# UNDERLYING GRAPHS OF 3-QUASI-TRANSITIVE DIGRAPHS AND 3-TRANSITIVE DIGRAPHS 

Ruixia Wang and Shiying Wang<br>School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi, 030006, PR China<br>e-mail: wangrx@sxu.edu.cn<br>shiying@sxu.edu.cn


#### Abstract

A digraph is 3 -quasi-transitive (resp. 3 -transitive), if for any path $x_{0} x_{1}$ $x_{2} x_{3}$ of length $3, x_{0}$ and $x_{3}$ are adjacent (resp. $x_{0}$ dominates $x_{3}$ ). César Hernández-Cruz conjectured that if $D$ is a 3 -quasi-transitive digraph, then the underlying graph of $D, U G(D)$, admits a 3 -transitive orientation. In this paper, we shall prove that the conjecture is true.


Keywords: graph orientation, 3 -quasi-transitive digraph, 3 -transitive digraph.
2010 Mathematics Subject Classification: 05C20.

## 1. Terminology and Introduction

We only consider finite graphs and digraphs without loops and multiple edges or multiple arcs. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A complete graph is a graph in which any two vertices are adjacent. A complete bipartite graph $G$ is a graph in which the vertices of $G$ can be partitioned into two partite sets such that every partite set is an independent set and for every pair $x, y$ of vertices from distinct partite sets, $(x, y) \in E(G)$.

Let $D$ be a digraph with vertex set $V(D)$ and arc set $A(D)$. For any $x, y \in$ $V(D)$, we will write $\overrightarrow{x y}$ or $x \rightarrow y$ if $x y \in A(D)$, and also, we will write $\overline{x y}$ if $\overrightarrow{x y}$ or $\overrightarrow{y x}$. For disjoint subsets $X$ and $Y$ of $V(D)$ or subdigraphs of $D, X \rightarrow Y$ means that every vertex of $X$ dominates every vertex of $Y, X \Rightarrow Y$ means that there is no arc from $Y$ to $X$ and $X \mapsto Y$ means that both of $X \rightarrow Y$ and $X \Rightarrow Y$ hold. Let $D^{\prime}$ be a subdigraph of $D$ and $x \in V(D)-V\left(D^{\prime}\right)$. We say that $x$ and $D^{\prime}$ are adjacent if $x$ and some vertex of $D^{\prime}$ are adjacent.

For any digraph $D$, we can associate a graph $G$ on the same vertex set simply by replacing each arc by an edge with the same ends. This graph is the underlying graph of $D$, denoted $U G(D)$. By a path of a digraph $D$, we mean a directed path of $D$. The length of a path is the number of its arcs. A path of length $k$ is called a $k$-path; the path is odd or even according to the parity of $k$. A digraph $D$ is said to be strongly connected or just strong, if for every pair $x, y$ of vertices of $D$, there is a path from $x$ to $y$. A strong component of a digraph $D$ is a maximal induced subdigraph of $D$ which is strong. The strong component digraph $S C(D)$ of $D$ is obtained by contracting strong components of $D$ and deleting any parallel arcs obtained in this process.

A digraph $D$ is semicomplete if there is at least one arc between any pair of distinct vertices of $D$. A tournament is a semicomplete digraph with no cycle of length 2. A digraph $D$ is semicomplete bipartite, if the vertices of $D$ can be partitioned into two partite sets such that every partite set is an independent set and for every pair $x, y$ of vertices from distinct partite sets, $x y$ or $y x$ (or both) is in $D$. A bipartite tournament is a semicomplete bipartite digraph with no cycle of length 2 . A digraph is $k$-quasi-transitive, where $k \geq 2$, if for any path $x_{0} x_{1} x_{2} \ldots x_{k}$ of length $k, x_{0}$ and $x_{k}$ are adjacent. A 2-quasi-transitive digraph is also called a quasi-transitive digraph. A 3 -quasi-transitive digraph is also called a quasi-arc-transitive digraph (see [7]). A digraph is $k$-transitive, where $k \geq 2$, if for any path $x_{0} x_{1} x_{2} \ldots x_{k}$ of length $k, x_{0}$ dominates $x_{k}$. A 2 -transitive digraph is also called a transitive digraph. $k$-transitive digraphs and $k$-quasi-transitive digraphs have been studied by several authors. See $[1,2,3,5,6,7]$. For a graph $G$, a digraph $D$ is called an orientation of $G$ if $D$ is obtained from $G$ by replacing each edge $(x, y)$ of $G$ by either $x y$ or $y x$.

In [4], see also [2], Ghouila-Houri proved the following theorem.
Theorem 1 [2, 4]. A graph $G$ has a quasi-transitive orientation if and only if it has a transitive orientation.
It seems natural to consider an analogue of Theorem 1 for 3 -quasi-transitive digraphs and 3 -transitive digraphs.

In [3], the author proposed the following conjecture.
Conjecture 2 [3]. Let $D$ be a 3-quasi-transitive oriented graph. Then the underlying graph of $D, U G(D)$, admits a 3 -transitive orientation.
In Section 2, we will prove that the conjecture is true.

## 2. Main Result

We begin with some useful lemmas. Let $F_{n}$ be the digraph with vertex set $\left\{x_{0}, x_{1}, \ldots, x_{n}\right\}$ and $\operatorname{arc}$ set $\left\{x_{0} x_{1}, x_{1} x_{2}, x_{2} x_{0}\right\} \cup\left\{x_{0} x_{i+3}, x_{i+3} x_{1}: i=0,1, \ldots, n-\right.$ $3\}$, where $n \geq 3$.

Lemma 3 [5]. Let $D$ be a strong 3-quasi-transitive digraph of order $n$. Then $D$ is either a semicomplete digraph, a semicomplete bipartite digraph or isomorphic to $F_{n}$.

Lemma 4 [7]. Let $D^{\prime}$ be a non-trivial strong induced subdigraph of a 3-quasitransitive digraph $D$. For any $s \in V(D)-V\left(D^{\prime}\right)$, if there exists a directed path between $s$ and $D^{\prime}$, then $s$ and $D^{\prime}$ are adjacent.

Lemma 5 [5]. Let $D$ be a 3-quasi-transitive digraph. For a pair $x, y$ of $V(D)$, if there exists an $(x, y)$-path of odd length, then $x$ and $y$ are adjacent.

Lemma 6 [7]. Let $D^{\prime}$ be a non-trivial strong induced subdigraph of a 3-quasitransitive digraph $D$ and let $s \in V(D)-V\left(D^{\prime}\right)$ with at least one arc from s to $D^{\prime}$ and $s \Rightarrow D^{\prime}$. Then each of the following holds:
(a) If $D^{\prime}$ is a bipartite digraph with bipartition $(X ; Y)$ and $s$ dominates a vertex of $X$, then $s \mapsto X$.
(b) If $D^{\prime}$ is a non-bipartite digraph, then $s \mapsto D^{\prime}$.

Lemma 7 [7]. Let $D^{\prime}$ be a non-trivial strong induced subdigraph of a 3 -quasitransitive digraph $D$ and let $s \in V(D)-V\left(D^{\prime}\right)$ with at least one arc from $D^{\prime}$ to $s$ and $D^{\prime} \Rightarrow s$. Then each of the following holds:
(a) If $D^{\prime}$ is a bipartite digraph with bipartition $(X, Y)$ and there exists a vertex of $X$ which dominates $s$, then $X \mapsto s$.
(b) If $D^{\prime}$ is a non-bipartite digraph, then $D^{\prime} \mapsto s$.

Lemma 8 [7]. Let $D_{1}$ and $D_{2}$ be two distinct non-trivial strong components of a 3-quasi-transitive digraph with at least one arc from $D_{1}$ to $D_{2}$. Then either $D_{1} \mapsto D_{2}$ or $D_{1} \cup D_{2}$ is a semicomplete bipartite digraph.

Lemma 9. If a graph $G$ has a strong 3-quasi-transitive orientation, then it has a transitive orientation.

Proof. Let $D$ be a strong 3-quasi-transitive orientation of $G$ and let the order of $D$ be $n$. By Lemma 3, $D$ is either a tournament, a bipartite tournament or isomorphic to $F_{n}$ because $D$ has no cycle of length 2. If $D$ is a tournament (resp. a bipartite tournament), then $G$ is a complete graph (resp. a complete bipartite graph). Any acyclic orientation of a complete graph is transitive and orienting the edges of a complete bipartite graph from one side of the bipartition to the other results in a transitive orientation. Now suppose that $D$ is isomorphic to $F_{n}$ with vertex set $\left\{x_{0}, x_{1}, \ldots, x_{n}\right\}$ and arc set $\left\{x_{0} x_{1}, x_{1} x_{2}, x_{2} x_{0}\right\} \cup\left\{x_{0} x_{i+3}, x_{i+3} x_{1}\right.$ : $i=0,1, \ldots, n-3\}$, where $n \geq 3$. We reorient $D$ as a digraph $D^{\prime}$ with arc set $\left\{x_{0} x_{1}\right\} \cup\left\{x_{i+2} \rightarrow x_{0}, x_{i+2} \rightarrow x_{1}: i=0,1, \ldots, n-2\right\}$. Clearly, $D^{\prime}$ is a transitive digraph. The proof of Lemma 9 is complete.

Lemma 10 [3]. If $D$ is a $k$-transitive digraph with $k \geq 2$, then $D$ is $(k+n(k-1))$ transitive for any $n \geq 1$ such that $k+n(k-1) \leq \operatorname{diam}(D)$, where $\operatorname{diam}(D)$ is the diameter of $D$.

The following theorem is our main result.
Theorem 11. A graph $G$ has a 3-quasi-transitive orientation if and only if it has a 3-transitive orientation.

Proof. Since every 3-transitive digraph is also a 3-quasi-transitive digraph, the sufficiency is trivial.

We shall prove the necessity below. Suppose that $D$ is a 3 -quasi-transitive orientation of $G$ and $D$ is not a 3 -transitive orientation. If $D$ is strong, then, by Lemmas 9 and 10, we are done. Suppose now that $D$ is non-strong and $D_{1}, D_{2}, \ldots, D_{t}$ are its strong components. Note that every $D_{i}, i=1,2, \ldots, t$, is also a strong 3 -quasi-transitive digraph. Hence, according to Lemma 9, every $D_{i}$, for $i=1,2, \ldots, t$, can be reoriented as a transitive digraph. Now, we reorient every $D_{i}$ as a transitive digraph $D_{i}^{\prime}$ as in the proof of Lemma 9 and keep the directions of remaining arcs in $D$. Denote the resulting digraph by $D^{\prime}$. From Lemma 10, we know that if a digraph is transitive, then it must be 3 -transitive. Hence $D_{i}^{\prime}$ is 3-transitive, $i=1,2, \ldots, t$. Now we shall show that $D^{\prime}$ is 3 -transitive. It suffices to prove that for any path $x_{0} x_{1} x_{2} x_{3}$ in $D^{\prime}, x_{0} \rightarrow x_{3}$ in $D^{\prime}$. By the definition of $D^{\prime}$, we can see that $D^{\prime}$ is acyclic. Hence it is sufficient to show that $\overline{x_{0} x_{3}}$ in $D^{\prime}$. Observe that $\overline{x_{0} x_{3}}$ in $D^{\prime}$ if and only if $\overline{x_{0} x_{3}}$ in $D$. Hence we shall prove that $\overline{x_{0} x_{3}}$ in $D$ or $\overline{x_{0} x_{3}}$ in $D^{\prime}$. Furthermore, in order to show that $\overline{x_{0} x_{3}}$ in $D$, by Lemma 5 , we only need to prove that there is an odd path from $x_{0}$ to $x_{3}$ in $D$.

If $x_{0}$ and $x_{3}$ belong to the same strong component in $D$, say $D_{i}$, then $x_{1}$ and $x_{2}$ both belong to $D_{i}$, otherwise, assume, without loss of generality, that $x_{1} \in$ $V\left(D_{j}\right)$ where $i \neq j$. Because the arcs of $D$ between distinct strong components are not reoriented, it would be the case that $D_{i}$ can reach $D_{j}$ and $D_{j}$ also can reach $D_{i}$ in $D$, contradicting that they are distinct strong components. Since $x_{k} \in V\left(D_{i}\right)$, for $k=0,1,2,3$ and $D_{i}^{\prime}$ is 3 -transitive, we have $x_{0} \rightarrow x_{3}$.

Now assume that $x_{0}$ and $x_{3}$ belong to distinct strong components and assume, without loss of generality, that $x_{0} \in V\left(D_{i}\right)$ and $x_{3} \in V\left(D_{j}\right)$ with $1 \leq i \neq j \leq t$. The following two claims will be useful.

Claim 1. $D_{j}$ is reachable from $D_{i}$ in $D$.
Proof. It suffices to show that there exists a path from $D_{i}$ to $D_{j}$ in $D$. If $x_{2} \in V\left(D_{i}\right)\left(x_{1} \in V\left(D_{j}\right)\right)$, then $x_{2} x_{3}\left(x_{0} x_{1}\right)$ is the desired path. So suppose $x_{2} \notin$ $V\left(D_{i}\right)$ and $x_{1} \notin V\left(D_{j}\right)$. If $x_{2} \in V\left(D_{j}\right)$, then since $x_{1} \notin V\left(D_{j}\right), x_{1} x_{2} \in A(D)$. If $x_{0} x_{1} \in A(D)$, then $x_{0} x_{1} x_{2}$ is the desired path; if not, then $x_{1} \in V\left(D_{i}\right)$ and so
$x_{1} x_{2}$ is the desired path. Thus suppose $x_{2} \in V\left(D_{s}\right)$, with $1 \leq s \leq t$ and $s \neq i, j$. So $x_{2} x_{3} \in A(D)$. If $x_{1} \in V\left(D_{i}\right)$, then $x_{1} x_{2} x_{3}$ is a path from $D_{i}$ to $D_{j}$ in $D$. Thus we may assume that $x_{1} \notin V\left(D_{i}\right)$ which implies $x_{0} x_{1} \in A(D)$. If $x_{1} x_{2} \in A(D)$, then $x_{0} x_{1} x_{2} x_{3}$ is the desired path; if not, then $x_{1}$ and $x_{2}$ both belong to $D_{s}$. Since $D_{s}$ is strong, there exists a path $P$ from $x_{1}$ to $x_{2}$ in $D$ and then $x_{0} x_{1} P x_{2} x_{3}$ is the desired path.

Claim 2. If $x_{1}, x_{2} \notin V\left(D_{i}\right) \cup V\left(D_{j}\right)$, then there exists an odd path from $x_{0}$ to $x_{3}$ in $D$.

Proof. Since $x_{1}, x_{2} \notin V\left(D_{i}\right) \cup V\left(D_{j}\right), x_{0} x_{1}, x_{2} x_{3} \in A(D)$. If $x_{1} x_{2} \in A(D)$, then $x_{0} x_{1} x_{2} x_{3}$ is the desired path. Now assume that $x_{1} x_{2} \notin A(D)$. Then by the definition of $D^{\prime}$, we have that $x_{2} x_{1} \in A(D)$ and $x_{1}, x_{2}$ belong to the same strong component in $D$, say $D_{k}$. If $D_{k}$ is a non-bipartite digraph, then by Lemmas 6 and $7, x_{0} \mapsto D_{k}$ and $D_{k} \mapsto x_{3}$ and in particular, $x_{0} \rightarrow x_{2}$ and $x_{1} \rightarrow x_{3}$ in $D$. Note that $x_{0} x_{2} x_{1} x_{3}$ is a path length 3 in $D$. If $D_{k}$ is a bipartite digraph, then $x_{1}$ and $x_{2}$ belong to different partite sets. Again since $D_{k}$ is a strong bipartite digraph, there exists an odd path $P$ from $x_{1}$ to $x_{2}$ in $D$. Then we have that $x_{0} x_{1} P x_{2} x_{3}$ is an odd path from $x_{0}$ to $x_{3}$ in $D$. Thus the claim holds. The proof of Claim 2 is complete.

We consider two cases.
Case 1. At least one of $D_{i}$ and $D_{j}$ is trivial, say $D_{i}$. Since $D_{i}$ is trivial, by the definition of $D^{\prime}, x_{0} x_{1} \in V(D)$. If $V\left(D_{j}\right)$ is also trivial, then $x_{1}, x_{2} \notin$ $V\left(D_{i}\right) \cup V\left(D_{j}\right)$. By Claim 2, we are done. Now assume that $D_{j}$ is non-trivial. By Claim 1, Lemma 4 and the definition of strong components, there exists at least an arc from $x_{0}$ to $D_{j}$. If $D_{j}$ is a non-bipartite digraph, then by Lemma 6 , $x_{0} \mapsto D_{j}$. In particular, $x_{0} \rightarrow x_{3}$ and so we are done.

Now suppose that $D_{j}$ is a bipartite digraph. Assume that $\left(X_{j}, Y_{j}\right)$ is the bipartition of $D_{j}$ and assume, without loss of generality, that $x_{3} \in X_{j}$. By Lemma 6, $x_{0} \mapsto X_{j}$ or $x_{0} \mapsto Y_{j}$. If $x_{0} \mapsto X_{j}$, then $x_{0} \rightarrow x_{3}$ and so we are done. Suppose that $x_{0} \mapsto Y_{j}$.

Subcase 1.1. $x_{2} \in V\left(D_{j}\right)$. Since $x_{2}$ and $x_{3}$ are adjacent, we have $x_{2} \in Y_{j}$. Since $x_{1}$ and $x_{2}$ are adjacent, we have that $x_{1} \notin Y_{j}$. If $x_{1} \in X_{j}$, then by $x_{0} x_{1} \in A(D)$ and Lemma $6, x_{0} \mapsto X_{j}$. In particular, $x_{0} \rightarrow x_{3}$ and so we are done. Now assume that $x_{1} \notin V\left(D_{j}\right)$ and so $x_{1} x_{2} \in A(D)$. Since $D_{j}$ is a strong bipartite digraph, there exists an odd path $P$ from $x_{2}$ to $x_{3}$ in $D_{j}$. Then $x_{0} x_{1} x_{2} P x_{3}$ is an odd path from $x_{0}$ to $x_{3}$.

Subcase 1.2. $x_{2} \notin V\left(D_{j}\right)$. Since $x_{2} \notin V\left(D_{j}\right)$, we have $x_{2} x_{3} \in A(D)$. By the definition of strong components, $x_{1} \notin V\left(D_{j}\right)$. Combining this with Claim 2, there exists an odd path from $x_{0}$ to $x_{3}$ and so we are done.

Case 2. $D_{i}$ and $D_{j}$ are both non-trivial. By Claim 1 and Lemma 4, there exists at least an arc from $D_{i}$ to $D_{j}$. By Lemma 8, we have $D_{i} \mapsto D_{j}$ or $D_{i} \cup D_{j}$ is a bipartite tournament. If $D_{i} \mapsto D_{j}$, then $x_{0} \rightarrow x_{3}$ in $D$ and so we are done. If $D_{i} \cup D_{j}$ is a bipartite tournament, then $D_{i}$ and $D_{j}$ are both bipartite. Assume that the bipartitions of $D_{i}$ and $D_{j}$ are $\left(X_{i}, Y_{i}\right)$ and $\left(X_{j}, Y_{j}\right)$, respectively and the bipartition of $D_{i} \cup D_{j}$ is $\left(X_{i} \cup X_{j}, Y_{i} \cup Y_{j}\right)$. Assume, without loss of generality, that $x_{0} \in X_{i}$. If $x_{3} \in Y_{j}$, then $x_{0} x_{3} \in A(D)$ and so we are done. Suppose that $x_{3} \in X_{j}$.

Subcase 2.1. $x_{2} \in V\left(D_{j}\right)$. Since $x_{2}$ and $x_{3}$ are adjacent, $x_{2} \in Y_{j}$. This implies that $x_{1} \notin V\left(D_{j}\right) \cup V\left(D_{i}\right)$, which follows from the fact that $D_{i} \cup D_{j}$ is bipartite. Thus we have $x_{0} x_{1}, x_{1} x_{2} \in A(D)$. Since $D_{j}$ is a strong bipartite digraph, there is an odd path $P$ from $x_{2}$ to $x_{3}$ in $D_{j}$. Then $x_{0} x_{1} x_{2} P x_{3}$ is an odd path from $x_{0}$ to $x_{3}$ in $D$ and so we are done.

Subcase 2.2. $x_{2} \notin V\left(D_{j}\right)$. So $x_{2} x_{3} \in A(D)$. By the definition of strong components, $x_{1} \notin V\left(D_{j}\right)$.

If $x_{2} \in V\left(D_{i}\right)$, then since $x_{2}$ and $x_{3}$ are adjacent, $x_{2} \in Y_{i}$. Since $x_{1}$ and $x_{0}, x_{2}$ are both adjacent, $x_{1} \notin V\left(D_{i}\right)$, which implies that $x_{0} x_{1}, x_{1} x_{2} \in A(D)$, a contradiction to the definition of strong components. Thus $x_{2} \notin V\left(D_{i}\right)$.

If $x_{1} \in V\left(D_{i}\right)$, then since $x_{0}$ and $x_{1}$ are adjacent, $x_{1} \in Y_{i}$ and $x_{1} x_{2} \in A(D)$. Since $D_{i}$ is a strong bipartite digraph, there is an odd path $P$ from $x_{0}$ to $x_{1}$ in $D_{i}$. Then $x_{0} P x_{1} x_{2} x_{3}$ is an odd path from $x_{0}$ to $x_{3}$ in $D$ and so we are done. Assume that $x_{1} \notin A\left(D_{i}\right)$.

Note that now $x_{1}, x_{2} \notin V\left(D_{i}\right) \cup V\left(D_{j}\right)$. By Claim 2, there exists an odd path from $x_{0}$ to $x_{3}$ in $D$ and so we are done.

We have considered all the cases. The proof of Theorem 11 is complete.
Conjecture 2 then is an immediate consequence of Theorem 11.

## Acknowledgement

The authors thank the anonymous referees for several helpful comments.

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