

## UNDERLYING GRAPHS OF 3-QUASI-TRANSITIVE DIGRAPHS AND 3-TRANSITIVE DIGRAPHS

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### Abstract

A digraph is 3-quasi-transitive (resp. 3-transitive), if for any path  $x_0x_1x_2x_3$  of length 3,  $x_0$  and  $x_3$  are adjacent (resp.  $x_0$  dominates  $x_3$ ). César Hernández-Cruz conjectured that if  $D$  is a 3-quasi-transitive digraph, then the underlying graph of  $D$ ,  $UG(D)$ , admits a 3-transitive orientation. In this paper, we shall prove that the conjecture is true.

**Keywords:** graph orientation, 3-quasi-transitive digraph, 3-transitive digraph.

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### 1. TERMINOLOGY AND INTRODUCTION

We only consider finite graphs and digraphs without loops and multiple edges or multiple arcs. Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . A *complete graph* is a graph in which any two vertices are adjacent. A *complete bipartite graph*  $G$  is a graph in which the vertices of  $G$  can be partitioned into two partite sets such that every partite set is an independent set and for every pair  $x, y$  of vertices from distinct partite sets,  $(x, y) \in E(G)$ .

Let  $D$  be a digraph with vertex set  $V(D)$  and arc set  $A(D)$ . For any  $x, y \in V(D)$ , we will write  $\overrightarrow{xy}$  or  $x \rightarrow y$  if  $xy \in A(D)$ , and also, we will write  $\overleftarrow{xy}$  if  $\overrightarrow{yx}$  or  $\overrightarrow{yx}$ . For disjoint subsets  $X$  and  $Y$  of  $V(D)$  or subdigraphs of  $D$ ,  $X \rightarrow Y$  means that every vertex of  $X$  dominates every vertex of  $Y$ ,  $X \Rightarrow Y$  means that there is no arc from  $Y$  to  $X$  and  $X \mapsto Y$  means that both of  $X \rightarrow Y$  and  $X \Rightarrow Y$  hold. Let  $D'$  be a subdigraph of  $D$  and  $x \in V(D) - V(D')$ . We say that  $x$  and  $D'$  are *adjacent* if  $x$  and some vertex of  $D'$  are adjacent.

For any digraph  $D$ , we can associate a graph  $G$  on the same vertex set simply by replacing each arc by an edge with the same ends. This graph is the *underlying graph* of  $D$ , denoted  $UG(D)$ . By a *path* of a digraph  $D$ , we mean a directed path of  $D$ . The length of a path is the number of its arcs. A path of length  $k$  is called a *k-path*; the path is odd or even according to the parity of  $k$ . A digraph  $D$  is said to be *strongly connected* or just *strong*, if for every pair  $x, y$  of vertices of  $D$ , there is a path from  $x$  to  $y$ . A *strong component* of a digraph  $D$  is a maximal induced subdigraph of  $D$  which is strong. The *strong component digraph*  $SC(D)$  of  $D$  is obtained by contracting strong components of  $D$  and deleting any parallel arcs obtained in this process.

A digraph  $D$  is *semicomplete* if there is at least one arc between any pair of distinct vertices of  $D$ . A *tournament* is a semicomplete digraph with no cycle of length 2. A digraph  $D$  is *semicomplete bipartite*, if the vertices of  $D$  can be partitioned into two partite sets such that every partite set is an independent set and for every pair  $x, y$  of vertices from distinct partite sets,  $xy$  or  $yx$  (or both) is in  $D$ . A *bipartite tournament* is a semicomplete bipartite digraph with no cycle of length 2. A digraph is *k-quasi-transitive*, where  $k \geq 2$ , if for any path  $x_0x_1x_2 \dots x_k$  of length  $k$ ,  $x_0$  and  $x_k$  are adjacent. A 2-quasi-transitive digraph is also called a *quasi-transitive digraph*. A 3-quasi-transitive digraph is also called a *quasi-arc-transitive digraph* (see [7]). A digraph is *k-transitive*, where  $k \geq 2$ , if for any path  $x_0x_1x_2 \dots x_k$  of length  $k$ ,  $x_0$  dominates  $x_k$ . A 2-transitive digraph is also called a *transitive digraph*.  $k$ -transitive digraphs and  $k$ -quasi-transitive digraphs have been studied by several authors. See [1, 2, 3, 5, 6, 7]. For a graph  $G$ , a digraph  $D$  is called an *orientation* of  $G$  if  $D$  is obtained from  $G$  by replacing each edge  $(x, y)$  of  $G$  by either  $xy$  or  $yx$ .

In [4], see also [2], Ghouila-Houri proved the following theorem.

**Theorem 1** [2, 4]. *A graph  $G$  has a quasi-transitive orientation if and only if it has a transitive orientation.*

It seems natural to consider an analogue of Theorem 1 for 3-quasi-transitive digraphs and 3-transitive digraphs.

In [3], the author proposed the following conjecture.

**Conjecture 2** [3]. *Let  $D$  be a 3-quasi-transitive oriented graph. Then the underlying graph of  $D$ ,  $UG(D)$ , admits a 3-transitive orientation.*

In Section 2, we will prove that the conjecture is true.

## 2. MAIN RESULT

We begin with some useful lemmas. Let  $F_n$  be the digraph with vertex set  $\{x_0, x_1, \dots, x_n\}$  and arc set  $\{x_0x_1, x_1x_2, x_2x_0\} \cup \{x_0x_{i+3}, x_{i+3}x_1 : i = 0, 1, \dots, n-3\}$ , where  $n \geq 3$ .

**Lemma 3** [5]. *Let  $D$  be a strong 3-quasi-transitive digraph of order  $n$ . Then  $D$  is either a semicomplete digraph, a semicomplete bipartite digraph or isomorphic to  $F_n$ .*

**Lemma 4** [7]. *Let  $D'$  be a non-trivial strong induced subdigraph of a 3-quasi-transitive digraph  $D$ . For any  $s \in V(D) - V(D')$ , if there exists a directed path between  $s$  and  $D'$ , then  $s$  and  $D'$  are adjacent.*

**Lemma 5** [5]. *Let  $D$  be a 3-quasi-transitive digraph. For a pair  $x, y$  of  $V(D)$ , if there exists an  $(x, y)$ -path of odd length, then  $x$  and  $y$  are adjacent.*

**Lemma 6** [7]. *Let  $D'$  be a non-trivial strong induced subdigraph of a 3-quasi-transitive digraph  $D$  and let  $s \in V(D) - V(D')$  with at least one arc from  $s$  to  $D'$  and  $s \Rightarrow D'$ . Then each of the following holds:*

- (a) *If  $D'$  is a bipartite digraph with bipartition  $(X; Y)$  and  $s$  dominates a vertex of  $X$ , then  $s \mapsto X$ .*
- (b) *If  $D'$  is a non-bipartite digraph, then  $s \mapsto D'$ .*

**Lemma 7** [7]. *Let  $D'$  be a non-trivial strong induced subdigraph of a 3-quasi-transitive digraph  $D$  and let  $s \in V(D) - V(D')$  with at least one arc from  $D'$  to  $s$  and  $D' \Rightarrow s$ . Then each of the following holds:*

- (a) *If  $D'$  is a bipartite digraph with bipartition  $(X, Y)$  and there exists a vertex of  $X$  which dominates  $s$ , then  $X \mapsto s$ .*
- (b) *If  $D'$  is a non-bipartite digraph, then  $D' \mapsto s$ .*

**Lemma 8** [7]. *Let  $D_1$  and  $D_2$  be two distinct non-trivial strong components of a 3-quasi-transitive digraph with at least one arc from  $D_1$  to  $D_2$ . Then either  $D_1 \mapsto D_2$  or  $D_1 \cup D_2$  is a semicomplete bipartite digraph.*

**Lemma 9.** *If a graph  $G$  has a strong 3-quasi-transitive orientation, then it has a transitive orientation.*

**Proof.** Let  $D$  be a strong 3-quasi-transitive orientation of  $G$  and let the order of  $D$  be  $n$ . By Lemma 3,  $D$  is either a tournament, a bipartite tournament or isomorphic to  $F_n$  because  $D$  has no cycle of length 2. If  $D$  is a tournament (resp. a bipartite tournament), then  $G$  is a complete graph (resp. a complete bipartite graph). Any acyclic orientation of a complete graph is transitive and orienting the edges of a complete bipartite graph from one side of the bipartition to the other results in a transitive orientation. Now suppose that  $D$  is isomorphic to  $F_n$  with vertex set  $\{x_0, x_1, \dots, x_n\}$  and arc set  $\{x_0x_1, x_1x_2, x_2x_0\} \cup \{x_0x_{i+3}, x_{i+3}x_1 : i = 0, 1, \dots, n-3\}$ , where  $n \geq 3$ . We reorient  $D$  as a digraph  $D'$  with arc set  $\{x_0x_1\} \cup \{x_{i+2} \rightarrow x_0, x_{i+2} \rightarrow x_1 : i = 0, 1, \dots, n-2\}$ . Clearly,  $D'$  is a transitive digraph. The proof of Lemma 9 is complete. ■

**Lemma 10** [3]. *If  $D$  is a  $k$ -transitive digraph with  $k \geq 2$ , then  $D$  is  $(k+n(k-1))$ -transitive for any  $n \geq 1$  such that  $k+n(k-1) \leq \text{diam}(D)$ , where  $\text{diam}(D)$  is the diameter of  $D$ .*

The following theorem is our main result.

**Theorem 11.** *A graph  $G$  has a 3-quasi-transitive orientation if and only if it has a 3-transitive orientation.*

**Proof.** Since every 3-transitive digraph is also a 3-quasi-transitive digraph, the sufficiency is trivial.

We shall prove the necessity below. Suppose that  $D$  is a 3-quasi-transitive orientation of  $G$  and  $D$  is not a 3-transitive orientation. If  $D$  is strong, then, by Lemmas 9 and 10, we are done. Suppose now that  $D$  is non-strong and  $D_1, D_2, \dots, D_t$  are its strong components. Note that every  $D_i$ ,  $i = 1, 2, \dots, t$ , is also a strong 3-quasi-transitive digraph. Hence, according to Lemma 9, every  $D_i$ , for  $i = 1, 2, \dots, t$ , can be reoriented as a transitive digraph. Now, we reorient every  $D_i$  as a transitive digraph  $D'_i$  as in the proof of Lemma 9 and keep the directions of remaining arcs in  $D$ . Denote the resulting digraph by  $D'$ . From Lemma 10, we know that if a digraph is transitive, then it must be 3-transitive. Hence  $D'_i$  is 3-transitive,  $i = 1, 2, \dots, t$ . Now we shall show that  $D'$  is 3-transitive. It suffices to prove that for any path  $x_0x_1x_2x_3$  in  $D'$ ,  $x_0 \rightarrow x_3$  in  $D'$ . By the definition of  $D'$ , we can see that  $D'$  is acyclic. Hence it is sufficient to show that  $\overline{x_0x_3}$  in  $D'$ . Observe that  $\overline{x_0x_3}$  in  $D'$  if and only if  $\overline{x_0x_3}$  in  $D$ . Hence we shall prove that  $\overline{x_0x_3}$  in  $D$  or  $\overline{x_0x_3}$  in  $D'$ . Furthermore, in order to show that  $\overline{x_0x_3}$  in  $D$ , by Lemma 5, we only need to prove that there is an odd path from  $x_0$  to  $x_3$  in  $D$ .

If  $x_0$  and  $x_3$  belong to the same strong component in  $D$ , say  $D_i$ , then  $x_1$  and  $x_2$  both belong to  $D_i$ , otherwise, assume, without loss of generality, that  $x_1 \in V(D_j)$  where  $i \neq j$ . Because the arcs of  $D$  between distinct strong components are not reoriented, it would be the case that  $D_i$  can reach  $D_j$  and  $D_j$  also can reach  $D_i$  in  $D$ , contradicting that they are distinct strong components. Since  $x_k \in V(D_i)$ , for  $k = 0, 1, 2, 3$  and  $D'_i$  is 3-transitive, we have  $x_0 \rightarrow x_3$ .

Now assume that  $x_0$  and  $x_3$  belong to distinct strong components and assume, without loss of generality, that  $x_0 \in V(D_i)$  and  $x_3 \in V(D_j)$  with  $1 \leq i \neq j \leq t$ . The following two claims will be useful.

**Claim 1.**  *$D_j$  is reachable from  $D_i$  in  $D$ .*

**Proof.** It suffices to show that there exists a path from  $D_i$  to  $D_j$  in  $D$ . If  $x_2 \in V(D_i)$  ( $x_1 \in V(D_j)$ ), then  $x_2x_3$  ( $x_0x_1$ ) is the desired path. So suppose  $x_2 \notin V(D_i)$  and  $x_1 \notin V(D_j)$ . If  $x_2 \in V(D_j)$ , then since  $x_1 \notin V(D_j)$ ,  $x_1x_2 \in A(D)$ . If  $x_0x_1 \in A(D)$ , then  $x_0x_1x_2$  is the desired path; if not, then  $x_1 \in V(D_i)$  and so

$x_1x_2$  is the desired path. Thus suppose  $x_2 \in V(D_s)$ , with  $1 \leq s \leq t$  and  $s \neq i, j$ . So  $x_2x_3 \in A(D)$ . If  $x_1 \in V(D_i)$ , then  $x_1x_2x_3$  is a path from  $D_i$  to  $D_j$  in  $D$ . Thus we may assume that  $x_1 \notin V(D_i)$  which implies  $x_0x_1 \in A(D)$ . If  $x_1x_2 \in A(D)$ , then  $x_0x_1x_2x_3$  is the desired path; if not, then  $x_1$  and  $x_2$  both belong to  $D_s$ . Since  $D_s$  is strong, there exists a path  $P$  from  $x_1$  to  $x_2$  in  $D$  and then  $x_0x_1Px_2x_3$  is the desired path.  $\square$

**Claim 2.** *If  $x_1, x_2 \notin V(D_i) \cup V(D_j)$ , then there exists an odd path from  $x_0$  to  $x_3$  in  $D$ .*

**Proof.** Since  $x_1, x_2 \notin V(D_i) \cup V(D_j)$ ,  $x_0x_1, x_2x_3 \in A(D)$ . If  $x_1x_2 \in A(D)$ , then  $x_0x_1x_2x_3$  is the desired path. Now assume that  $x_1x_2 \notin A(D)$ . Then by the definition of  $D'$ , we have that  $x_2x_1 \in A(D)$  and  $x_1, x_2$  belong to the same strong component in  $D$ , say  $D_k$ . If  $D_k$  is a non-bipartite digraph, then by Lemmas 6 and 7,  $x_0 \mapsto D_k$  and  $D_k \mapsto x_3$  and in particular,  $x_0 \rightarrow x_2$  and  $x_1 \rightarrow x_3$  in  $D$ . Note that  $x_0x_2x_1x_3$  is a path length 3 in  $D$ . If  $D_k$  is a bipartite digraph, then  $x_1$  and  $x_2$  belong to different partite sets. Again since  $D_k$  is a strong bipartite digraph, there exists an odd path  $P$  from  $x_1$  to  $x_2$  in  $D$ . Then we have that  $x_0x_1Px_2x_3$  is an odd path from  $x_0$  to  $x_3$  in  $D$ . Thus the claim holds. The proof of Claim 2 is complete.  $\square$

We consider two cases.

*Case 1.* At least one of  $D_i$  and  $D_j$  is trivial, say  $D_i$ . Since  $D_i$  is trivial, by the definition of  $D'$ ,  $x_0x_1 \in V(D)$ . If  $V(D_j)$  is also trivial, then  $x_1, x_2 \notin V(D_i) \cup V(D_j)$ . By Claim 2, we are done. Now assume that  $D_j$  is non-trivial. By Claim 1, Lemma 4 and the definition of strong components, there exists at least an arc from  $x_0$  to  $D_j$ . If  $D_j$  is a non-bipartite digraph, then by Lemma 6,  $x_0 \mapsto D_j$ . In particular,  $x_0 \rightarrow x_3$  and so we are done.

Now suppose that  $D_j$  is a bipartite digraph. Assume that  $(X_j, Y_j)$  is the bipartition of  $D_j$  and assume, without loss of generality, that  $x_3 \in X_j$ . By Lemma 6,  $x_0 \mapsto X_j$  or  $x_0 \mapsto Y_j$ . If  $x_0 \mapsto X_j$ , then  $x_0 \rightarrow x_3$  and so we are done. Suppose that  $x_0 \mapsto Y_j$ .

*Subcase 1.1.*  $x_2 \in V(D_j)$ . Since  $x_2$  and  $x_3$  are adjacent, we have  $x_2 \in Y_j$ . Since  $x_1$  and  $x_2$  are adjacent, we have that  $x_1 \notin Y_j$ . If  $x_1 \in X_j$ , then by  $x_0x_1 \in A(D)$  and Lemma 6,  $x_0 \mapsto X_j$ . In particular,  $x_0 \rightarrow x_3$  and so we are done. Now assume that  $x_1 \notin V(D_j)$  and so  $x_1x_2 \in A(D)$ . Since  $D_j$  is a strong bipartite digraph, there exists an odd path  $P$  from  $x_2$  to  $x_3$  in  $D_j$ . Then  $x_0x_1x_2Px_3$  is an odd path from  $x_0$  to  $x_3$ .

*Subcase 1.2.*  $x_2 \notin V(D_j)$ . Since  $x_2 \notin V(D_j)$ , we have  $x_2x_3 \in A(D)$ . By the definition of strong components,  $x_1 \notin V(D_j)$ . Combining this with Claim 2, there exists an odd path from  $x_0$  to  $x_3$  and so we are done.

*Case 2.*  $D_i$  and  $D_j$  are both non-trivial. By Claim 1 and Lemma 4, there exists at least an arc from  $D_i$  to  $D_j$ . By Lemma 8, we have  $D_i \mapsto D_j$  or  $D_i \cup D_j$  is a bipartite tournament. If  $D_i \mapsto D_j$ , then  $x_0 \rightarrow x_3$  in  $D$  and so we are done. If  $D_i \cup D_j$  is a bipartite tournament, then  $D_i$  and  $D_j$  are both bipartite. Assume that the bipartitions of  $D_i$  and  $D_j$  are  $(X_i, Y_i)$  and  $(X_j, Y_j)$ , respectively and the bipartition of  $D_i \cup D_j$  is  $(X_i \cup X_j, Y_i \cup Y_j)$ . Assume, without loss of generality, that  $x_0 \in X_i$ . If  $x_3 \in Y_j$ , then  $x_0 x_3 \in A(D)$  and so we are done. Suppose that  $x_3 \in X_j$ .

*Subcase 2.1.*  $x_2 \in V(D_j)$ . Since  $x_2$  and  $x_3$  are adjacent,  $x_2 \in Y_j$ . This implies that  $x_1 \notin V(D_j) \cup V(D_i)$ , which follows from the fact that  $D_i \cup D_j$  is bipartite. Thus we have  $x_0 x_1, x_1 x_2 \in A(D)$ . Since  $D_j$  is a strong bipartite digraph, there is an odd path  $P$  from  $x_2$  to  $x_3$  in  $D_j$ . Then  $x_0 x_1 x_2 P x_3$  is an odd path from  $x_0$  to  $x_3$  in  $D$  and so we are done.

*Subcase 2.2.*  $x_2 \notin V(D_j)$ . So  $x_2 x_3 \in A(D)$ . By the definition of strong components,  $x_1 \notin V(D_j)$ .

If  $x_2 \in V(D_i)$ , then since  $x_2$  and  $x_3$  are adjacent,  $x_2 \in Y_i$ . Since  $x_1$  and  $x_0, x_2$  are both adjacent,  $x_1 \notin V(D_i)$ , which implies that  $x_0 x_1, x_1 x_2 \in A(D)$ , a contradiction to the definition of strong components. Thus  $x_2 \notin V(D_i)$ .

If  $x_1 \in V(D_i)$ , then since  $x_0$  and  $x_1$  are adjacent,  $x_1 \in Y_i$  and  $x_1 x_2 \in A(D)$ . Since  $D_i$  is a strong bipartite digraph, there is an odd path  $P$  from  $x_0$  to  $x_1$  in  $D_i$ . Then  $x_0 P x_1 x_2 x_3$  is an odd path from  $x_0$  to  $x_3$  in  $D$  and so we are done. Assume that  $x_1 \notin A(D_i)$ .

Note that now  $x_1, x_2 \notin V(D_i) \cup V(D_j)$ . By Claim 2, there exists an odd path from  $x_0$  to  $x_3$  in  $D$  and so we are done.

We have considered all the cases. The proof of Theorem 11 is complete. ■

Conjecture 2 then is an immediate consequence of Theorem 11.

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### REFERENCES

- [1] J. Bang-Jensen, *Kings in quasi-transitive digraphs*, Discrete Math. **185** (1998) 19–27.  
doi:10.1016/S0012-365X(97)00179-9
- [2] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications* (Springer, London, 2000).
- [3] C. Hernández-Cruz, *3-transitive digraphs*, Discuss. Math. Graph Theory **32** (2012) 205–219.  
doi:10.7151/dmgt.1613

- [4] A. Ghouila-Houri, *Caractérisation des graphes non orientés dont on peut orienter les arrêtes de manière à obtenir le graphe d'une relation d'ordre*, Comptes Rendus de l'Académie des Sciences Paris **254** (1962) 1370–1371.
- [5] H. Galeana-Sánchez, I.A. Goldfeder and I. Urrutia, *On the structure of strong 3-quasi-transitive digraphs*, Discrete Math. **310** (2010) 2495–2498.  
doi:10.1016/j.disc.2010.06.008
- [6] H. Galeana-Sánchez and C. Hernández-Cruz, *k-kernels in k-transitive and k-quasi-transitive digraphs*, Discrete Math. **312** (2012) 2522–2530.  
doi:10.1016/j.disc.2012.05.005
- [7] S. Wang and R. Wang, *Independent sets and non-augmentable paths in arc-locally in-semicomplete digraphs and quasi-arc-transitive digraphs*, Discrete Math. **311** (2011) 282–288.  
doi:10.1016/j.disc.2010.11.009

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