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THE CROSSING NUMBERS OF PRODUCTS OF PATH WITH GRAPHS OF ORDER SIX

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Abstract

The crossing numbers of Cartesian products of paths, cycles or stars with all graphs of order at most four are known. For the path P_n of length n, the crossing numbers of Cartesian products $G\Box P_n$ for all connected graphs G on five vertices are also known. In this paper, the crossing numbers of Cartesian products $G\Box P_n$ for graphs G of order six are studied. Let H denote the unique tree of order six with two vertices of degree three. The main contribution is that the crossing number of the Cartesian product $H\Box P_n$ is $2(n-1)$. In addition, the crossing numbers of $G\Box P_n$ for fourty graphs G on six vertices are collected.

Keywords: graph, drawing, crossing number, Cartesian product, path, tree.

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1. Introduction

The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with vertex set $V(G)$ and edge set $E(G)$ is defined as the minimum possible number of edge crossings in a drawing of G in the plane. A drawing with minimum number of crossings (an optimal drawing) must be a *good* drawing; that is, each two edges have at most one point in common, which is either a common end-vertex or a crossing. Moreover, no three edges cross in a point. Let D be a good drawing of the graph G. We

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denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote by $\mathrm{cr}_D(G_i, G_j)$ the number of crossings between edges of G_i and edges of G_j, and by $\mathrm{cr}_D(G_i)$ the number of crossings among edges of G_i in D.

The investigation on the crossing number of graphs is a classical but very difficult problem. According to their special structure, Cartesian products of special graphs are one of few graph classes for which the exact values of crossing numbers were obtained. Let G_1 and G_2 be simple graphs with vertex sets $V(G_1)$ and $V(G_2)$, and edge sets $E(G_1)$ and $E(G_2)$, respectively. The Cartesian product $G_1 \Box G_2$ of the graphs G_1 and G_2 has vertex set $V(G_1 \Box G_2) = V(G_1) \times V(G_2)$ and two vertices (u, u') and (v, v') are adjacent in $G_1 \Box G_2$ if and only if either $u = v$ and u' is adjacent with v' in G_2, or $u' = v'$ and u is adjacent with v in G_1.

Let C_n be the cycle of length n, P_n be the path of length n, and S_n be the star isomorphic to $K_{1,n}$. Beineke and Ringeisen in [1] started to study the crossing numbers of Cartesian products of cycles with all graphs of order at most four. In [3], [4], and [5], the crossing numbers of Cartesian products of cycles, paths and stars with all graphs of order four are given. The crossing numbers of Cartesian products of paths with all graphs of order five are collected in [8]. It seems natural to enquire about crossing numbers of Cartesian products of paths with other graphs. There are known the crossing numbers of products $G \Box P_n$ for some graphs G on six vertices, see [9], [10], [11], [12], and [13]. In the paper, we extend these results by giving the exact values of crossing numbers for Cartesian products of paths with several graphs of order six. We consider graphs G_i, $i = 1, 2, \ldots, 40$, on six vertices which are collected in Table 1 in the last section of the paper. All known results concerning crossing numbers of Cartesian products of these graphs with paths are presented in Table 1.

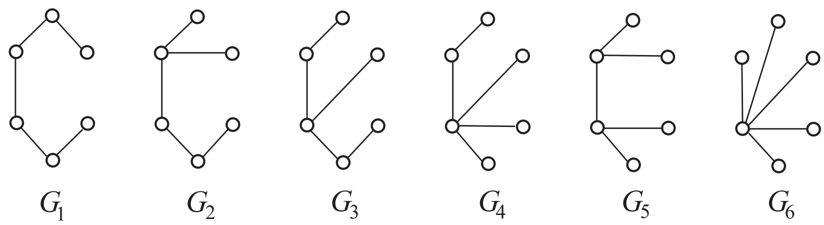


Figure 1. All trees of order six.

2. Trees on Six Vertices

In this section, we give the crossing numbers of Cartesian products of paths with all trees on six vertices. There are six trees of order six shown in Figure 1. The graph $G_1 \Box P_n = P_5 \Box P_n$ is planar. The graph G_6 is isomorphic with the star

S_5. It was proved in [2] that $\mathrm{cr}(S_m \square P_n) = (n-1)\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. So, $\mathrm{cr}(G_6 \square P_n) = 4(n-1)$. As both graphs G_2 and G_3 contain the star S_3 as a subgraph, the Cartesian product $S_3 \square P_n$ is a subgraph of both graphs $G_2 \square P_n$ and $G_3 \square P_n$. Thus, $\mathrm{cr}(G_2 \square P_n) \geq n-1$ and $\mathrm{cr}(G_3 \square P_n) \geq n-1$, because $\mathrm{cr}(S_3 \square P_n) = n-1$, see [3]. On the other hand, in Figure 2(a) and Figure 2(b) there are drawings of the graphs $G_2 \square P_n$ and $G_3 \square P_n$ with $n-1$ crossings. This implies that $\mathrm{cr}(G_2 \square P_n) \leq n-1$ and $\mathrm{cr}(G_3 \square P_n) \leq n-1$ and therefore, $\mathrm{cr}(G_2 \square P_n) = \mathrm{cr}(G_3 \square P_n) = n-1$. The drawing in Figure 2(c) shows the graph $G_4 \square P_n$ with $2(n-1)$ crossings. As the graph $G_4 \square P_n$ contains $S_4 \square P_n$ as a subgraph and $\mathrm{cr}(S_4 \square P_n) = 2(n-1)$, see [4], the crossing number of the graph $G_4 \square P_n$ is $2(n-1)$. The aim of the rest of this section is to establish the crossing number of the graph $G_5 \square P_n$.

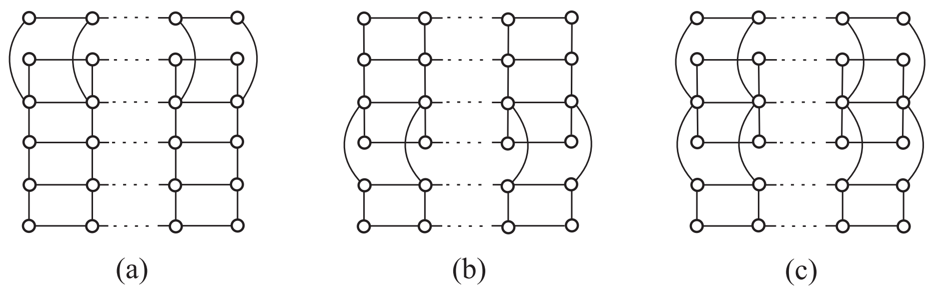


Figure 2. The graphs $G_2 \square P_n$, $G_3 \square P_n$ and $G_4 \square P_n$.

We assume $n \geq 1$ and find it convenient to consider the graph $G_5 \square P_n$ in the following way: it has $6(n+1)$ vertices and edges that are the edges in $n+1$ copies G_5^i, $i = 0, 1, \ldots, n$, and in six paths of length n. For $i = 0, 1, \ldots, n$, let a_i, b_i, e_i, and f_i be the vertices of G_5^i of degree one, c_i and d_i the vertices of degree three (see Figure 3). Thus, for $x \in \{a, b, c, d, e, f\}$, the path P_n^x is induced by the vertices $x_0, x_1, \ldots, x_n$. For $i = 1, 2, \ldots, n$, let H^i denote the subgraph of $G_5 \square P_n$ containing the vertices of G_5^{i-1} and G_5^i and the six edges joining G_5^{i-1} to G_5^i. Let Q^i, $i = 1, 2, \ldots, n-1$, denote the subgraph of $G_5 \square P_n$ induced by $V(G_5^{i-1}) \cup V(G_5^i) \cup V(G_5^{i+1})$. So, $Q^i = G_5^{i-1} \cup H^i \cup G_5^i \cup H^{i+1} \cup G_5^{i+1}$. Let us denote by Q_{ab}^i the subgraph of Q^i obtained from Q^i by removing six vertices e_j and f_j for $j = i-1, i, i+1$ and two edges $\{c_{i-1}, c_i\}$ and $\{c_i, c_{i+1}\}$. Likewise, let Q_{ef}^i be the subgraph of Q^i obtained by removing six vertices a_j and b_j for $j = i-1, i, i+1$ and two edges $\{d_{i-1}, d_i\}$ and $\{d_i, d_{i+1}\}$. It is easy to see that both subgraphs Q_{ab}^i and Q_{ef}^i are subdivisions of the graph $K_{3,3}$.
The graph $G_5 \square P_1$ is planar. In the next lemma, the crossing number of the graph $G_5 \square P_2$ is determined.

Lemma 1. $\mathrm{cr}(G_5 \square P_2) = 2$.

Proof. It can be seen from the drawing in Figure 3 that $\mathrm{cr}(G_5 \square P_2) \leq 2$. To

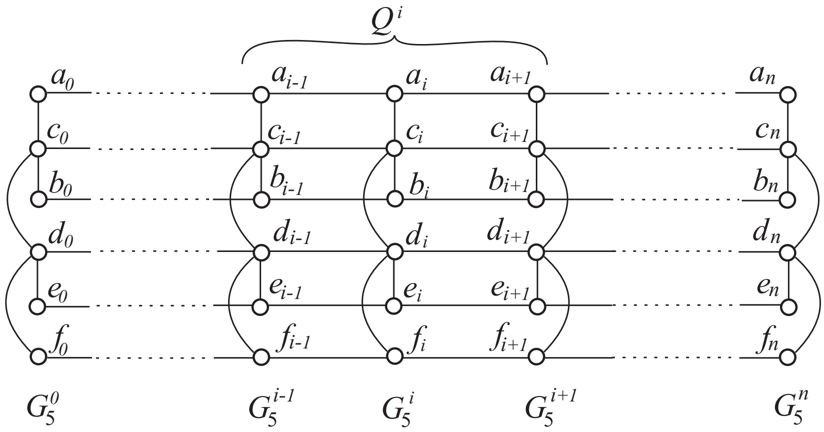


Figure 3. The drawing of the graph $G_5 \Box P_n$ with $2(n-1)$ crossings.

prove the reverse inequality, assume that there is a drawing of the graph $G_5 \Box P_2$ with less than two crossings. The graph $G_5 \Box P_2$ can be consider as the graph Q^1 defined above. As the subgraph Q^1_{ab} of Q^1 is a subdivision of $K_{3,3}$, at least one crossing appears among the edges of Q^1_{ab}. This implies that $\mathrm{cr}(G_5 \Box P_2) \geq 1$. Our assumption of the considered drawing with less than two crossings forces that none of the edges incident with the vertices e_i and f_i, $i = 0, 1, 2$, is crossed. But the unique planar drawing of the subgraph induced by the edges incident with the vertices e_0, e_1, e_2, f_0, f_1, and f_2 divides the plane into two hexagonal regions and one octagonal region in such a way that at most two of the vertices d_0, d_1, and d_2 are contained on a boundary of one region. Hence, the edge $\{c_1, d_1\}$ or at least one of the paths $c_1c_0d_0$ and $c_1c_2d_2$ joining the vertex c_1 with the vertices d_0, d_1, and d_2 crosses the edges incident with the vertices e_0, e_1, e_2, f_0, f_1, and f_2. Thus, at least two crossings appear in any drawing of the graph $G_5 \Box P_2$. This completes the proof. ■

Lemma 2. *If D is a good drawing of the graph $G_5 \Box P_n$, $n \geq 3$, in which every of the subgraphs $G_5^0 \cup H^1$, $G_5^n \cup H^n$ and G_5^i, $i = 1, 2, \ldots, n-1$, has at most one crossing on its edges, then in D there are at least $2(n-1)$ crossings.*

Proof. The proof is based on counting the total force of crossings in a drawing of a graph. This concept was introduced by Beineke and Ringeisen in [1]. Let us consider the following types of possible crossings on the edges of Q^i in a drawing of the graph $G_5 \Box P_n$:

(1) a crossing of an edge in $H^i \cup H^{i+1}$ with an edge in G_5^i,

(2) a crossing of an edge in $G_5^{i-1} \cup H^i$ with an edge in $G_5^{i+1} \cup H^{i+1}$,

(3) a crossing of an edge in $G_5^{i-1} \cup G_5^{i+1}$ with an edge in G_5^i,

(4) a crossing of an edge in $G_5^{i-1} \cup H^i$ with an edge in $G_5^{i+2} \cup H^{i+2}$,

(5) a crossing of an edge in $G_5^{i+1} \cup H^{i+1}$ with an edge in $G_5^{i-2} \cup H^{i-1}$.

It is readily seen that every crossing of types (1) and (2) appears in a good drawing of the graph $G_5 \Box P_n$ only on the edges of the subgraph Q^i. For $i \in \{2, 3, \ldots, n-1\}$, a crossing of type (3) in Q^i between an edge of G_5^{i-1} and an edge of G_5^i appears only in Q^{i-1} as a crossing of type (3), and a crossing of type (5) in Q^i appears only in Q^{i-1} as a crossing of type (4). For $i \in \{1, 2, \ldots, n-2\}$, a crossing between an edge of G_5^{i+1} and an edge of G_5^i appears only in Q^{i+1} as a crossing of type (3), and a crossing of type (4) in Q^i appears only as a crossing of type (5) in Q^{i+1}.

In a good drawing of $G_5 \Box P_n$, we define the *force* $f(Q^i)$ of Q^i in the following way: every crossing of type (1) or (2) contributes the value 1 to $f(Q^i)$ and every crossing of type (3), (4) or (5) contributes the value $\frac{1}{2}$ to $f(Q^i)$ (and $\frac{1}{2}$ to Q^{i-1} or $\frac{1}{2}$ to Q^{i+1}). The *total force* of the drawing is the sum of $f(Q^i)$. As every crossing of type (1) or (2) is counted only once and every crossing of type (3), (4) or (5) is counted at most twice and no other crossing contributes to the total force of the drawing, the number of crossings in the drawing is not less than the total force of the drawing. So, the aim of this proof is to show that if every of the subgraphs $G_5^0 \cup H^1$, $G_5^n \cup H^n$ and G_5^i, $i = 1, 2, \ldots, n-1$, has at most one crossing on its edges, then $f(Q^i) \geq 2$ for all $i = 1, 2, \ldots, n-1$.

Consider the good drawing D of $G_5 \Box P_n$ assumed in Lemma 2 and let D_{ab}^i be the subdrawing of the subgraph Q_{ab}^i induced by D. Any drawing of $K_{3,3}$ contains a pair of edges that cross each other and do not meet in a vertex. The graph Q_{ab}^i can be obtained by elementary subdivision of six edges of $K_{3,3}$. So, in D_{ab}^i there is a *forced* crossing between an edge in $G_5^{i-1} \cup H^i$ and an edge in $G_5^{i+1} \cup H^{i+1}$, or between an edge in G_5^i and an edge in $H^i \cup H^{i+1}$, or between an edge in G_5^i and an edge in $G_5^{i-1} \cup G_5^{i+1}$. Every of the first two considered types of crossings contributes the value 1 to $f(Q^i)$ and the last one contributes the value $\frac{1}{2}$. Hence, the minimal contribution of the subdrawing D_{ab}^i to $f(Q^i)$ is $\frac{1}{2}$, but, an edge of G_5^i is crossed by an edge of G_5^{i-1} or by an edge of G_5^{i+1} if $f(Q^i) = \frac{1}{2}$. The same consideration can be repeated for the subdrawing D_{ef}^i of Q_{ef}^i. Only three edges $\{c_{i-1}, d_{i-1}\}$, $\{c_i, d_i\}$ and $\{c_{i+1}, d_{i+1}\}$ appear in both subgraphs Q_{ab}^i and Q_{ef}^i. As two adjacent edges cannot cross each other in $K_{3,3}$, a possible crossing between two of these three edges cannot be the forced crossing in the subdrawing D_{ab}^i of Q_{ab}^i. The same holds for the subdrawing D_{ef}^i of Q_{ef}^i. Thus, there are at least two forced crossings in the subdrawing D^i of Q^i induced by D. As there is at most one crossing on the edges of G_5^i, at least one forced crossing contributes 1 to $f(Q^i)$. If both forced crossings among the edges of Q^i contribute 1 to $f(Q^i)$, then $f(Q^i) \geq 2$.

For $i \in \{2, 3, \ldots, n-2\}$, assume that one of the forced crossings among the edges of Q^i, say in D^i_{ab}, contributes only $\frac{1}{2}$ to $f(Q^i)$. Without loss of generality suppose that an edge of G^i_5 is crossed by an edge of G^{i-1}_5 in D^i_{ab}. As no two adjacent edges cross in a drawing of the graph $K_{3,3}$, it is easy to see that neither a crossing between the edges $\{a_i, c_i\}$ and $\{a_{i-1}, c_{i-1}\}$ nor a crossing between the edges $\{b_i, c_i\}$ and $\{b_{i-1}, c_{i-1}\}$ is forced in D^i_{ab}. Thus, the forced crossing in D^i_{ab} is one of the following: $\{a_i, c_i\}$ crosses $\{b_{i-1}, c_{i-1}\}$, $\{b_i, c_i\}$ crosses $\{a_{i-1}, c_{i-1}\}$, $\{d_i, c_i\}$ crosses $\{a_{i-1}, c_{i-1}\}$ or $\{b_{i-1}, c_{i-1}\}$, and $\{d_{i-1}, c_{i-1}\}$ crosses $\{a_i, c_i\}$ or $\{b_i, c_i\}$. Up to the symmetry it is enough to consider only three case: the edge $\{a_i, c_i\}$ is crossed by $\{b_{i-1}, c_{i-1}\}$, the edge $\{d_i, c_i\}$ is crossed by $\{b_{i-1}, c_{i-1}\}$, and the edge $\{d_{i-1}, c_{i-1}\}$ is crossed by $\{a_i, c_i\}$. Since in D there is no other crossing either on the edges of G^i_5 or on the edges of G^{i-1}_5, one can find in Figure 3 that in all three cases two vertex disjoint cycles $a_i a_{i+1} c_{i+1} d_{i+1} e_{i+1} e_i d_i c_i a_i$ and $b_{i-1} b_{i-2} c_{i-2} d_{i-2} e_{i-2} e_{i-1} d_{i-1} c_{i-1} b_{i-1}$ cross each other in D at least two times in such a way that the path $a_i a_{i+1} c_{i+1} d_{i+1} e_{i+1} e_i$ crosses the path $b_{i-1} b_{i-2} c_{i-2} d_{i-2}$ $e_{i-2} e_{i-1}$. This crossing of type (5) contributes $\frac{1}{2}$ to $f(Q^i)$, and hence, $f(Q^i) \geq 2$.

Consider now the subgraph Q^1 induced on the vertices of G^0_5, G^1_5, and G^2_5. In the subdrawing D^1 of Q^1 induced by D there are at least two forced crossings. If $\mathrm{cr}_D(G^0_5,\, G^1_5) = \mathrm{cr}_D(G^1_5,\, G^2_5) = 0$, then $f(Q^1) \geq 2$. If the subgraphs G^1_5 and G^2_5 cross each other, then the analysis in the previous paragraph implies that a crossing of type (4) between $G^0_5 \cup H^1$ and $G^3_5 \cup H^2$ is necessary. Hence, $f(Q^1) \geq 2$ in this case. As a crossing between the edges $\{c_0, d_0\}$ and $\{c_1, d_1\}$ is not a forced crossing in D^1, one of the edges $\{c_0, a_0\}$, $\{c_0, b_0\}$, $\{d_0, e_0\}$, and $\{d_0, f_0\}$ must be crossed if $\mathrm{cr}(G^0_5,\, G^1_5) = 1$. Without loss of generality let $\{c_0, a_0\}$ is crossed by an edge of G^1_5. Assume now the subgraph of Q^1_{ef} induced by the edges incident with the vertices $c_0, d_0, e_0, f_0, c_1, d_1, e_1$ and f_1. As, by hypothesis, no other crossing appear on the edges of $G^0_5 \cup H^1 \cup G^1_5$, the unique planar subdrawing of the considered subgraph divides the plane into three hexagonal regions in such a way that at most two of the vertices c_1, e_1, and f_1 are placed on a boundary of one region. But, in this case, in the subdrawing D^1_{ef} at least one of the paths $d_2 c_2 c_1$, $d_2 e_2 e_1$, and $d_2 f_2 f_1$ crosses an edge of $G^0_5 \cup H^1 \cup G^1_5$. This contradiction with the assumption of Lemma 2 implies that both forced crossings in D^1 contribute 1 to $f(Q^1)$ an therefore, $f(Q^1) \geq 2$. A similar analysis for the subgraph Q^{n-1} gives that $f(Q^{n-1}) \geq 2$ as well. Hence, the total force of the drawing D is at least $2(n-1)$, and in D there are at least $2(n-1)$ crossings. This completes the proof. ■

Theorem 3. $\mathrm{cr}(G_5 \Box P_n) = 2(n-1)$ *for* $n \geq 1$.

Proof. The drawing in Figure 3 shows that $\mathrm{cr}(G_5 \Box P_n) \leq 2(n-1)$, because every copy of G^i_5, $i = 1, 2, \ldots, n-1$, is crossed two times and there is no other crossings in the drawing. We prove the reverse inequality by induction on n. It is easy to see that the graph $G_5 \Box P_1$ is planar and, by Lemma 1, $\mathrm{cr}(G_5 \Box P_2) = 2$. So, the result is true for $n = 1$ and $n = 2$. Assume that it is true for $n = k$, $k \geq 2$, and suppose that there is a good drawing of $G_5 \Box P_{k+1}$ with fewer than $2k$ crossings. By Lemma 2, some of the subgraphs $G_5^0 \cup H^1$, $G_5^{k+1} \cup H^{k+1}$ and G_5^i, $i = 1, 2, \ldots, k$, must be crossed at least twice. If $G_5^0 \cup H^1$ has at least two crossings on its edges, the deletion of all vertices of G_5^0 results in a drawing of the graph $G_5 \Box P_k$ with fewer than $2(k-1)$ crossings. This contradicts the induction hypothesis. The same contradiction is obtained, if at least two crossings appear on the edges of $G_5^{k+1} \cup H^{k+1}$. If some G_5^i, $i = 1, 2, \ldots, k$, is crossed at least twice, by the removal of all edges of this G_5^i, a subdivision of $G_5 \Box P_k$ with fewer than $2(k-1)$ crossings is obtained. This contradiction with the induction hypothesis completes the proof. ∎

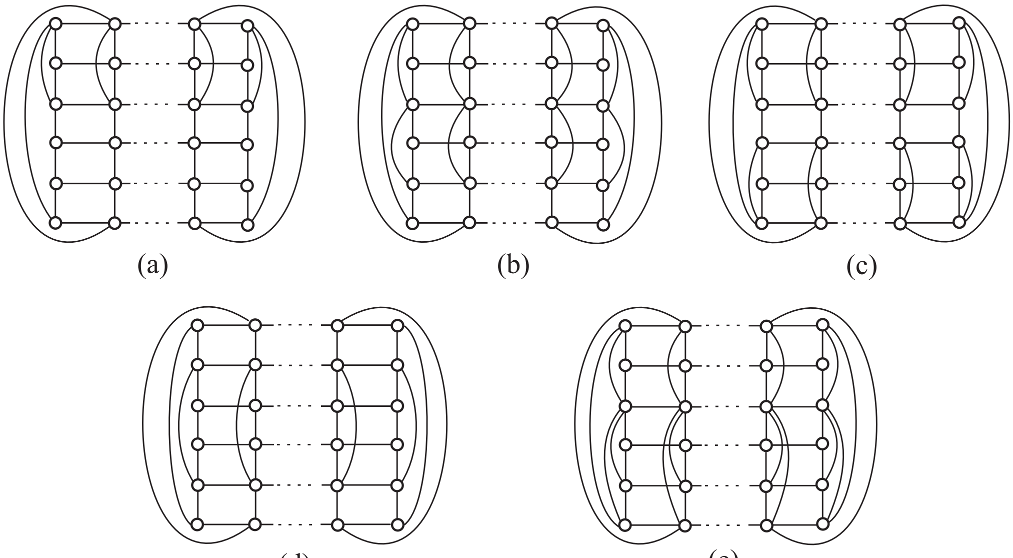
3. The Collection of $\mathrm{cr}(G_i \Box P_n)$ for Graphs G_i on Six Vertices

The aim of this section is to collect Cartesian products of graphs of order six with paths for which the crossing numbers are known. As for a disconnected graph G, the Cartesian product $G \Box P_n$ is disconnected, we are interesting only of connected graphs on six vertices. There are 112 connected graphs on six vertices. At present, we are able to summarise the crossing numbers of $G_i \Box P_n$ for fourty connected graphs G_i of order six shown in the Table 1.

In the previous section, the crossing numbers of Cartesian products of paths with all trees on six vertices are collected. These results enable us to determine the exact values of crossing numbers for Cartesian products of paths with some other graphs. It is easy to see that the graph $C_m \Box P_n$ is planar. As the graph G_7 is isomorphic to the cycle C_6, $\mathrm{cr}(G_7 \Box P_n) = 0$. The graphs G_8, G_9, G_{12}, and G_{18} contain S_3 as a subgraph. Thus, all Cartesian products $G_i \Box P_n$, $i = 8, 9, 12, 18$, contain $S_3 \Box P_n$ as a subgraph. It was proved in [3] that $\mathrm{cr}(S_3 \Box P_n) = n - 1$. This implies that $\mathrm{cr}(G_i \Box P_n) \geq n - 1$ for $i = 8, 9, 12, 18$. On the other hand, the graphs $G_8 \Box P_n$, $G_9 \Box P_n$, and $G_{12} \Box P_n$ are subgraphs of the graph $G_{18} \Box P_n$. In Figure 4(a) there is a drawing of the graph $G_{18} \Box P_n$ with $n - 1$ crossings and therefore, $\mathrm{cr}(G_{18} \Box P_n) \leq n - 1$. Hence, $\mathrm{cr}(G_i \Box P_n) = n - 1$ for the graphs G_i, $i = 8, 9, 12, 18$.

Figure 4(b) shows the drawing of the graph $G_{27} \Box P_n$ with $2(n-1)$ crossings. The graph $G_{27} \Box P_n$ contains $G_{11} \Box P_n$, $G_{15} \Box P_n$, $G_{16} \Box P_n$, $G_{19} \Box P_n$, and $G_{25} \Box P_n$ as subgraphs. Thus, $\mathrm{cr}(G_i \Box P_n) \leq 2(n-1)$ for $i = 11, 15, 16, 19, 25$, and 27. As $\mathrm{cr}(S_4 \Box P_n) = 2(n-1)$, see [4], $\mathrm{cr}(G_{11} \Box P_n) = \mathrm{cr}(G_{15} \Box P_n) = \mathrm{cr}(G_{16} \Box P_n) = \mathrm{cr}(G_{19} \Box P_n) = \mathrm{cr}(G_{25} \Box P_n) = \mathrm{cr}(G_{27} \Box P_n) = 2(n-1)$, because each of these graphs contains $S_4 \Box P_n$ as a subgraph.

By Theorem 3, $\mathrm{cr}(G_5 \Box P_n) = 2(n-1)$. The graph $G_5 \Box P_n$ is a subgraph of all



(a) (b) (c) (d) (e)

Figure 4. The graphs $G_{18}\Box P_n$, $G_{27}\Box P_n$, $G_{31}\Box P_n$, $G_{17}\Box P_n$, and $G_{35}\Box P_n$.

graphs $G_i\Box P_n$ for $i = 10, 14, 17, 21, 23, 31$ and therefore, the crossing number of all these graphs is at least $2(n-1)$. To show the reverse inequality, we need suitable drawings of two of the considered six graphs. Except of the graph $G_{17}\Box P_n$, all other graphs $G_i\Box P_n$, $i = 10, 14, 21, 23$, are subgraphs of the graph $G_{31}\Box P_n$. In Figure 4(c) and Figure 4(d) one can find the drawings of the graphs $G_{31}\Box P_n$ and $G_{17}\Box P_n$, respectively, both with $2(n-1)$ crossings. This implies that for $i = 10, 14, 17, 21, 23, 31$, the crossing number of the graphs $G_i\Box P_n$ is $2(n-1)$.

The drawing of the graph $G_{35}\Box P_n$ with $4(n-1)$ crossings is shown in Figure 4(e). Thus, $\mathrm{cr}(G_{35}\Box P_n) \le 4(n-1)$. As $G_{35}\Box P_n$ contains all graphs $G_i\Box P_n$, $i = 13, 22, 24, 26, 28$, as subgraphs, the value $4(n-1)$ is the upper bound for crossing numbers of these graphs. On the other hand, each of the graphs $G_i\Box P_n$, $i = 13, 22, 24, 26, 28, 35$, contains $S_5\Box P_n$ as a subgraph. Bokal in [2] proved that $\mathrm{cr}(S_5\Box P_n) = 4(n-1)$. Hence, $\mathrm{cr}(G_{13}\Box P_n) = \mathrm{cr}(G_{22}\Box P_n) = \mathrm{cr}(G_{24}\Box P_n) = \mathrm{cr}(G_{26}\Box P_n) = \mathrm{cr}(G_{28}\Box P_n) = \mathrm{cr}(G_{35}\Box P_n) = 4(n-1)$.

In [6], the crossing number of the Cartesian product $K_{2,3}\Box P_n$ is given. Namely, $\mathrm{cr}(K_{2,3}\Box P_n) = 2n$. We use these result and we give the values of crossing numbers of two other Cartesian products of paths with graphs of order six. The graph G_{20} is a subdivision of the complete bipartite graph $K_{2,3}$ and the graph G_{29} contains a subdivision of $K_{2,3}$ as a subgraph. Hence, the crossing number of both Cartesian products $G_{20}\Box P_n$ and $G_{29}\Box P_n$ is at least $2n$. In Figure 5(a) there is a drawing of $G_{29}\Box P_n$ with $2n$ crossings. Thus, $\mathrm{cr}(G_{29}\Box P_n) \le 2n$ and therefore, $\mathrm{cr}(G_{29}\Box P_n) = 2n$. Moreover, as $G_{20}\Box P_n$ is a subgraph of $G_{29}\Box P_n$, the

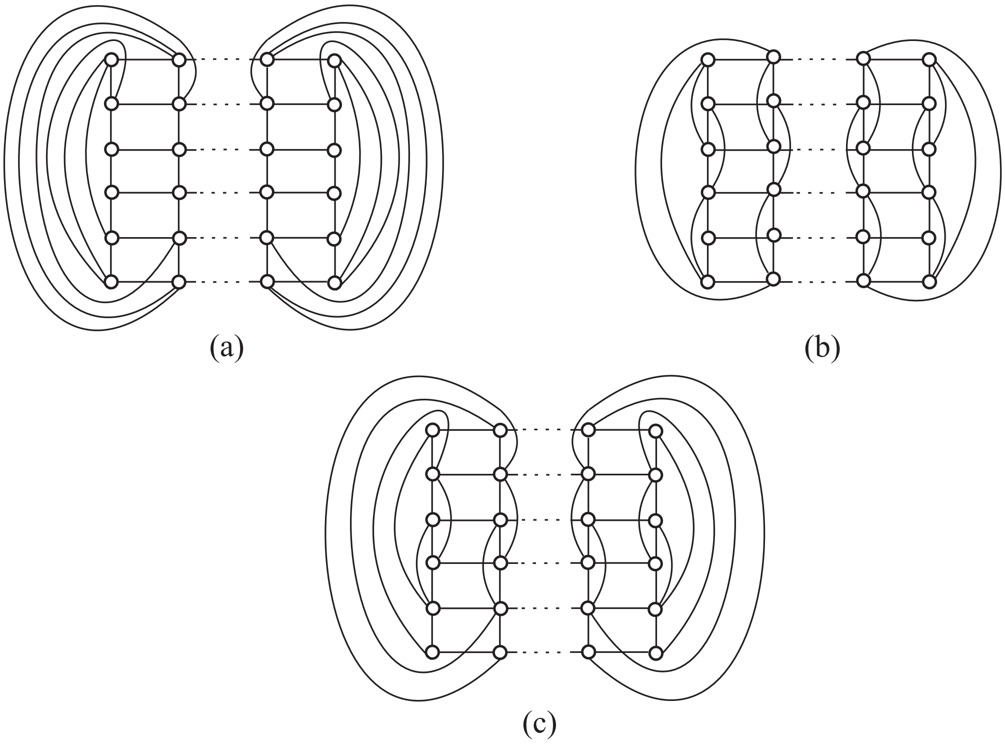


Figure 5. The graphs $G_{29}\Box P_n$, $G_{36}\Box P_n$, and $G_{34}\Box P_n$.

crossing number of the graph $G_{20}\Box P_n$ is $2n$ too.

Let H_5 be the graph obtained from the complete graph on five vertices K_5 by deleting three edges incident with the same vertex. It was shown in [7] that $\mathrm{cr}(H_5\Box P_n) = 3n-1$. Both graphs G_{30} and G_{36} contain a subdivision of the graph H_5 as a subgraph. This implies that the crossing number of both Cartesian products $G_{30}\Box P_n$ and $G_{36}\Box P_n$ is greater or equal $3n-1$, which is the crossing number of the graph $H_5\Box P_n$. The graph $G_{30}\Box P_n$ is a subgraph of $G_{36}\Box P_n$ and therefore, $\mathrm{cr}(G_{30}\Box P_n) \leq \mathrm{cr}(G_{36}\Box P_n)$. In the drawing of the graph $G_{36}\Box P_n$ in Figure 5(b) it is easy to see that $\mathrm{cr}(G_{36}\Box P_n) \leq 3n-1$. Thus, $\mathrm{cr}(G_{30}\Box P_n) = \mathrm{cr}(G_{36}\Box P_n) = 3n-1$.

Recently, some few results concerning crossing numbers of Cartesian products of paths with graphs on six vertices were obtained. For the graph $G_{33} = P(3,1)$, Peng and Yiew proved that the Cartesian product $G_{33}\Box P_n$ has crossing number $4n$, see [10]. The graph G_{37} is isomorphic with the second power of the path of length five denoted by P_5^2. It was proved in [9] that $\mathrm{cr}(P_5^2\Box P_n) = \mathrm{cr}(G_{37}\Box P_n) = 4(n-1)$. For two other graphs, namely for G_{38} and G_{39}, the crossing numbers are also known. In [12] one can find that $\mathrm{cr}(G_{38}\Box P_n) = 4n$ and $\mathrm{cr}(G_{39}\Box P_n) = 6n$.

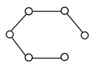
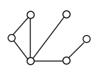
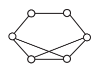
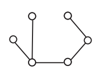
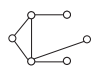
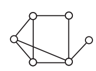
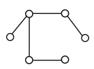
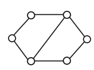
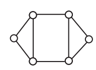
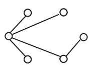
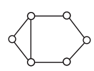
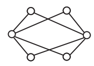
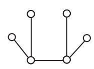
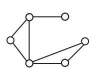
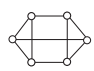
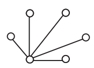
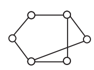
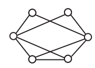
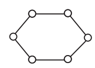
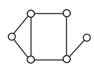
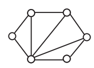
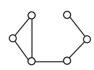
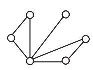
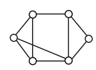
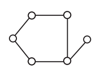
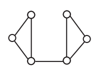
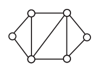
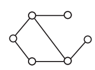
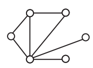
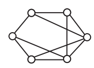
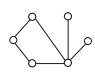
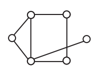
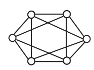
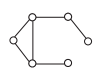
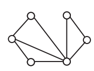
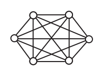
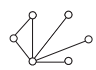
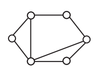
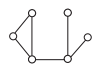
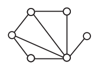
G_i	$\mathrm{cr}(G_i \square P_n)$	G_i	$\mathrm{cr}(G_i \square P_n)$	G_i	$\mathrm{cr}(G_i \square P_n)$
G_1 	0	G_{15} 	$2(n - 1)$	G_{29} 	$2n$
G_2 	$n - 1$	G_{16} 	$2(n - 1)$	G_{30} 	$3n - 1$
G_3 	$n - 1$	G_{17} 	$2(n - 1)$	G_{31} 	$2(n - 1)$
G_4 	$2(n - 1)$	G_{18} 	$n - 1$	G_{32} 	$4n$
G_5 	$2(n - 1)$	G_{19} 	$2(n - 1)$	G_{33} 	$4n$
G_6 	$4(n - 1)$	G_{20} 	$2n$	G_{34} 	$4n$
G_7 	0	G_{21} 	$2(n - 1)$	G_{35} 	$4(n - 1)$
G_8 	$n - 1$	G_{22} 	$4(n - 1)$	G_{36} 	$3n - 1$
G_9 	$n - 1$	G_{23} 	$2(n - 1)$	G_{37} 	$4(n - 1)$
G_{10} 	$2(n - 1)$	G_{24} 	$4(n - 1)$	G_{38} 	$4n$
G_{11} 	$2(n - 1)$	G_{25} 	$2(n - 1)$	G_{39} 	$6n$
G_{12} 	$n - 1$	G_{26} 	$4(n - 1)$	G_{40} 	$15n + 3$
G_{13} 	$4(n - 1)$	G_{27} 	$2(n - 1)$		
G_{14} 	$2(n - 1)$	G_{28} 	$4(n - 1)$		

Table 1. The known crossing numbers of $G_i \square P_n$ for graphs G_i on six vertices.

For the complete graph on six vertices, it is shown in [13] that the crossing number of its Cartesian product with the path P_n is $15n+3$. Thus, we have that $\mathrm{cr}(K_6 \square P_n) = \mathrm{cr}(G_{40} \square P_n) = 15n + 3$. The last known result one can find in [11].

It is shown that the crossing number of the graph $G_{32}\Box P_n = K_{2,4}\Box P_n$ is $4n$. This result we use to establish the crossing number of the Cartesian product $G_{34}\Box P_n$. The graph G_{34} contains a subgraph $K_{2,4}$ and therefore, $\mathrm{cr}(G_{34}\Box P_n) \geq 4n$. On the other hand, in Figure 5(c) there is a drawing of the graph $G_{34}\Box P_n$ with $4n$ crossings. This confirms that $\mathrm{cr}(G_{34}\Box P_n) = 4n$. All known results concerning crossing numbers of Cartesian products of paths with graphs on six vertices are collected in Table 1.

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