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Note

THE PATH-DISTANCE-WIDTH OF HYPERCUBES

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Abstract

The *path-distance-width* of a connected graph G is the minimum integer w satisfying that there is a nonempty subset of $S \subseteq V(G)$ such that the number of the vertices with distance i from S is at most w for any nonnegative integer i.

In this note, we determine the path-distance-width of hypercubes.

Keywords: path-distance-width, hypercube.

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1. Introduction

Path-distance-width is a graph parameter to measure how close a graph is to a path [11]. There are several other such graph parameters including pathwidth and bandwidth. It is known that for any connected graph, its pathwidth is bounded by its bandwidth, and its bandwidth is less than 2 times its path-distance-width [5, 11].

Yamazaki *et al.* [11] introduced the concept of path-distance-width to study the graph isomorphism problem. They showed that determining the path-distance-width is NP-hard even for trees. Furthermore, Yamazaki [10] showed that approximating the path-distance-width of trees within a ratio better than 4/3 is NP-hard. Recently, Otachi *et al.* [8] have studied the path-distance-width of AT-free graphs. They showed that it is NP-hard to determine the path-distance-width of AT-free graphs. On the other hand, they also showed that the path-distance-width of AT-free graphs can be approximated within factor 3, by using only one vertex as an initial set.

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In this short note, we exactly determine the path-distance-width of hypercubes. Our optimal initial set consists of only one vertex. Indeed, it can be *any* vertex since hypercubes are vertex transitive.

2. Preliminaries

In this note, graphs are finite, undirected, simple, and connected. Let G be a connected graph. The *distance* between two vertices in G is the length of a shortest path connecting the vertices. The *distance* between a vertex and a vertex set in G is the length of a shortest path connecting the vertex and a vertex in the set. For $S \subseteq V(G)$, let $\partial(S)$ denote the set of vertices adjacent to an element of S but not included in S. Let $\beta(s) = \min_{S \subseteq V(G),\, |S|=s} |\partial(S)|$.

The *distance structure* of G with an *initial set* $S \subseteq V(G)$ is the sequence $(D_0, \dots, D_t)$ such that D_i is the set of vertices of distance i from S (thus $D_0 = S$), and t is the largest integer such that $D_t \neq \emptyset$. The *width* of a distance structure $\mathcal{D} = (D_0, \dots, D_t)$ is $\max\{|D_i| : 0 \leq i \leq t\}$, and denoted by $w(\mathcal{D})$. The *path-distance-width* of a graph G, denoted by $\mathrm{pdw}(G)$, is defined as the minimum width over all distance structures of G.

The *d-dimensional hypercube* Q_d is the graph whose vertex set is the set of all binary strings of length d, where two vertices are adjacent if and only if the Hamming distance between the corresponding strings is exactly 1.

3. A Lower Bound and its Application for Hypercubes

It is known that several width parameters such as bandwidth [4] and treewidth [3] have lower bounds in terms of $\beta(s)$. In this section, we present a lower bound of similar type for path-distance-width, and apply the bound for hypercubes.

Lemma 3.1. *Let G be a connected graph and let k and x be positive integers. If $\beta(s) \geq k$ for $x \leq s \leq x + k - 2$, then $\mathrm{pdw}(G) \geq k$.*

Proof. Let $\mathcal{D} = (D_0, \dots, D_t)$ be a distance structure of G with $\mathrm{pdw}(G) = w(\mathcal{D})$. Observe that if there is some index i such that $|D_i| \geq k$, then $\mathrm{pdw}(G) = w(\mathcal{D}) \geq |D_i| \geq k$. Thus, in what follows, we assume that $|D_i| \leq k - 1$ for any i.

For convenience, let $D_{\leq i} = \bigcup_{j=0}^{i} D_j$. Note that $D_{i+1} = \partial(D_{\leq i})$ for any i. Let h be the smallest index such that $|D_{\leq h}| \geq x$. Since $|D_{\leq h-1}| \leq x - 1$ and $|D_h| \leq k - 1$, it follows that $|D_{\leq h}| = |D_{\leq h-1}| + |D_h| \leq x + k - 2$. Now, by the assumption, $\beta(|D_{\leq h}|) \geq k$, and thus $\mathrm{pdw}(G) = w(\mathcal{D}) \geq |D_{h+1}| = |\partial(D_{\leq h})| \geq \beta(|D_{\leq h}|) \geq k$. ■

Using Lemma 3.1, we determine the path-distance-width of hypercubes. Note that the initial set of the distance structure used in the proof consists of only one vertex.

Theorem 3.2. $\mathrm{pdw}(Q_d) = \binom{d}{\lfloor d/2 \rfloor}$.

Proof. First we show the upper bound. Let $W_i \subseteq V(Q_d)$ be the vertices of Hamming weight i. Clearly, $\mathcal{W} = (W_0, \ldots, W_d)$ is a distance structure of Q_d, and

$$w(\mathcal{W}) = \max\{|W_i| : 0 \leq i \leq d\} = \max\left\{\binom{d}{i} : 0 \leq i \leq d\right\} = \binom{d}{\lfloor d/2 \rfloor}.$$

Therefore, $\mathrm{pdw}(Q_d) \leq \binom{d}{\lfloor d/2 \rfloor}$.

Next we show the lower bound. Let $f(m) = \sum_{j=1}^{m/2} \binom{2j-1}{j}$ for even m. We now divide the proof into two cases according to the parity of d. The two cases are similar and use a result of Kleitman [6], who showed a lower bound on β for Q_d.

Case 1. d is even. In this case, $f(d) = \sum_{j=1}^{d/2} \binom{2j-1}{j} \geq \binom{d-1}{d/2} = \frac{1}{2}\binom{d}{d/2}$. Kleitman [6] showed that $\beta(s) \geq \binom{d}{d/2}$ for even d if $s \in S$, where

$$S = \left\{2^{d-1} - \tfrac{1}{2}\binom{d}{d/2} - f(d) + 1, \ldots, 2^{d-1} + f(d-2)\right\}.$$

Since $|S| = \frac{1}{2}\binom{d}{d/2} + f(d) + f(d-2) \geq \binom{d}{d/2}$, it follows that $\mathrm{pdw}(Q_d) \geq \binom{d}{d/2}$ by Lemma 3.1.

Case 2. d is odd. Then, $f(d+1) = \sum_{j=1}^{(d+1)/2} \binom{2j-1}{j} \geq \binom{d}{(d+1)/2} = \binom{d}{\lfloor d/2 \rfloor}$. Kleitman [6] showed that $\beta(s) \geq \binom{d}{\lfloor d/2 \rfloor}$ for odd d if $s \in S'$, where

$$S' = \left\{2^{d-1} - f(d+1) + 1, \ldots, 2^{d-1} + f(d-1)\right\}.$$

Thus $|S'| = f(d-1) + f(d+1) \geq \binom{d}{\lfloor d/2 \rfloor}$, and $\mathrm{pdw}(Q_d) \geq \binom{d}{\lfloor d/2 \rfloor}$ by Lemma 3.1. ∎

4. Concluding Remarks

We have determined the path-distance-width of hypercubes by using a general lower bound, which is also presented in this note. We believe our lower bound can be used to determine the path-distance-width of other graphs, such as d-dimensional grids [2, 7] and d-dimensional even tori [1, 9], for which the structure of $S \subseteq V(G)$ with $|\partial(S)| = \beta(|S|)$ is well studied for every $s = |S|$.

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