $f[v]$ for notational convenience. The function $f$ is defined in [1] to be a *signed dominating function* of $G$ if $f[v] = f(N[v]) \geq 1$ for every $v \in V(G)$. The *signed domination number* of $G$ is the minimum weight of a signed dominating function on $G$.

The function $f : V(G) \to \{-1, 1\}$ is defined in [7] to be a *signed 2-independence function* on $G$ if $f[v] = f(N[v]) \leq 1$ for every $v \in V(G)$. The *signed 2-independence number* $\alpha_s^2(G)$ of $G$ is the maximum weight of a signed 2-independence function on $G$. Hence the signed 2-independence number is a certain dual to the signed domination number of a graph. Results on the signed 2-independence number can be found in [4, 5, 7].

In this paper we continue the investigations of the signed 2-independence number. We mainly present upper bounds on $\alpha_s^2(G)$ for general graphs and $K_{r+1}$-free graphs. In addition, we prove the Nordhaus-Gaddum type inequality $\alpha_s^2(G) + \alpha_s^2(\overline{G}) \leq n+1$. Some of our results improve known bounds on the signed 2-independence numbers of graphs given by Henning [4] in 2002 and Shan, Sohn and Kang [5] in 2003.

Zelinka [7] determined the signed 2-independence number of complete graphs, and he established a sharp upper bound on $\alpha_s^2(G)$ for regular graphs $G$.

**Theorem 1** [7]. *If $G$ is isomorphic to the complete graph $K_n$, then $\alpha_s^2(G) = 0$ when $n$ is even and $\alpha_s^2(G) = 1$ when $n$ is odd.*

**Theorem 2** [7]. *If $G$ is an $r$-regular graph of order $n$, then $\alpha_s^2(G) \leq n/(r+1)$ when $r$ is even and $\alpha_s^2(G) \leq 0$ when $r$ is odd.*

## 2. Main Results

**Theorem 3.** *If $G$ is a graph of order $n$, then*

$$2 - n \leq \alpha_s^2(G) \leq n - 2\left\lceil \frac{\Delta}{2} \right\rceil.$$

***Proof.*** Let $w \in V(G)$ be a vertex of maximum degree $d(w) = \Delta$, and let $f$ be a signed 2-independence function on $G$ for which $f(V(G)) = \alpha_s^2(G)$. We define the two sets $P = \{v \in V(G) \,|\, f(v) = 1\}$ and $M = \{v \in V(G) \,|\, f(v) = -1\}$. If $|P| = p$ and $|M| = m$, then $n = p + m$ and $\alpha_s^2(G) = p - m = n - 2m$.

Assume first that $f(w) = 1$ and therefore $w \in P$. The condition $f[w] \le 1$ leads to the inequality $|N(w) \cap P| - |N(w) \cap M| \le 0$, and since $w$ is a vertex of maximum degree, we have $|N(w) \cap P| + |N(w) \cap M| = \Delta$. Combining the last two inequalities, we deduce that $m \ge |N(w) \cap M| \ge \lceil \Delta/2 \rceil$, and this yields to

$$\alpha_s^2(G) = n - 2m \le n - 2\left\lceil \frac{\Delta}{2} \right\rceil.$$

Assume second that $f(w) = -1$ and so $w \in M$. As $f[w] \le 1$ and $d(w) = \Delta$, we obtain $|N(w) \cap P| - |N(w) \cap M| \le 2$ and $|N(w) \cap P| + |N(w) \cap M| = \Delta$. Combining these two inequalities, we conclude that

$$m \ge |N(w) \cap M| + 1 = \frac{2|N(w) \cap M| + 2}{2} \ge \frac{\Delta}{2}$$

and thus $m \ge \lceil \Delta/2 \rceil$. This implies $\alpha_s^2(G) = n - 2m \le n - 2\lceil \Delta/2 \rceil$ as above, and the upper bound on $\alpha_s^2(G)$ is proved.

For the first inequality define $f : V(G) \to \{-1, 1\}$ by $f(v) = 1$ for an arbitrary vertex $v \in V(G)$ and $f(x) = -1$ for each vertex $x \in V(G) - \{v\}$. Obviously, $f$ is a signed 2-independence function on $G$ of weight $2 - n$ and thus $\alpha_s^2(G) \ge 2 - n$. ■

If $G$ is isomorphic to the star $K_{1,\Delta}$, then

$$\alpha_s^2(G) = n - 2\left\lceil \frac{\Delta}{2} \right\rceil,$$

and therefore the upper bound on $\alpha_s^2(G)$ in Theorem 3 is sharp.

**Corollary 4.** *If $G$ is a graph of order $n$, then $\alpha_s^2(G) = n$ if and only if $G$ is the empty graph.*

***Proof.*** If $G$ is the empty graph, then $f : V(G) \to \{-1, 1\}$ with $f(v) = 1$ for each vertex $v \in V(G)$ is a signed 2-independence function on $G$ of weight $n$ and thus $\alpha_s^2(G) = n$.

Conversely, assume that $\alpha_s^2(G) = n$. If we suppose that $G$ is not the empty graph, then $\Delta \ge 1$, and Theorem 3 leads to the contradiction $n = \alpha_s^2(G) \le n - 2$. Therefore $G$ is the empty graph, and the proof is complete. ■

Obviously, $\alpha_s^2(K_2) = 0 = n - 2$, and therefore equality in the left inequality of Theorem 3 is achieved. However, if $G$ is a graph of order $n \ge 3$, then the next result improves the lower bound in Theorem 3.

**Theorem 5.** *If $G$ is a graph of order $n \geq 3$, then $\alpha_s^2(G) \geq 4 - n$.*

***Proof.*** If $G$ has two non-adjacent vertices $u$ and $v$, then $f : V(G) \rightarrow \{-1, 1\}$ with $f(u) = f(v) = 1$ and $f(x) = -1$ for each $x \in V(G) - \{u, v\}$ is a signed 2-independence function on $G$ of weight $4 - n$ and thus $\alpha_s^2(G) \geq 4 - n$. Otherwise, $G$ is the complete graph. If $n \geq 4$, then it follows from Theorem 1 that $\alpha_s^2(G) \geq 0 \geq 4 - n$, and if $n = 3$, then Theorem 1 implies that $\alpha_s^2(G) = 1 = 4 - n$. ■

As an application of Theorems 1, 2, 3 and Corollary 4, we will prove the following Nordhaus-Gaddum type result.

**Theorem 6.** *If $G$ is a graph of order $n$, then*

$$\alpha_s^2(G) + \alpha_s^2(\overline{G}) \leq n + 1$$

*with equality if and only if $n$ is odd and $G = K_n$ or $\overline{G} = K_n$.*

***Proof.*** Theorem 3 implies that

$$\begin{aligned} \alpha_s^2(G) + \alpha_s^2(\overline{G}) &\leq n - \Delta(G) + n - \Delta(\overline{G}) \\ &= n - \Delta(G) + n - (n - \delta(G) - 1) \\ &= n + 1 - \Delta(G) + \delta(G), \end{aligned} \tag{1}$$

and the desired bound follows, since $\delta(G) - \Delta(G) \leq 0$. If $n$ is odd and $G = K_n$ or $\overline{G} = K_n$, then we deduce from Theorem 1 and Corollary 4 that $\alpha_s^2(G) + \alpha_s^2(\overline{G}) = n + 1$.

If $\Delta(G) - \delta(G) \geq 1$, then the inequality chain (1) leads to $\alpha_s^2(G) + \alpha_s^2(\overline{G}) \leq n$.

Assume now that $\Delta(G) = \delta(G) = \delta$, implying that $G$ is $\delta$-regular.

Assume first that $n$ is even. If $\delta$ is even, then $\delta(\overline{G}) = n - \delta - 1$ is odd, and Theorem 2 implies that $\alpha_s^2(\overline{G}) \leq 0$ and thus $\alpha_s^2(G) + \alpha_s^2(\overline{G}) \leq n$. If $\delta$ is odd, then Theorem 2 implies that $\alpha_s^2(G) \leq 0$ and thus $\alpha_s^2(G) + \alpha_s^2(\overline{G}) \leq n$.

Finally assume that $n$ is odd. The handshaking lemma implies that $\delta$ and $\delta(\overline{G})$ are even. If $\delta = 0$ or $\delta(\overline{G}) = 0$, then $\overline{G} = K_n$ or $G = K_n$ and thus $\alpha_s^2(G) + \alpha_s^2(\overline{G}) = n + 1$. In the remaining case that $\delta \geq 2$ and $\delta(\overline{G}) \geq 2$, Theorem 2 shows that

$$\alpha_s^2(G) + \alpha_s^2(\overline{G}) \leq \frac{n}{\delta + 1} + \frac{n}{\delta(\overline{G}) + 1} \leq \frac{2n}{3} < n,$$

and the proof of Theorem 6 is complete. ■

The following upper bound on $\alpha_s^2(G)$ was obtained by Henning [4] in 2002.

**Theorem 7** [4]**.** *If $G$ is a connected graph of order $n \geq 2$ and size $q$, then*

$$\alpha_s^2(G) \leq \frac{4q - n}{5}.$$

We now improve the bound in Theorem 7.

**Theorem 8.** *If $G$ is a connected graph of order $n \geq 2$ and size $q$, then*

$$\alpha_s^2(G) \leq \frac{4q + (2 - \delta - 2\lceil\delta/2\rceil)n}{2 + \delta + 2\lceil\delta/2\rceil}.$$

***Proof.*** Let $f$ be a signed 2-independence function on $G$ for which $f(V(G)) = \alpha_s^2(G)$, and let $P, M, p$ and $m$ be defined as in the proof of Theorem 3. Then $n = p + m$ and $\alpha_s^2(G) = p - m = 2p - n$. The condition $f[v] \leq 1$ implies that $|N(v) \cap P| \leq |N(v) \cap M|$ for $v \in P$ and $|N(v) \cap P| \leq |N(v) \cap M| + 2$ for $v \in M$. Thus we obtain

$$\delta \leq d(v) = |N(v) \cap P| + |N(v) \cap M| \leq 2|N(v) \cap M|$$

and so $|N(v) \cap M| \geq \lceil \frac{\delta}{2} \rceil$ for each $v \in P$. Hence we deduce that

$$e(P, M) = \sum\nolimits_{v \in P} |N(v) \cap M| \geq p \left\lceil \frac{\delta}{2} \right\rceil = (n - m) \left\lceil \frac{\delta}{2} \right\rceil. \tag{2}$$

In addition, we have

$$\begin{aligned} e(P, M) &= \sum\nolimits_{v \in M} |N(v) \cap P| \leq \sum\nolimits_{v \in M} (|N(v) \cap M| + 2) \\ &= 2|E(G[M])| + 2m. \end{aligned} \tag{3}$$

Combining (2) and (3), we find that

$$2|E(G[M])| \geq p \left\lceil \frac{\delta}{2} \right\rceil - 2m. \tag{4}$$

Furthermore, we deduce from (2) that

$$\begin{aligned} e(P, M) + |E(G[P])| &= \sum\nolimits_{v \in P} |N(v) \cap M| + \frac{1}{2} \sum\nolimits_{v \in P} |N(v) \cap P| \\ &= \frac{1}{2} \sum\nolimits_{v \in P} |N(v) \cap M| + \frac{1}{2} \sum\nolimits_{v \in P} |N(v) \cap M| \\ &+ \frac{1}{2} \sum\nolimits_{v \in P} |N(v) \cap P| \\ &= \frac{1}{2} \sum\nolimits_{v \in P} |N(v)| + \frac{1}{2} \sum\nolimits_{v \in P} |N(v) \cap M| \\ &\geq \frac{1}{2} p\delta + \frac{1}{2} p \left\lceil \frac{\delta}{2} \right\rceil. \end{aligned} \tag{5}$$

According to (4) and (5), we have

$$\begin{aligned} 2q &= 2e(P, M) + 2|E(G[P])| + 2|E(G[M])| \\ &\geq p\delta + 2p \left\lceil \frac{\delta}{2} \right\rceil - 2m = p \left( 2 + \delta + 2 \left\lceil \frac{\delta}{2} \right\rceil \right) - 2n. \end{aligned}$$

Hence

$$p \leq \frac{2q + 2n}{2 + \delta + 2\lceil \delta/2 \rceil},$$

and so we obtain the desired bound as follows

$$\alpha_s^2(G) = 2p - n \leq \frac{4q + (2 - \delta - 2\lceil \delta/2 \rceil)n}{2 + \delta + 2\lceil \delta/2 \rceil}.$$

■

Note that

$$\frac{4q + (2 - \delta - 2\lceil \delta/2 \rceil)n}{2 + \delta + 2\lceil \delta/2 \rceil} \leq \frac{4q - n}{5}$$

for $\delta \geq 1$, and therefore Theorem 8 is an improvement of Theorem 7.

For the next result, we use the famous theorem of Turán [6].

**Theorem 9** [6]**.** *Let $r \geq 1$ be an integer. If $G$ is a $K_{r+1}$-free graph of order $n$, then*

$$|E(G)| \leq \frac{r-1}{2r} n^2.$$

**Theorem 10.** *If $G$ is a $K_{r+1}$-free graph of order $n$ with $r \geq 2$ and minimum degree $\delta \geq 1$, then*

$$\alpha_s^2(G) \leq n + \frac{r(2 + \lceil \delta/2 \rceil)}{r-1} - \sqrt{\left(\frac{r(2 + \lceil \delta/2 \rceil)}{r-1}\right)^2 + \frac{4rn\lceil \delta/2 \rceil}{r-1}}.$$

***Proof.*** Let $f$, $P$, $M$, $p$ and $m$ be defined as in the proof of Theorem 3. By (2) and (3), we obtain

$$(n - m)\lceil \delta/2 \rceil \leq e(P, M) \leq 2|E(G[M])| + 2m. \tag{6}$$

Since $G$ is $K_{r+1}$-free, the induced subgraph $G[M]$ is also $K_{r+1}$-free, and hence it follows from Theorem 9 that $|E(G[M])| \leq (r-1)m^2/2r$. Using (6), we obtain

$$(n - m)\lceil \delta/2 \rceil \leq e(P, M) \leq \frac{r-1}{r} m^2 + 2m$$

and so

$$m^2 + \frac{r}{r-1}(2 + \lceil \delta/2 \rceil)m - \frac{r}{r-1} n\lceil \delta/2 \rceil \geq 0.$$

This yields

$$m \geq -\frac{r}{2(r-1)}(2 + \lceil \delta/2 \rceil) + \sqrt{\left(\frac{r}{2(r-1)}(2 + \lceil \delta/2 \rceil)\right)^2 + \frac{r}{r-1} n\lceil \delta/2 \rceil},$$

and we obtain the desired bound as follows

$$\alpha_s^2(G) = n - 2m \le n + \frac{r(2+\lceil\delta/2\rceil)}{r-1} - \sqrt{\left(\frac{r(2+\lceil\delta/2\rceil)}{r-1}\right)^2 + \frac{4rn\lceil\delta/2\rceil}{r-1}}.$$

■

Since

$$\frac{r(2+\lceil\delta/2\rceil)}{r-1} - \sqrt{\left(\frac{r(2+\lceil\delta/2\rceil)}{r-1}\right)^2 + \frac{4rn\lceil\delta/2\rceil}{r-1}} \le \frac{3r}{r-1} - \sqrt{\left(\frac{3r}{r-1}\right)^2 + \frac{4rn}{r-1}}$$

for $\delta \ge 1$, the next known result is an immediate consequence of Theorem 10.

**Corollary 11** [5]**.** *If $G$ is an $r$-partite graph of order $n$ with $r \ge 2$, then*

$$\alpha_s^2(G) \le n + \frac{3r}{r-1} - \sqrt{\left(\frac{3r}{r-1}\right)^2 + \frac{4rn}{r-1}}.$$

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