Discussiones Mathematicae
Graph Theory 33 (2013) 411–428
doi:10.7151/dmgt.1678

ON CLOSED MODULAR COLORINGS OF TREES

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Abstract

Two vertices u and v in a nontrivial connected graph G are twins if u and v have the same neighbors in $V(G) - \{u, v\}$. If u and v are adjacent, they are referred to as true twins; while if u and v are nonadjacent, they are false twins. For a positive integer k, let $c : V(G) \to \mathbb{Z}_k$ be a vertex coloring where adjacent vertices may be assigned the same color. The coloring c induces another vertex coloring $c' : V(G) \to \mathbb{Z}_k$ defined by $c'(v) = \sum_{u \in N[v]} c(u)$ for each $v \in V(G)$, where $N[v]$ is the closed neighborhood of v. Then c is called a closed modular k-coloring if $c'(u) \neq c'(v)$ in $\mathbb{Z}_k$ for all pairs u, v of adjacent vertices that are not true twins. The minimum k for which G has a closed modular k-coloring is the closed modular chromatic number $\overline{\mathrm{mc}}(G)$ of G. The closed modular chromatic number is investigated for trees and determined for several classes of trees. For each tree T in these classes, it is shown that $\overline{\mathrm{mc}}(T) = 2$ or $\overline{\mathrm{mc}}(T) = 3$. A closed modular k-coloring c of a tree T is called nowhere-zero if $c(x) \neq 0$ for each vertex x of T. It is shown that every tree of order 3 or more has a nowhere-zero closed modular 4-coloring.

Keywords: trees, closed modular k-coloring, closed modular chromatic number.

2010 Mathematics Subject Classification: 05C05, 05C15.

1. Introduction

In 1986, at the 250th Anniversary of Graph Theory Conference held at Indiana University-Purdue University Fort Wayne, a weighting (or edge labeling with positive integers) of a connected graph G was introduced for the purpose of producing a weighted graph whose degrees (obtained by adding the weights of the incident edges of each vertex) were distinct. Such a weighted graph was called *irregular*. This concept could be looked at in another manner, however. In particular, let $\mathbb{N}$ denote the set of positive integers and let E_v denote the set of edges of G incident with a vertex v. An edge coloring $c : E(G) \to \mathbb{N}$, where adjacent edges may be colored the same, is said to be *vertex-distinguishing* if the coloring $c' : V(G) \to \mathbb{N}$ induced by c and defined by $c'(v) = \sum_{e \in E_v} c(e)$ has the property that $c'(x) \neq c'(y)$ for every two distinct vertices x and y of G. A paper [2] on this concept appeared in the proceedings of this conference. The main emphasis of this research dealt with minimizing the largest color assigned to the edges of the graph to produce an irregular graph. Vertex-distinguishing colorings have received increased attention during the past 25 years (see [7]).

Two decades earlier, in 1968, Rosa [13] introduced a vertex labeling that induces an *edge-distinguishing* labeling defined by subtracting labels. In particular, for a graph G of size m, a vertex labeling (an injective function) $f : V(G) \to \{0, 1, \ldots, m\}$ was called a *β-valuation* by Rosa if the induced edge labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ defined by $f'(uv) = |f(u) - f(v)|$ is bijective. In 1972 Golomb [10] called a β-valuation a *graceful labeling* and a graph possessing a graceful labeling a *graceful graph*. It is this terminology that became standard. Much research has been done on graceful graphs. A popular conjecture in graph theory, due to Kotzig and Ringel, is the following.

The Graceful Tree Conjecture. *Every nontrivial tree is graceful.*

In 1991 Gnana Jothi [9] introduced a concept that, in a certain sense, reverses the roles of vertices and edges in graceful labelings (also see [8]). For a connected graph G of order $n \geq 3$, let $f : E(G) \to \mathbb{Z}_n$ be an edge labeling of G that induces a bijective function $f' : V(G) \to \mathbb{Z}_n$ defined by $f'(v) = \sum_{e \in E_v} f(e)$ for each vertex v of G. Such a labeling f is called a *modular edge-graceful labeling*, while a graph possessing such a labeling is called *modular edge-graceful*. Verifying a conjecture by Gnana Jothi on trees, Jones, Kolasinski and Zhang [11] showed not only that every tree of order $n \geq 3$ is modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$ but a connected graph of order $n \geq 3$ is modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$.

Many of these weighting or labeling concepts were later interpreted as coloring concepts with the resulting vertex-distinguishing labeling becoming a vertex-distinguishing coloring. A *neighbor-distinguishing coloring* is a coloring in which every pair of adjacent vertices are colored differently. Such a coloring is more commonly called a *proper coloring*. The minimum number of colors in a proper vertex coloring of a graph G is its chromatic number $\chi(G)$.

In 2004 a neighbor-distinguishing edge coloring $c : E(G) \to \{1, 2, \ldots, k\}$ of a graph G was introduced (see [6, p. 385]) in which an induced vertex coloring $s : V(G) \to \mathbb{N}$ is defined by $s(v) = \sum_{e \in E_v} c(e)$ for each vertex v of G. The minimum k for which such a neighbor-distinguishing coloring exists is called the *sum distinguishing index*, denoted by $sd(G)$ of G. This is therefore the proper coloring analogue of the irregular weighting mentioned earlier. It was shown in [12] that if $\chi(G) \leq 3$, then $sd(G) \leq 3$. In [1] it was shown for every connected graph G of order at least 3 that $sd(G) \leq 4$. In fact, Karoński, Łuczak and Thomason [12] made the following conjecture, which has acquired a name used by many.

The 1-2-3 Conjecture. *If G is a connected graph of order 3 or more, then $sd(G) \leq 3$.*

Consequently, if the 1-2-3 Conjecture is true, then for every connected graph G of order 3 or more, it is possible to assign each edge of G one of the colors 1, 2, 3 in such a way that if u and v are adjacent vertices of G, then the sums of the colors of the incident edges of u and v are different.

A number of neighbor-distinguishing vertex colorings different from standard proper colorings have been introduced in the literature (see [6, p.379–385], for example). In 2010 a neighbor-distinguishing vertex coloring of a graph was introduced based on sums of colors (see [3]). For a nontrivial connected graph G, let $c : V(G) \to \mathbb{N}$ be a vertex coloring of G where adjacent vertices may be colored the same. If k colors are used by c, then c is a *k-coloring* of G. The *color sum* $\sigma(v)$ of a vertex v is defined by $\sigma(v) = \sum_{u \in N(v)} c(u)$ where $N(v)$ denotes the neighborhood of v (the set of vertices adjacent to v). If $\sigma(u) \neq \sigma(v)$ for every two adjacent vertices u and v of G, then c is neighbor-distinguishing and is called a *sigma coloring* of G. The minimum number of colors required in a sigma coloring of a graph G is called the *sigma chromatic number* of G and is denoted by $\sigma(G)$. It was shown in [3] that for each pair a, b of positive integer with $a \leq b$, there is a connected graph G with $\sigma(G) = a$ and $\chi(G) = b$.

In 2011 another neighbor-distinguishing vertex coloring was introduced in [4] that is closely related to colorings discussed above. We describe this next.

2. Closed Modular Colorings of Graphs

For a nontrivial connected graph G, let $c : V(G) \to \mathbb{Z}_k$ ($k \geq 2$) be a vertex coloring where adjacent vertices may be assigned the same color. The coloring c induces another vertex coloring $c' : V(G) \to \mathbb{Z}_k$, where

$$c'(v) = \sum_{u \in N[v]} c(u), \tag{1}$$

$N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v and the sum in (1) is performed in $\mathbb{Z}_k$. A coloring c of G is called a *closed modular k-coloring* if for every pair x, y of adjacent vertices in G either $c'(x) \neq c'(y)$ or $N[x] = N[y]$, in the latter case of which we must have $c'(x) = c'(y)$. Closed modular colorings of graphs were introduced in [4] and inspired by a domination problem. The minimum k for which G has a closed modular k-coloring is called the *closed modular chromatic number* of G and is denoted by $\overline{\text{mc}}(G)$. It was observed in [4] that the nontrivial complete graphs are the only nontrivial connected graphs G for which $\overline{\text{mc}}(G) = 1$.

Two vertices u and v in a connected graph G are *twins* if u and v have the same neighbors in $V(G) - \{u, v\}$. If u and v are adjacent, they are referred to as *true twins*; while if u and v are nonadjacent, they are *false twins*. If u and v are adjacent vertices of a graph G such that $N[u] = N[v]$ (that is, u and v are true twins), then $c'(u) = c'(v)$ for every vertex coloring c of G. The following result appeared in [4].

Proposition 2.1. *If G is a nontrivial connected graph, then $\overline{\text{mc}}(G)$ exists. Furthermore, if G contains no true twins, then $\overline{\text{mc}}(G) \geq \chi(G)$.*

To illustrate these concepts, consider the bipartite graph G of Figure 1. Since $\chi(G) = 2$ and G has no true twins, it follows that $\overline{\text{mc}}(G) \geq 2$ by Proposition 2.1. We show that $\overline{\text{mc}}(G) = 3$. Figure 1 shows a closed modular 3-coloring of G (where the color of a vertex is placed within the vertex) together with the color $c'(v)$ for each vertex v of G (where the color $c'(v)$ of a vertex is placed next to the vertex). Thus $\overline{\text{mc}}(G) \leq 3$. Assume, to the contrary, that there exists a closed modular 2-coloring c of G. Since $U = \{u_1, u_2, u_3, u_4\}$ and $V = \{v_1, v_2\}$ are the partite sets of G and the induced coloring c' is a proper 2-coloring of G, the color classes of c' are U and V.

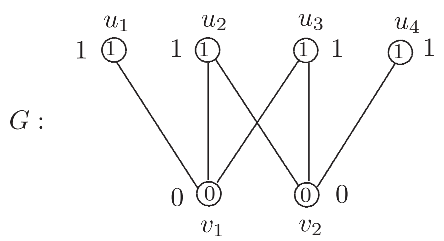

Figure 1. A graph G with $\chi(G) = 2$ and $\overline{\text{mc}}(G) = 3$.

Assume first that $c'(x) = 0$ for all $x \in U$ and $c'(x) = 1$ for all $x \in V$. Since $c'(u_1) = 0$, either $c(u_1) = c(v_1) = 0$ or $c(u_1) = c(v_1) = 1$. In either case, it follows because $c'(v_1) = 1$ that $\{c(u_2), c(u_3)\} = \{0, 1\}$ and so $c(u_2) \neq c(u_3)$. However then, since $c(u_2) \neq c(u_3)$ and $N(u_2) = N(u_3)$, we have $c'(u_2) \neq c'(u_3)$, a contradiction. Consequently, $c'(x) = 1$ for all $x \in U$ and $c'(x) = 0$ for all $x \in V$. Since $c'(u_1) = 1$, it follows that $\{c(u_1), c(v_1)\} = \{0, 1\}$. Because $c'(v_1) = 0$, we have $\{c(u_2), c(u_3)\} = \{0, 1\}$. Again, $c(u_2) \neq c(u_3)$ and so $c'(u_2) \neq c'(u_3)$, once again a contradiction. Therefore, $\overline{\text{mc}}(G) = 3$, as claimed. Observe for the false twins u_2 and u_3 in the graph G of Figure 1 that if $c(u_2) \neq c(u_3)$, then $c'(u_2) \neq c'(u_3)$. This example illustrates the following useful result (see [4]).

Proposition 2.2. *Let c be a closed modular coloring of a connected graph G and let u and v be false twins in G. Then $c(u) = c(v)$ if and only if $c'(u) = c'(v)$.*

By Proposition 2.1, if G is a nontrivial connected graph that contains no true twins, then $\overline{\mathrm{mc}}(G) \geq \chi(G)$. On the other hand, if G contains true twins, then it is possible that $\overline{\mathrm{mc}}(G) < \chi(G)$. In fact, it was shown in [4] that for each pair a, b of positive integers with $a \leq b$ and $b \geq 2$, there is a connected graph G such that $\overline{\mathrm{mc}}(G) = a$ and $\chi(G) = b$.

For an edge uv of a graph G, the graph G/uv obtained from G by *contracting the edge* uv has the vertex set $V(G)$ in which u and v are identified. If we denote the vertex $u = v$ in G/uv by w, then $V(G/uv) = (V(G) \cup \{w\}) - \{u, v\}$ and the edge set of G/uv is

$$\begin{aligned} E(G/uv) &= \{xy : \ xy \in E(G),\ x, y \in V(G) - \{u, v\}\} \\ &\cup \ \{wx : \ ux \in E(G) \text{ or } vx \in E(G), x \in V(G) - \{u, v\}\}. \end{aligned}$$

The graph G/uv is referred to as an *elementary contraction* of G. For a nontrivial connected graph G, define the *true twins closure* $TC(G)$ of G as the graph obtained from G by a sequence of elementary contractions of pairs of true twins in G until no such pair remains. In particular, if G contains no true twins, then $TC(G) = G$. Thus $TC(G)$ is a minor of G. It was shown in [4] that $\overline{\mathrm{mc}}(G) = \overline{\mathrm{mc}}(TC(G))$ for every nontrivial connected graph G. Therefore, it suffices to consider nontrivial connected graphs containing no true twins.

Closed modular chromatic numbers were determined for several classes of regular graphs in [4]. In particular, it was shown that for each integer $k \geq 2$, if G is a regular complete k-partite graph such that each of its partite sets has at least $2k + 1$ vertices, then $\overline{\mathrm{mc}}(G) \leq 2\chi(G) - 1$ and this bound is sharp. In this work, we investigate the closed modular chromatic numbers of trees. We refer to [5] for graph theory notation and terminology not described in this paper.

3. Some Results on Trees

Since a tree T of order at least 3 contains no true twins, it follows by Proposition 2.1 that $\overline{\mathrm{mc}}(T) \geq \chi(T) = 2$. Closed modular chromatic numbers of paths and stars were determined in [4]. We state this result below.

Proposition 3.1. *For each integer $n \geq 3$*

$$\overline{\mathrm{mc}}(P_n) = \begin{cases} 3 & \text{if } n \equiv 2 \pmod 6, \\ 2 & \text{otherwise.} \end{cases} \tag{2}$$

$$\overline{\mathrm{mc}}(K_{1,n-1}) = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 3 & \text{if } n \text{ is even.} \end{cases} \tag{3}$$

A *double star* is a tree whose diameter is 3.

Proposition 3.2. *For integers $a, b \geq 2$, let $S_{a,b}$ be the double star of order $a+b$ whose central vertices have degrees a and b, respectively. Then*

$$\overline{\mathrm{mc}}(S_{a,b}) = \begin{cases} 2 & \text{if at least one of } a \text{ and } b \text{ is even,} \\ 3 & \text{if } a \text{ and } b \text{ are both odd.} \end{cases}$$

Proof. Let $G = S_{a,b}$ be a double star with central vertices u and v such that $\deg u = a$ and $\deg v = b$, let U be the set of end-vertices adjacent to u and let V be the set of end-vertices adjacent to v. We consider two cases.

Case 1. *At least one of a and b is even.* Since $\chi(G) = 2$, it remains only to show that G has a closed modular 2-coloring. If a and b are both even, then define $c : V(G) \to \mathbb{Z}_2$ by $c(x) = 0$ for all $x \in V$ and $c(x) = 1$ for $x \notin V$. Then $c'(x) = 0$ if $x \in U \cup \{v\}$ and $c'(x) = 1$ if $x \in \{u\} \cup V$. If a and b are of opposite parity, say a is odd and b is even, then define $c : V(G) \to \mathbb{Z}_2$ by $c(x) = 0$ if $x \in U$ and $c(x) = 1$ if $x \notin U$. Then $c'(x) = 0$ if $x \in \{u\} \cup V$ and $c'(x) = 1$ if $x \in U \cup \{v\}$. In each case, c is a closed modular 2-coloring of G and so $\overline{\mathrm{mc}}(G) = 2$.

Case 2. *Both a and b are odd.* Let $U = \{u_1, u_2, \ldots, u_{a-1}\}$ and $V = \{v_1, v_2, \ldots, v_{b-1}\}$. Define $c_0 : V(G) \to \mathbb{Z}_3$ by $c_0(x) = 0$ if $x \in (U - \{u_1\}) \cup V$ and $c_0(x) = 1$ if $x \in \{u, u_1, v\}$. Then $c_0'(u) = 0$, $c_0'(u_1) = c_0'(v) = 2$ and $c_0'(x) = 1$ for $x \in (U - \{u_1\}) \cup V$. Since c_0 is a closed modular 3-coloring of G, it follows that $\overline{\mathrm{mc}}(G) \leq 3$. Assume, to the contrary, that $\overline{\mathrm{mc}}(G) = 2$. Let $c : V(G) \to \mathbb{Z}_2$ be a closed modular 2-coloring of G. Then the induced vertex coloring c' is a proper 2-coloring of G. Thus, we may assume, without loss of generality, that $c'(x) = 0$ if $x \in U \cup \{v\}$ and $c'(x) = 1$ if $x \in \{u\} \cup V$. First, assume that $c(u) = 0$. Since $c'(u) = 1$ and $c'(u_i) = 0$ for $1 \leq i \leq a-1$, it follows that $c(u_i) = 0$ $(1 \leq i \leq a-1)$ and $c(v) = 1$. Since $c'(v_j) = 1$ for $1 \leq j \leq b-1$ and $c(v) = 1$, we have $c(v_j) = 0$ for $1 \leq j \leq b-1$. However then, $c'(v) = 1$, a contradiction. Next, assume that $c(u) = 1$. Since $c'(u) = 1$ and $c'(u_i) = 0$ for $1 \leq i \leq a-1$, it follows that $c(u_i) = 1$ $(1 \leq i \leq a-1)$ and $c(v) = 0$. Since $c'(v_j) = 1$ for $1 \leq j \leq b-1$ and $c(v) = 0$, it follows that $c(v_j) = 1$ for $1 \leq j \leq b-1$. However then, $c'(v) = \deg v = 1$ in $\mathbb{Z}_2$, a contradiction. Therefore, $\overline{\mathrm{mc}}(G) = 3$. ∎

By (3) and Proposition 3.2, if T is a tree of diameter 2 or 3 containing only odd vertices, then $\overline{\mathrm{mc}}(T) = 3$. In fact, this is true in general.

Theorem 3.3. *If T is a tree of order at least 4 each of whose vertices is odd, then $\overline{\mathrm{mc}}(T) = 3$.*

Proof. First, we show that $\overline{\mathrm{mc}}(T) \geq 3$. Assume, to the contrary, that this statement is false. Among all trees each of whose vertices is odd and having closed modular chromatic number 2, let T be one of minimum order n. Thus n

is even. By (3) and Proposition 3.2, T is neither a star nor a double star and so $n \geq 8$. Let v be an end-vertex of T with maximum eccentricity and let u be the vertex adjacent to v in T that is not an end-vertex of T. Since T contains no vertex of degree 2, it follows that u is adjacent to at least two end-vertices in T. Suppose that $u_1, u_2, \ldots, u_k$ $(k \geq 2)$ are end-vertices of T that are adjacent to u. Let $c : V(T) \to \mathbb{Z}_2$ be a closed modular 2-coloring of T. Since the induced vertex coloring c' is a proper 2-coloring of T, it follows that either $c'(u_i) = 0$ for $1 \leq i \leq k$ or $c'(u_i) = 1$ for $1 \leq i \leq k$. This implies that either $c(u_i) = 0$ for $1 \leq i \leq k$ or $c(u_i) = 1$ for $1 \leq i \leq k$. However then, the restriction of c to the tree $T' = T - u_1 - u_2$ is a closed modular 2-coloring of T' and so $\overline{\mathrm{mc}}(T') = 2$. Since each vertex of T' has odd degree, this contradicts the defining property of T.

To show that $\overline{\mathrm{mc}}(T) \leq 3$, we proceed by induction on the even order of trees each of whose vertices is odd. By (3) and Proposition 3.2, the result is true for all trees of order 4 and 6. Suppose that if T' is a tree of order n for some even $n \geq 6$ each of whose vertices is odd, then $\overline{\mathrm{mc}}(T') \leq 3$. Let T be a tree of order $n+2$ each of whose vertices is odd. Note that T contains a vertex u that is adjacent to at least two end-vertices of T, say u is adjacent to the end-vertices x and y in T. Then $T_0 = T - x - y$ is a tree of order n each of whose vertices is odd. By the induction hypotheses, $\overline{\mathrm{mc}}(T_0) = 2$ or $\overline{\mathrm{mc}}(T_0) = 3$. We consider these two cases.

Case 1. $\overline{\mathrm{mc}}(T_0) = 2$. Let $c_0 : V(T_0) \to \mathbb{Z}_2$ be a closed modular 2-coloring of T_0. We extend c_0 to a closed modular 2-coloring c of T. There are two subcases, according to whether $c_0'(u) = 0$ or $c_0'(u) = 1$.

Subcase 1.1. $c_0'(u) = 0$. If $c_0(u) = 0$, then define $c(v) = c_0(v)$ for $v \in V(T_0)$ and $c(x) = c(y) = 1$. Thus $c'(v) = c_0'(v)$ for $v \in V(T_0)$ and $c'(x) = c'(y) = 1$. If $c_0(u) = 1$, then define $c(v) = c_0(v)$ for $v \in V(T_0)$ and $c(x) = c(y) = 0$. Thus $c'(v) = c_0'(v)$ for $v \in V(T_0)$ and $c'(x) = c'(y) = 1$.

Subcase 1.2. $c_0'(u) = 1$. If $c_0(u) = 0$, then define $c(v) = c_0(v)$ for $v \in V(T_0)$ and $c(x) = c(y) = 0$. Thus $c'(v) = c_0'(v)$ for $v \in V(T_0)$ and $c'(x) = c'(y) = 0$. If $c_0(u) = 1$, then define $c(v) = c_0(v)$ for $v \in V(T_0)$ and $c(x) = c(y) = 1$. Thus $c'(v) = c_0'(v)$ for $v \in V(T_0)$ and $c'(x) = c'(y) = 0$.

Case 2. $\overline{\mathrm{mc}}(T_0) = 3$. Let $c_0 : V(T_0) \to \mathbb{Z}_3$ be a closed modular 3-coloring of T_0. We extend c_0 to a closed modular 3-coloring c of T. Since $c_0'(u) \in \{0, 1, 2\}$, we consider these subcases. As in Case 1, in each of these subcases, define $c(v) = c_0(v)$ for $v \in V(T_0)$.

Subcase 2.1. $c_0'(u) = 0$. If $c_0(u) = 0$, then define $c(x) = 1$ and $c(y) = 2$. Thus $c'(v) = c_0'(v)$ for for $v \in V(T_0)$ and $c'(x) = 1$ and $c'(y) = 2$. If $c_0(u) = i$ where $i = 1, 2$ then define $c(x) = c(y) = 0$. Thus $c'(v) = c_0'(v)$ for $v \in V(T_0)$ and $c'(x) = c'(y) = i$ for $i = 1, 2$.

Subcase 2.2. $c'_0(u) = 1$. If $c_0(u) = i$ for $i = 0, 2$, then define $c(x) = c(y) = 0$. Thus $c'(v) = c'_0(v)$ for $v \in V(T_0)$ and $c'(x) = c'(y) = i$ for $i = 0, 2$. If $c_0(u) = 1$, then define $c(x) = 1$ and $c(y) = 2$. Thus $c'(v) = c'_0(v)$ for $v \in V(T_0)$ and $c'(x) = 2$ and $c'(y) = 0$.

Subcase 2.3. $c'_0(u) = 2$. If $c_0(u) = i$ for $i = 0, 1$, then define $c(x) = c(y) = 0$. Thus $c'(v) = c'_0(v)$ for $v \in V(T_0)$ and $c'(x) = c'(y) = i$ for $i = 0, 1$. If $c_0(u) = 2$, then define $c(x) = 1$ and $c(y) = 2$. Thus $c'(v) = c'_0(v)$ for $v \in V(T_0)$ and $c'(x) = 0$ and $c'(y) = 1$. ■

By Theorem 3.1, if $n \geq 8$ is even and $n \equiv 2 \pmod 3$, then $\overline{\mathrm{mc}}(P_n) = 3$. Thus the converse of Theorem 3.3 is not true. Furthermore, Theorem 3.3 does not hold for bipartite graphs in general. For example, each vertex is odd in the bipartite graph G of Figure 2 but $\overline{\mathrm{mc}}(G) = 2$. A closed modular 2-coloring of G is also shown in Figure 2.

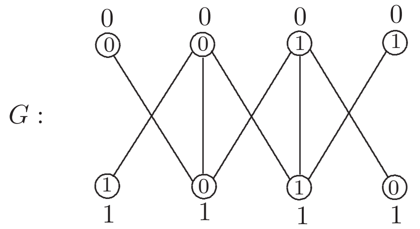

Figure 2. A bipartite graph G with $\overline{\mathrm{mc}}(G) = 2$.

We next consider a well-known class of trees, namely caterpillars. A *caterpillar* is a tree of order 3 or more, the removal of whose end-vertices produces a path called the *spine* of the caterpillar. Thus every path and star (of order at least 3) and every double star is a caterpillar. By (3) and Proposition 3.2, if T is a star or a double star, then $\overline{\mathrm{mc}}(G) \leq 3$. In fact, this is true for all caterpillars.

Theorem 3.4. *If T is a caterpillar of order at least* 3*, then* $\overline{\mathrm{mc}}(T) \leq 3$.

Proof. Let T be a caterpillar of order at least 3 and let $P_k = (v_1, v_2, \ldots, v_k)$ be the spine of T. Since $\overline{\mathrm{mc}}(T) \leq 3$ if T is a star or a double star, we may assume that $k \geq 3$. Define a coloring $c : V(T) \to \mathbb{Z}_3$ by $c(v_i) = 1$ if i is odd and $1 \leq i \leq k$, $c(v_i) = 2$ if i is even and $2 \leq i \leq k$ and $c(x) = 0$ for all end-vertices x of T. Let $s_{c'} = (c'(v_1), c'(v_2), \ldots, c'(v_k))$ be the color sequence of the induced coloring c' on the spine P_k of T. Then

$$\begin{aligned} s_{c'} &= (0,1,2,1,2,\ldots,1,2,1,0) \ \text{ if } k \text{ is odd,} \\ &= (0,1,2,1,2,\ldots,1,2,0) \ \text{ if } k \text{ is even.} \end{aligned}$$

Let x and y be two adjacent vertices of T. If $x, y \in V(P_k)$, then $c'(x) \neq c'(y)$. Thus, we may assume that x is an end-vertex of T and $y = v_i$ for some i with

$1 \le i \le s$. If $i \in \{1, k\}$, then $c'(x) = c(v_i) \neq 0$ and $c'(v_i) = 0$; if $2 \le i \le k-1$ and i is even, then $c'(x) = c(v_i) = 2$ and $c'(y) = 1$; if $3 \le i \le k-1$ and i is odd, then $c'(x) = c(v_i) = 1$ and $c'(y) = 2$. In any case, $c'(x) \neq c'(y)$. Hence c is a closed modular 3-coloring of T and so $\overline{\mathrm{mc}}(G) \le 3$. ■

If T is a caterpillar with an added property, then the exact value of $\overline{\mathrm{mc}}(T)$ can be determined.

Theorem 3.5. *Let T be a caterpillar of order at least 3 where $P_k = (v_1, v_2, \ldots, v_k)$ is the spine of T. For each i with $1 \le i \le k$, let W_i be the set of end-vertices adjacent to v_i in T.*

(a) *If $|W_i|$ is even for $1 \le i \le k$, then*

$$\overline{\mathrm{mc}}(T) = \begin{cases} 3 & \text{if } k \text{ is even and } k \equiv 2 \pmod 3, \\ 2 & \text{otherwise.} \end{cases}$$

(b) *If $|W_i|$ is odd for $1 \le i \le k$, then*

$$\overline{\mathrm{mc}}(T) = \begin{cases} 3 & \text{if } k \equiv 1 \pmod 4, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. We first verify (a) which says that $\overline{\mathrm{mc}}(T) = \overline{\mathrm{mc}}(P_k)$ by Proposition 3.1. Since $\overline{\mathrm{mc}}(P_k) = 2$ or $\overline{\mathrm{mc}}(P_k) = 3$, we consider these two cases.

Case 1. $\overline{\mathrm{mc}}(P_k) = 2$. Let c_{P_k} be a closed modular 2-coloring of P_k. Define a coloring $c : V(T) \to \mathbb{Z}_2$ by $c(v_i) = c_{P_k}(v_i)$ for $1 \le i \le k$, $c(w) = 1$ if $w \in W_i$ $(1 \le i \le k)$ such that $c_{P_k}(v_i) = c'_{P_k}(v_i)$ in $\mathbb{Z}_2$ and $c(w) = 0$ if $w \in W_i$ $(1 \le i \le k)$ such that $c_{P_k}(v_i) \neq c'_{P_k}(v_i)$ in $\mathbb{Z}_2$. We show that c is a closed modular 2-coloring of T. Let x and y be two adjacent vertices of T. Since $c'(v_i) = c'_{P_k}(v_i)$ for $1 \le i \le k$, it follows that $c'(x) \neq c'(y)$ if $x, y \in V(P_k)$. Thus we may assume that x is an end-vertex and $y = v_i$ for some i with $1 \le i \le k$. It follows by the definition of c that (1) if $c(x) = 0$, then $c(x) = c(v_i) = c_{P_k}(v_i) \neq c'_{P_k}(v_i) = c'(v_i)$ in $\mathbb{Z}_2$ and (2) if $c(x) = 1$, then $c(x) = c(v_i) + 1 = c_{P_k}(v_i) + 1 \neq c'_{P_k}(v_i) + 1 \neq c'_{P_k}(v_i) = c'(v_i)$ in $\mathbb{Z}_2$. Thus c is a closed modular 2-coloring of T and so $\overline{\mathrm{mc}}(T) = 2$.

Case 2. $\overline{\mathrm{mc}}(P_k) = 3$. Assume, to the contrary, that $\overline{\mathrm{mc}}(T) = 2$. Let c be a closed modular 2-coloring of T. Since the induced vertex coloring c' is a proper 2-coloring of T, it follows that for each i with $1 \le i \le k$, if $x, y \in W_i$, then $c'(x) = c'(y)$. By Proposition 2.2, $c(x) = c(y)$ for all $x, y \in W_i$ where $1 \le i \le k$. Thus $c'(v_i) = c(v_{i-1}) + c(v_i) + c(v_{i+1}) + \sum_{w \in W_i} c(w)$, where we define $c(v_{i-1}) = 0$ if $i = 1$ and $c(v_{i+1}) = 0$ if $i = k$. Since $|W_i| \equiv 0 \pmod 2$, it follows that $\sum_{w \in W_i} c(w) = 0$ in $\mathbb{Z}_2$ and so $c'(v_i) = c(v_{i-1}) + c(v_i) + c(v_{i+1})$ for $1 \le i \le k$. However then, this implies that if we restrict c to P_k, then we obtain a closed modular 2-coloring of P_k, which is a contradiction. Therefore, $\overline{\mathrm{mc}}(T) = 3$.

Next, we verify (b). First, suppose that $k \equiv 1 \pmod 4$. Since the result is true for stars by (3), we may assume that $k \ge 5$. Assume, to the contrary, that there is a closed modular 2-coloring c of T. Since the induced vertex coloring c'

is a proper 2-coloring, it follows that for each i with $1 \leq i \leq k$, if $x, y \in W_i$, then $c'(x) = c'(y)$. By Proposition 2.2, $c(x) = c(y)$ for all $x, y \in W_i$ where $1 \leq i \leq k$. First, we claim that

$$\begin{aligned} &c(v_1) \neq c(v_1) + c(v_2) = c^*(v_1), \\ &c(v_i) \neq c(v_{i-1}) + c(v_i) + c(v_{i+1}) = c^*(v_i) \text{ for } 2 \leq i \leq k-1, \\ &c(v_k) \neq c(v_{k-1}) + c(v_k) = c^*(v_k). \end{aligned}$$

For otherwise, suppose that there is i with $1 \leq i \leq k$ such that $c(v_i) = c^*(v_i)$. If $c(w) = 0$ for all $w \in W_i$, then $c'(w) = c(v_i) = c^*(v_i) = c'(v_i)$ in $\mathbb{Z}_2$, which is a contradiction. Thus, we may assume that $c(w) = 1$ for all $w \in W_i$. Since $|W_i|$ is odd, $c'(w) = c(v_i) + 1 = c^*(v_i) + 1 = c'(v_i)$ in $\mathbb{Z}_2$, which is a contradiction. Thus, as claimed, $c(v_i) \neq c^*(v_i)$ for $1 \leq i \leq k$.

Let $s = (c(v_1), c(v_2), \ldots, c(v_k))$ be the color sequence of c on the spine P_k of T, where $c(v_i) \in \mathbb{Z}_2$ for $1 \leq i \leq k$. First, suppose that $c(v_1) = 0$. Then $c^*(v_1) = 1$. This implies that $c(v_2) = 1$ and $c^*(v_2) = 0$. Since $c(v_1) = 0$, $c(v_2) = 1$ and $c^*(v_2) = 0$, it follows that $c(v_3) = 1$ and $c^*(v_3) = 0$. Continuing in this manner, we obtain that

$$s = (0, 1, 1,\ \underline{0, 0, 1, 1},\ \underline{0, 0, 1, 1}, \ldots, \underline{0, 0, 1, 1}).$$

However, this implies that $k \equiv 3 \pmod 4$, a contradiction. Next, suppose that $c(v_1) = 1$. Then $c^*(v_1) = 0$. This implies that $c(v_2) = 1$ and $c^*(v_2) = 0$. Then $c(v_3) = 0$ and $c^*(v_3) = 1$, which implies that $c(v_4) = 0$ and $c^*(v_4) = 1$. Continuing in this manner, we obtain that s must be one of the following two sequences

$$\begin{aligned} s_1 &= (\underline{1, 1, 0, 0},\ \underline{1, 1, 0, 0}, \ldots, \underline{1, 1, 0, 0},\ 1, 1), \\ s_2 &= (\underline{1, 1, 0, 0},\ \underline{1, 1, 0, 0}, \ldots, \underline{1, 1, 0, 0},\ 1, 1, 0). \end{aligned}$$

However then, this implies that $k \equiv 2 \pmod 4$ or $k \equiv 3 \pmod 4$, a contradiction. Therefore, $\overline{\text{mc}}(T) \neq 2$ and so $\overline{\text{mc}}(T) = 3$ by Theorem 3.4.

Next, suppose that $k \not\equiv 1 \pmod 4$. We show $\overline{\text{mc}}(T) = 2$ by providing a closed modular 2-coloring of T. First, suppose that $k \equiv 0 \pmod 4$. Define the coloring $c : V(T) \to \mathbb{Z}_2$ by $c(v_i) = 0$ if $i \equiv 0, 1 \pmod 4$, $c(v_i) = 1$ if $i \equiv 2, 3 \pmod 4$, $c(w) = 0$ if $w \in W_i$ and $i \equiv 0, 3 \pmod 4$ and $c(w) = 1$ if $w \in W_i$ and $i \equiv 1, 2 \pmod 4$. The color sequence $s_{c'} = (c'(v_1), c'(v_2), \ldots, c'(v_k))$ of the induced coloring c' on the spine P_k of T is $s_{c'} = (0, 1, 0, 1, \ldots, 0, 1)$. Furthermore, $c'(w) = 1$ if $w \in W_i$ and i is odd and $c'(w) = 0$ if $w \in W_i$ and i is even. Thus $c'(x) \neq c'(y)$ for every pair x, y of adjacent vertices of T. Hence c is a closed modular 2-coloring of T. Next, suppose that $k \equiv 2 \pmod 4$ or $k \equiv 3 \pmod 4$. Define the coloring $c : V(T) \to \mathbb{Z}_2$ by $c(v_i) = 0$ if $i \equiv 0, 3 \pmod 4$, $c(v_i) = 1$ if $i \equiv 1, 2 \pmod 4$, $c(w) = 0$ if $w \in W_i$ and $i \equiv 2, 3 \pmod 4$ and $c(w) = 1$ if $w \in W_i$ and $i \equiv 0, 1 \pmod 4$. For $k \equiv 2 \pmod 4$, the color sequence of the induced coloring c' on the spine P_k of T is $s_{c'} = (1, 0, 1, 0, \ldots, 1, 0)$; while for $k \equiv 3 \pmod 4$, the color sequence of the induced coloring c' on the spine P_k of T is $s_{c'} = (1, 0, 1, 0, \ldots, 1, 0, 1)$. In each case, $c'(w) = 0$ if $w \in W_i$ and i is odd and $c'(w) = 1$ if $w \in W_i$ and i is even. Thus $c'(x) \neq c'(y)$ for every pair x, y of adjacent vertices of T. Hence c is a closed modular 2-coloring of T. Therefore, $\overline{\mathrm{mc}}(T) = 2$ if $k \not\equiv 1 (\mathrm{mod}\ 4)$. ■

4. A Four Color Theorem

While the closed modular chromatic number of every tree considered thus far is either 2 or 3, no upper bound for $\overline{\mathrm{mc}}(T)$ has been determined for trees of order at least 3 in general. We now show that $\overline{\mathrm{mc}}(T) \leq 4$ for every such tree. In fact, we show that every tree of order at least 3 has a special type of closed modular 4-coloring. A closed modular k-coloring $c : V(T) \to \mathbb{Z}_k$ of a tree T of order 3 or more is a *nowhere-zero* coloring if $c(x) \neq 0$ for each vertex x of T.

Lemma 4.1. *Every star of order at least* 3 *has a nowhere-zero closed modular* 4*-coloring.*

Proof. Let $T = K_{1,k}$ be a star with $V(T) = \{v, v_1, v_2, \ldots, v_k\}$ where v is the central vertex of T and $k \geq 2$. First assume that $k \equiv 0, 2, 3\ (\mathrm{mod}\ 4)$. Define the coloring $c : V(G) \to \mathbb{Z}_4$ by $c(x) = 1$ for each $x \in V(T)$. Then $c'(v_i) = 2$ for $1 \leq i \leq k$. If $k \equiv 0\ (\mathrm{mod}\ 4)$, then $c'(v) = 1$; if $k \equiv 2\ (\mathrm{mod}\ 4)$, then $c'(v) = 3$; and if $k \equiv 3\ (\mathrm{mod}\ 4)$, then $c'(v) = 0$. Next assume that $k \equiv 1\ (\mathrm{mod}\ 4)$. Define the coloring $c : V(G) \to \mathbb{Z}_4$ by $c(v_1) = c(v_2) = 2$ and $c(x) = 1$ for each $x \in V(T) - \{v_1, v_2\}$. Then $c'(v) = 0$ and $c'(v_1) = c'(v_2) = 3$ and $c'(v_i) = 2$ for $3 \leq i \leq k$. In each case, c is a nowhere-zero closed modular 4-coloring. ■

Theorem 4.2. *Every tree of order at least* 3 *has a nowhere-zero closed modular* 4*-coloring.*

Proof. We proceed by strong induction on the order of a tree. By Lemma 4.1, the base step of induction holds. Assume for some integer $n \geq 4$ that every tree of order at least 3 and at most n has a nowhere-zero closed modular 4-coloring. Let T be a tree of order $n+1$. We show that T has a nowhere-zero closed modular 4-coloring. By Lemma 4.1, we may assume that T is not a star.

Let z be a peripheral vertex of T and so z is an end-vertex of T. Suppose that z is adjacent to the vertex u in T. Hence each vertex adjacent to u is an end-vertex of T with exactly one exception. Let $V = \{z = v_1, v_2, \ldots, v_k\}$ be the set of end-vertices of T that are adjacent to u. Then $T^* = T - V$ is a tree of order at least 3 and u is an end-vertex of T^*. By the induction hypothesis, T^* has a nowhere-zero closed modular 4-coloring $c : V(T^*) \to \mathbb{Z}_4$. Next, we show that the coloring c can be extended to a nowhere-zero closed modular 4-coloring $c_T : V(T) \to \mathbb{Z}_4$ of T; that is, $c_T(x) = c(x)$ for each $x \in V(T^*)$ and so $c'_T(x) = c'(x)$ for each $x \in V(T^*) - \{u\}$. Suppose that u is adjacent to the vertex w in T^*. Since c is a nowhere-zero closed modular coloring, it follows that $c(u), c(w) \in \{1, 2, 3\}$. We consider three cases, according to the values of $c(u)$.

Case 1. $c(u) = 1$. If $c(w) = 1$, then $c'(u) = 2$; if $c(w) = 2$, then $c'(u) = 3$; and if $c(w) = 3$, then $c'(u) = 0$. Hence there are three subcases.

Subcase 1.1. $c'(u) = 2$. Then $c'(w) \in \{0, 1, 3\}$. We consider these three subcases.

Subcase 1.1.1. $c'(w) = 0$. Define the coloring c_T on V as follows. If $k \equiv 1, 3 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Hence $c'_T(v) = 2$ for each $v \in V$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 1 \pmod 4$, then $c'_T(u) = 3$, while if $k \equiv 3 \pmod 4$, then $c'_T(u) = 1$. If $k \equiv 2, 0 \pmod 4$, then c_T assigns the color 2 to each vertex in V. Hence $c'_T(v) = 3$ for each $v \in V$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*)$.

Subcase 1.1.2. $c'(w) = 1$. Define the coloring c_T on V as follows. If $k \not\equiv 3 \pmod 4$, then define c_T as in Subcase 1.1.1 (since $c'_T(u) \neq 1$ in this coloring). If $k \equiv 3 \pmod 4$, then define the coloring c_T on V by assigning 2 to each vertex in V. Hence $c'_T(v) = 3$ for each $v \in V$, $c'_T(u) = 0$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$.

Subcase 1.1.3. $c'(w) = 3$. Define the coloring c_T on V as follows. If $k \equiv 1 \pmod 4$, then c_T assigns the color 2 to each vertex in V. Hence $c'_T(v) = 3$, $c'_T(u) = 0$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \not\equiv 1 \pmod 4$, then define c_T as in Subcase 1.1.2 (since $c'_T(u) \neq 3$ in this coloring).

Subcase 1.2. $c'(u) = 0$. Then $c'(w) \in \{1, 2, 3\}$. We consider these three subcases.

Subcase 1.2.1. $c'(w) = 1$. Define the coloring c_T on V as follows. If $k \equiv 1 \pmod 4$, then c_T assigns the color 3 to each vertex in V. Hence $c'_T(v) = 0$ for each $v \in V$, $c'_T(u) = 3$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 2 \pmod 4$, then c_T assigns the color 2 to each vertex in V. Hence $c'_T(v) = 3$ for each $v \in V$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*)$. If $k \equiv 3, 0 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Hence $c'_T(v) = 2$ for each $v \in V$. If $k \equiv 3 \pmod 4$, then $c'_T(u) = 3$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$; while if $k \equiv 0 \pmod 4$, then $c'_T(x) = c'(x)$ for all $x \in V(T^*)$.

Subcase 1.2.2. $c'(w) = 2$. Define the coloring c_T on V as the same in Subcase 1.2.1 (since $c'_T(u) \neq 2$ in this coloring).

Subcase 1.2.3. $c'(w) = 3$. Define the coloring c_T on V as follows. If $k \equiv 1 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Hence $c'_T(v) = 2$ for each $v \in V$, $c'_T(u) = 1$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 3 \pmod 4$, then c_T assigns the color 2 to each vertex in V. Hence $c'_T(v) = 3$ for each $v \in V$, $c'_T(u) = 2$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 0, 2 \pmod 4$, then define c_T as in Subcase 1.2.1 (since $c'(u) \neq 3$ in this coloring).

Subcase 1.3. $c'(u) = 3$. Then $c'(w) \in \{0, 1, 2\}$ and so we consider these three subcases.

Subcase 1.3.1. $c'(w) = 0$. Define the coloring c_T on V as follows. If $k \equiv 1, 3 \pmod 4$, then c_T assigns the color 2 to each vertex in V. Hence $c'_T(v) = 3$ for each $v \in V$, $c'_T(u) = 1$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 2 \pmod 4$, then c_T assigns the color 2 to each vertex in V. Hence $c'_T(v) = 3$ for each $v \in V$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*)$. If $k \equiv 0 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Hence $c'_T(v) = 2$ for each $v \in V$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*)$.

Subcase 1.3.2. $c'(w) = 1$. Define the coloring c_T on V as follows. If $k \equiv 1 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Hence $c'_T(v) = 2$ for each $v \in V$, $c'_T(u) = 0$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 3 \pmod 4$, then c_T assigns the color 1 to v_1 and assigns the color 2 to each vertex in $V - \{v_1\}$. Hence $c'_T(v_1) = 2$, $c'_T(v) = 3$ for each $v \in V - \{v_1\}$, $c'_T(u) = 0$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 0, 2 \pmod 4$, then define c_T as in Subcase 1.2.1 (since $c'_T(u) \neq 1$ in this coloring).

Subcase 1.3.3. $c'(w) = 2$. Define c_T on V as in Subcase 1.2.2 (since $c'_T(u) \neq 2$ in this coloring).

Case 2. $c(u) = 2$. If $c(w) = 1$, then $c'(u) = 3$; if $c(w) = 2$, then $c'(u) = 0$; and if $c(w) = 3$, then $c'(u) = 1$. We consider these three subcases, according to the values of $c'(u)$.

Subcase 2.1. $c'(u) = 0$. Then $c'(w) \in \{1, 2, 3\}$. There are three subcases.

Subcase 2.1.1. $c'(w) = 1$. Define the coloring c_T on V as follows. If $k \equiv 1 \pmod 4$, then c_T assigns the color 3 to each vertex in V. Hence $c'_T(v) = 1$ for each $v \in V$, $c'_T(u) = 3$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 2 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Hence $c'_T(v) = 3$ for each $v \in V$, $c'_T(u) = 2$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 3 \pmod 4$, then c_T assigns the color 2 to each vertex in V. Hence $c'_T(v) = 0$ for each $v \in V$, $c'_T(u) = 2$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 0 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Hence $c'_T(v) = 3$ for each $v \in V$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*)$.

Subcase 2.1.2. $c'(w) = 2$. Define the coloring c_T on V as follows. If $k \equiv 1, 0 \pmod 4$, then define c_T as in Subcase 2.1.1 (since $c'_T \neq 2$ in this coloring). If $k \equiv 2 \pmod 4$, then c_T assigns the color 1 to v_1 and 3 each vertex in $V - \{v_1\}$. Hence $c'_T(v_1) = 3$, $c'_T(v) = 1$ for each $v \in V - \{v_1\}$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*)$. If $k \equiv 3 \pmod 4$, then c_T assigns the color 1 to v_1 and the color 2 to each vertex in $V - \{v_1\}$. Hence $c'_T(v_1) = 3$, $c'_T(v) = 0$ for each $v \in V - \{v_1\}$, $c'_T(u) = 1$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$.

Subcase 2.1.3. $c'(w) = 3$. Define the coloring c_T on V as follows. If $k \equiv 1 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Then $c'_T(v) = 3$ for each

$v \in V$, $c'_T(u) = 1$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \not\equiv 1 \pmod 4$, then define c_T as in Subcase 2.1.2 (since $c'_T(u) \neq 3$ in this coloring).

Subcase 2.2. $c'(u) = 1$. Then $c'(w) \in \{0, 2, 3\}$. There are three subcases.

Subcase 2.2.1. $c'(w) = 0$. Define the coloring c_T on V as follows. If $k \equiv 1$ (mod 4), then c_T assigns the color 2 to each vertex in V. Hence $c'_T(v) = 0$ for each $v \in V$, $c'_T(u) = 3$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 2 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Hence $c'_T(v) = 3$ for each $v \in V$, $c'_T(u) = 3$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 3 \pmod 4$, then c_T assigns the color 2 to each vertex in V. Hence $c'_T(v) = 0$ for each $v \in V$, $c'_T(u) = 3$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 0 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Hence $c'_T(v) = 3$ for each $v \in V$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*)$.

Subcase 2.2.2. $c'(w) = 2$. Define the coloring c_T on V as in Subcase 2.2.1 (since $c'_T(u) \neq 2$ in this coloring).

Subcase 2.2.3. $c'(w) = 3$. Define the coloring c_T on V as follows. If $k \equiv 1 \pmod 4$, then c_T assigns the color 3 to each vertex in V. Hence $c'_T(v) = 1$ for each $v \in V$, $c'_T(u) = 0$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 2 \pmod 4$, then c_T assigns the color 2 to each vertex in V. Hence $c'_T(v) = 0$ for each $v \in V$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*)$. If $k \equiv 3 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Hence $c'_T(v) = 3$ for each $v \in V$, $c'_T(u) = 0$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 0 \pmod 4$, then c_T assigns the color 1 to each vertex in V as in Subcase 2.2.1 (since $c'_T(u) \neq 3$ in this coloring).

Subcase 2.3. $c'(u) = 3$. Then $c'(w) \in \{0, 1, 2\}$. There are three subcases.

Subcase 2.3.1. $c'(w) = 0$. Define the coloring c_T assigns the color 2 to each vertex in V. Then $c'_T(v) = 0$ for each $v \in V$. If $k \equiv 1, 3 \pmod 4$, $c'_T(u) = 1$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 0, 2 \pmod 4$, then $c'_T(x) = c'(x)$ for all $x \in V(T^*)$.

Subcase 2.3.2. $c'(w) = 1$. Define the coloring c_T as follows. If $k \equiv 1 \pmod 4$, the coloring c_T assigns the color 3 to each vertex in V. Hence $c'_T(v) = 1$ for each $v \in V$, $c'_T(u) = 2$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 3 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Hence $c'_T(v) = 3$ for each $v \in V$, $c'_T(u) = 2$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 0, 2 \pmod 4$, then define c_T as Subcase 2.3.1 (since $c'_T(u) \neq 1$ in this coloring).

Subcase 2.3.3. $c'(w) = 2$. Define the coloring c_T as in Subcase 2.3.1 (since $c'_T(u) \neq 2$ in this coloring).

Case 3. $c(u) = 3$. If $c(w) = 1$, then $c'(u) = 0$; if $c(w) = 2$, then $c'(u) = 1$ and if $c(w) = 3$, then $c'(u) = 2$. We consider these three subcases, according to the values of $c'(u)$.

Subcase 3.1. $c'(u) = 0$. Then $c'(w) \in \{1, 2, 3\}$. There are three subcases.

Subcase 3.1.1. $c'(w) = 1$. Define the coloring c_T on V as follows. If $k \equiv 1 \pmod 4$, then c_T assigns the color 3 to each vertex in V. Hence $c'_T(v) = 2$ for each $v \in V$, $c'_T(u) = 3$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \not\equiv 1 \pmod 4$, then c_T assigns the color 2 to each vertex in V. Hence $c'_T(v) = 1$ for each $v \in V$. If $k \equiv 2, 0 \pmod 4$, then $c'_T(x) = c'(x)$ for all $x \in V(T^*)$. If $k \equiv 3 \pmod 4$, then $c'_T(u) = 2$ and $c'_T(x) = c(x)$ for all $x \in V(T^*) - \{u\}$.

Subcase 3.1.2. $c'(w) = 2$. Define the coloring c_T on V as follows. If $k \not\equiv 3 \pmod 4$, then define c_T as in Subcase 3.1.1 (since $c'_T(u) \neq 2$ in this coloring). If $k \equiv 3 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Then $c'_T(v) = 0$ for each $v \in V$, $c'_T(u) = 3$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$.

Subcase 3.1.3. $c'(w) = 3$. Define the coloring c_T on V as follows. If $k \equiv 1 \pmod 4$, then c_T assigns the color 1 to each vertex in V_1. Hence $c'_T(v) = 0$ for each $v \in V$, $c'_T(u) = 1$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \not\equiv 1 \pmod 4$, then define c_T as in Subcase 3.1.1 (since $c'_T(u) \neq 3$ in this coloring).

Subcase 3.2. $c'(u) = 1$. Then $c'(w) \in \{0, 2, 3\}$. There are three subcases.

Subcase 3.2.1. $c'(w) = 0$. Define the coloring c_T on V as follows. If $k \equiv 1 \pmod 4$, then c_T assigns the color 2 to each vertex in V. Hence $c'_T(v) = 1$ for each $v \in V$, $c'_T(u) = 3$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 2 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Hence $c'_T(v) = 0$ for each $v \in V$, $c'_T(u) = 3$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 3 \pmod 4$, then c_T assigns the color 2 to each vertex in V. Hence $c'_T(v) = 1$ for each $v \in V$, $c'_T(u) = 3$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 0 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Hence $c'_T(v) = 0$ for each $v \in V$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*)$.

Subcase 3.2.2. $c'(w) = 2$. Define the coloring c_T on V as in Subcase 3.2.1 (since $c'_T(u) \neq 2$ in this coloring).

Subcase 3.2.3. $c'(w) = 3$. Define the coloring c_T on V as follows. If $k \equiv 1 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Hence $c'_T(v) = 0$ for each $v \in V$, $c'_T(u) = 2$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 2 \pmod 4$, then c_T assigns the color 1 to v_1 and assigns the color 3 each vertex in $V - \{v_1\}$. Hence $c'_T(v_1) = 0$, $c'_T(v) = 2$ for each $v \in V - \{v_1\}$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*)$. If $k \equiv 3 \pmod 4$, then c_T assigns the color 2 to v_1 and v_2 and assigns the color 3 to each vertex in $V - \{v_1, v_2\}$. Hence $c'_T(v_1) = c'_T(v_2) = 1$, $c'_T(v) = 2$ for each $v \in V - \{v_1, v_2\}$, $c'_T(u) = 0$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 0 \pmod 4$, then define the coloring c_T on V as in Subcase 3.2.1 (since $c'_T(u) \neq 3$ in this coloring).

Subcase 3.3. $c'(u) = 2$. Then $c'(w) \in \{0, 1, 3\}$. There are three subcases.

Subcase 3.3.1. $c'(w) = 0$. Define the coloring c_T on V as follows. If $k \equiv 1 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Hence $c'_T(v) = 0$ for each $v \in V$, $c'_T(u) = 3$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \equiv 2, 0$

(mod 4), then c_T assigns the color 2 to each vertex in V. Hence $c'_T(v) = 1$ for each $v \in V$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*)$. If $k \equiv 3 \pmod 4$, then c_T assigns the color 1 to each vertex in V. Hence $c'_T(v) = 0$ for each $v \in V$, $c'_T(u) = 1$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$.

Subcase 3.3.2. $c'(w) = 1$. Define the coloring c_T on V as follows. If $k \not\equiv 3 \pmod 4$, then define c_T as in Subcase 3.3.1 (since $c'_T(u) \neq 1$ in this coloring). If $k \equiv 3 \pmod 4$, then c_T assigns the color 3 to each vertex in V. Hence $c'_T(v) = 2$ for each $v \in V$, $c'_T(u) = 3$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$.

Subcase 3.3.3. $c'(w) = 3$. Define the coloring c_T on V as follows. If $k \equiv 1 \pmod 4$, then c_T assigns the color 2 to each vertex in V. Hence $c'_T(v) = 1$ for each $v \in V$, $c'_T(u) = 1$ and $c'_T(x) = c'(x)$ for all $x \in V(T^*) - \{u\}$. If $k \not\equiv 1 \pmod 4$, then define c_T as in Subcase 3.3.1 (since $c'_T(u) \neq 3$ in this coloring).

In each case, c is a nowhere-zero closed modular 4-coloring. ■

Theorem 4.2 cannot be improved as there is an infinite class of trees that do not have a nowhere-zero closed modular 3-coloring. In order to show this, we first present a lemma.

Lemma 4.3. *Suppose that T is a tree that contains a vertex u of degree 3 such that u is adjacent to two end-vertices u_1 and u_2 and one non-end-vertex w. If c is a nowhere-zero closed modular 3-coloring of T, then $c(u_1) = c(u_2)$ and $c'(w) \neq c(u)$.*

Proof. Assume, to the contrary, that there is a nowhere-zero closed modular 3-coloring $c : V(G) \to \mathbb{Z}_3 - \{0\}$ such that either $c(u_1) \neq c(u_2)$ or $c'(w) = c(u)$. First, suppose that $c(u_1) \neq c(u_2)$, where say $c(u_1) = 1$ and $c(u_2) = 2$. Then $c'(u_i) = c(u) + i$ for $i = 1, 2$ and $c'(u) = c(u) + c(w)$. Since $c(w) \in \{1, 2\}$, it follows that either $c'(u) = c'(u_1)$ or $c'(u) = c'(u_2)$, which is impossible. Thus, $c(u_1) = c(u_2)$. Next, suppose that $c'(w) = c(u)$. Since $c(u_1) = c(u_2) \in \{1, 2\}$, there are two cases.

Case 1. $c(u_i) = 1$ *for* $i = 1, 2$. Hence $c'(u_i) = 1 + c(u)$ for $i = 1, 2$ and $c'(u) = 2 + c(u) + c(w)$ in $\mathbb{Z}_3$. If $c(w) = 1$, then $c'(u) = c(u) = c'(w)$, which is impossible; while if $c(w) = 2$, then $c'(u) = c(u) + 1 = c'(u_i)$ for $i = 1, 2$, which again is impossible.

Case 2. $c(u_i) = 2$ *for* $i = 1, 2$. Hence $c'(u_i) = 2 + c(u)$ for $i = 1, 2$ and $c'(u) = 1 + c(u) + c(w)$ in $\mathbb{Z}_3$. If $c(w) = 1$, then $c'(u) = c(u) + 2 = c'(u_i)$ for $i = 1, 2$, which is impossible; while if $c(w) = 2$, then $c'(u) = c(u) = c'(w)$, which again is impossible. ■

Theorem 4.4. *There is an infinite class of trees that do not have a nowhere-zero closed modular 3-coloring.*

Proof. For each integer $k \geq 2$ with $k \equiv 2 \pmod 3$, we construct a tree T_k that does not have a nowhere-zero closed modular 3-coloring. We begin with the star $K_{1,k+1}$ with the central vertex v. Then the graph T_k is obtained by (1) subdividing exactly k edges of $K_{1,k+1}$ exactly once and (2) adding two pendant edges at each end-vertex of $K_{1,k+1}$. Suppose that $N(v) = \{u, w_1, w_2, \ldots, w_k\}$ where $\deg u = 3$, $\deg w_i = 2$ for $1 \leq i \leq k$, u is adjacent to two end-vertices u_1 and u_2 and k non-end-vertices $w_1, w_2, \ldots, w_2$, each vertex w_i is adjacent to v and x_i and each vertex x_i is adjacent to exactly two end-vertices for $1 \leq i \leq k$. Assume, to the contrary, that for some $k \geq 2$ with $k \equiv 2 \pmod 3$ the tree T_k has a nowhere-zero closed modular 3-coloring $c : V(T_k) \to \mathbb{Z}_3 - \{0\}$. For each i with $1 \leq i \leq k$, since $\deg x_i = 3$ and x_i is adjacent to two end-vertices and one non-end-vertex w_i, it follows by Lemma 4.3 that

$$c'(w_i) \neq c(x_i) \text{ for } 1 \leq i \leq k. \tag{4}$$

Furthermore, since $\deg u = 3$ and u is adjacent to two end-vertices u_1 and u_2 and one non-end-vertex v, it follows by Lemma 4.3 that $c(u_1) = c(u_2) \in \{1, 2\}$. We consider two cases.

Case 1. $c(u_1) = c(u_2) = 1$. Since $c(u) \in \{1, 2\}$, there are two possibilities. If $c(u) = 1$, then $c'(u_i) = 2$ for $i = 1, 2$. This forces $c(v) = 1$ and so $c'(u) = 1$. First, suppose that $c(w_i) = 2$ for some i with $1 \leq i \leq k$, then $c'(w_i) = c(v) + c(w_i) + c(x_i) = 1 + 2 + c(x_i) = c(x_i)$ in $\mathbb{Z}_3$, which contradicts (4). Thus $c(w_i) = 1$ for all i with $1 \leq i \leq k$. Since $k \equiv 2 \pmod 3$, it follows that $c'(v) = 1 = c'(u)$, which is a contradiction. If $c(u) = 2$, then $c'(u_i) = 0$ for $i = 1, 2$. This forces $c(v) = 1$ and so $c'(u) = 2$. By (4), an argument similar to the one in the case when $c(u) = 1$ shows that $c(w_i) = 1$ for all i with $1 \leq i \leq k$. Since $k \equiv 2 \pmod 3$, it follows that $c'(v) = 2 = c'(u)$, which is a contradiction.

Case 2. $c(u_1) = c(u_2) = 2$. Since $c(u) \in \{1, 2\}$, there are two possibilities. If $c(u) = 1$, then $c'(u_i) = 0$ for $i = 1, 2$. This forces $c(v) = 2$ and so $c'(u) = 1$. First, suppose that $c(w_i) = 1$ for some i with $1 \leq i \leq k$, then $c'(w_i) = c(v) + c(w_i) + c(x_i) = 1 + 2 + c(x_i) = c(x_i)$ in $\mathbb{Z}_3$, which contradicts (4). Thus $c(w_i) = 2$ for all i with $1 \leq i \leq k$. Since $k \equiv 2 \pmod 3$, it follows that $c'(v) = 1 = c'(u)$, which is a contradiction. If $c(u) = 2$, then $c'(u_i) = 1$ for $i = 1, 2$. This forces $c(v) = 2$ and so $c'(u) = 2$. By (4), an argument similar to the one used in the case when $(c(u) = 1$ shows that $c(w_i) = 2$ for all i with $1 \leq i \leq k$. Since $k \equiv 2 \pmod 3$, it follows that $c'(v) = 2 = c'(u)$, which is a contradiction. ■

We know of no trees T for which $\overline{\mathrm{mc}}(T) = 4$. Therefore, we close this section with the following conjecture.

Conjecture. *For every tree T of order at least* 3, $\overline{\mathrm{mc}}(T) \leq 3$.

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Received 11 January 2012
Accepted 11 June 2012