Discussiones Mathematicae
Graph Theory 33 (2013) 147–165
doi:10.7151/dmgt.1674

Dedicated to Mietek Borowiecki on the occasion of his seventieth birthday.

ON MAXIMUM WEIGHT OF A BIPARTITE GRAPH OF GIVEN ORDER AND SIZE [1]

Mirko Horňák

Stanislav Jendrol'

Institute of Mathematics
P.J. Šafárik University
Jesenná 5, 04001 Košice, Slovakia

e-mail: mirko.hornak@upjs.sk
stanislav.jendrol@upjs.sk

and

Ingo Schiermeyer

Institut für Diskrete Mathematik und Algebra
Technische Universität Bergakademie Freiberg
09596 Freiberg, Germany

e-mail: ingo.schiermeyer@tu-freiberg.de

Abstract

The weight of an edge xy of a graph is defined to be the sum of degrees of the vertices x and y. The weight of a graph G is the minimum of weights of edges of G. More than twenty years ago Erdős was interested in finding the maximum weight of a graph with n vertices and m edges. This paper presents a complete solution of a modification of the above problem in which a graph is required to be bipartite. It is shown that there is a function $w^*(n, m)$ such that the optimum weight is either $w^*(n, m)$ or $w^*(n, m) + 1$.

Keywords: weight of an edge, weight of a graph, bipartite graph.

2010 Mathematics Subject Classification: 05C35.

[1]This research was supported by the DAAD cooperation contract Freiberg - Košice and by the Agency of the Ministry of Education, Science, Research and Sport of the Slovak Republic for the Structural Funds of EU under the project ITMS 26110230056. The work of the first two authors was supported by the Slovak Research and Development Agency under the contract no. APVV-0023-10 and by the Slovak grant VEGA 1/0652/12.

Let G be a finite simple nonoriented graph. The *weight* $w_G(e)$ of an edge $e = xy \in E(G)$ is defined to be $\deg_G(x) + \deg_G(y)$. The concept of the weight of an edge was introduced by Kotzig [10] who proved that every planar 3-connected graph contains an edge of the weight not exceeding 13.

The mentioned result was further developed in various directions. Grünbaum [4], Jucovič [7], Borodin [1], Fabrici and Jendrol' [3] studied inequalities for the number of edges having weight at most 13 in planar 3-connected graphs. Ivančo [5] found an analogue of Kotzig's result for graphs with minimum degree at least 3 and embedded on orientable 2-manifolds. Another analogue of Kotzig's result, this time for triangulations of orientable 2-manifolds, can be found in Zaks [11]. The case of graphs embedded on nonorientable 2-manifolds was investigated by Jendrol' et al. [9].

In [3] it is proved that each 3-connected planar graph of maximum degree at least k contains a path on k vertices such that each of its vertices has degree at most $5k$; moreover, the bound $5k$ is the best possible. Enomoto and Ota [2] proved that each planar 3-connected graph of order at least k contains a connected subgraph on k vertices such that the degree sum of the vertices of this subgraph is at most $8k - 1$.

Let $p, q \in \mathbb{Z}$. Throughout the paper we shall use the notation

$$[p, q] := \{z \in \mathbb{Z} : p \leq z \leq q\},$$
$$[p, \infty) := \{z \in \mathbb{Z} : p \leq z\}$$

(for *integer* intervals).

Let the *weight* of a graph G, in symbols $w(G)$, be the minimum of weights of edges of G. At the Fourth Czechoslovak Symposium on Combinatorics held in Prachatice in 1990, Erdős posed the question: What is the maximum weight of an (n, m)-graph (having n vertices and m edges)? If $\mathcal{P}$ is a *graph property*, i.e., a set of (isomorphism classes of) finite simple nonoriented graphs, $n \in [2, \infty)$ and $m \in [1, \binom{n}{2}]$ is such that

$$\mathcal{P}(n, m) := \{G \in \mathcal{P} : |V(G)| = n, |E(G)| = m\} \neq \emptyset,$$

then the above problem can be naturally generalised:

Problem 1. Determine $w(\mathcal{P}, n, m) := \max\{w(G) : G \in \mathcal{P}(n, m)\}$.

Thus, Erdős was interested in finding $w(\mathcal{I}, n, m)$, where $\mathcal{I}$ is the set of all finite simple nonoriented graphs, $n \in [2, \infty)$ and $m \in [1, \binom{n}{2}]$. In [6] Ivančo and Jendrol' obtained some partial results. They observed that the weight of any edge e of a graph $G \in \mathcal{I}(n, m)$ cannot be larger than $m + 1$.

Proposition 2. *If* $n \in [2, \infty)$ *and* $m \in [1, n - 1]$, *then* $w(\mathcal{I}, n, m) = m + 1$ *and the bound is attained by the graph* $K_{1,m} \cup (n - m - 1)K_1$.

The case of very dense graphs is solved by the following theorem of [6].

Theorem 3. *If $n \in [2,\infty)$ and $m = \binom{n}{2} - r$ with $r \in [0, n-2]$, then*

$$w(\mathcal{I},n,m) = \begin{cases} 2n-2, & \text{if } r = 0,\\ 2n-3, & \text{if } r = 1,\\ 2n-4, & \text{if } r \in [2, \lfloor \frac{n}{2} \rfloor] \text{ or } r = 3,\\ 2n-5, & \text{if } r \in [\lfloor \frac{n}{2} \rfloor + 1, \lceil \frac{n+2}{2} \rceil] \text{ or } r = 6,\\ 2n-6, & \text{otherwise.} \end{cases}$$

Graphs that attain the extremal value can be obtained by taking K_n and removing from it r independent edges or edges of a triangle (if $r = 3$) in the cases when $w(\mathcal{I}, n, m) \in [2n-2, 2n-4]$. In the case of $w(\mathcal{I}, n, m) = 2n-5$ take K_n and remove from it either $r-3$ independent edges and edges of an independent triangle or edges of a K_4 (if $r = 6$). Finally, in the case of $w(\mathcal{I}, n, m) = 2n-6$, edges of a cycle of length r are deleted from K_n.

In [6] there was also found a lower bound for $w(\mathcal{I}, n, m)$. The result reads as follows:

Theorem 4. *Let $n \in [2,\infty)$, $m \in [1, \binom{n}{2}]$, $a = \left\lceil \frac{1}{2}(1 + \sqrt{1+8m}) \right\rceil$, $b = \frac{1}{2}(a^2 - a - 2m)$, $h = \left\lceil \frac{1}{2}(2n - 1 - \sqrt{(2n-1)^2 - 8m}) \right\rceil$ and let p, k be integers such that $hk + p = m$, $h + k \le n$ and $h(h-3) < 2p \le h(h-1)$. Let $f(n,m) = h + k + \lfloor \frac{2p}{h} \rfloor$ and let $g(n,m)$ be defined by*

$$g(n,m) = \begin{cases} 2a-2, & \text{if } b = 0,\\ 2a-3, & \text{if } b = 1,\\ 2a-4, & \text{if } 2 \le b \le \lfloor \frac{a}{2} \rfloor \text{ or } b = 3,\\ 2a-5, & \text{if either } \lfloor \frac{a}{2} \rfloor + 1 \le b \le \lceil \frac{a+2}{2} \rceil \text{ or } a = 8 \text{ and } b = 6,\\ 2a-6, & \text{otherwise.} \end{cases}$$

Then $w(\mathcal{I}, n, m) \ge \max\{f(n,m), g(n,m)\}$.

The authors of [6] conjectured that the lower bound of Theorem 4 is in fact equal to $w(\mathcal{I}, n, m)$. The conjecture was proved by Jendrol' and Schiermeyer in [8].

Theorem 5. *If $n \in [2,\infty)$, $m \in [1, \binom{n}{2}]$ and $f(n,m)$, $g(n,m)$ are functions defined in Theorem 4*, then $w(\mathcal{I}, n, m) = max\{f(n,m), g(n,m)\}$.

In this paper we are dealing with the graph property

$$\mathcal{B} := \{G \in \mathcal{I} : G \text{ is bipartite}\}$$

and we solve completely the corresponding "portion" of Problem 1. Namely, we prove that there is $w^*(n,m) \in [2,n]$ such that $w^*(n,m) \le w(\mathcal{B}, n, m) \le$

$w^*(n,m)+1$. Moreover, $w(\mathcal{B},n,m) \le n$ and $w(\mathcal{B},n,m) = w^*(n,m)+1$ implies $w(\mathcal{B},n,m) = n-1$.

It is well known that $\mathcal{B}(n,m) \neq \emptyset$ if and only if $n \in [2,\infty)$ and $m \in [1, \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil]$. Henceforth we shall suppose implicitly that n and m are fixed and $\mathcal{B}(n,m) \neq \emptyset$. Then $1 \le m \le \frac{n^2}{4}$ and $m = \frac{n^2-4k}{4}$ for some $k \in [0, \frac{n^2-4}{4}]$ provided that $n \equiv 0 \pmod 2$, while $n \equiv 1 \pmod 2$ means that $m = \frac{n^2-4k-1}{4}$ for some $k \in [0, \frac{n^2-5}{4}]$.

Let G be a bipartite graph with a bipartition $\{X, Y\}$. An edge $xy \in E(G)$, $x \in X$, $y \in Y$, is *universal in* G provided that $\deg_G(x) = |Y|$ and $\deg_G(y) = |X|$ (or, equivalently, if $N_G(x) = Y$ and $N_G(y) = X$).

Lemma 6. *If* $G \in \mathcal{B}(n,m)$ *and* $e \in E(G)$, *then* $w_G(e) \in [2,n]$. *Moreover,* $w_G(e) = n$ *if and only if* e *is universal in* G.

Proof. Suppose that $\{X, Y\}$ is a bipartition of G and $e = xy$ with $x \in X$ and $y \in Y$. Then $1 \le \deg_G(x) \le |Y|$, $1 \le \deg_G(y) \le |X|$ and $2 \le w_G(e) = \deg_G(x) + \deg_G(y) \le |Y| + |X| = n$. Moreover, $w_G(e) = n$ is equivalent to $\deg_G(x) = |Y|$ and $\deg_G(y) = |X|$. ■

Corollary 7. $w(\mathcal{B},n,m) \in [2,n]$.

Lemma 8. *Suppose that* $n \in [2,\infty)$ *and* $l \in [1, \lfloor \frac{n^2}{4} \rfloor]$. *Then* $\sqrt{n^2-4l}$ *is an integer if and only if there is* $k \in [1, \lfloor \frac{n}{2} \rfloor]$ *such that* $l = k(n-k)$.

Proof. If $\sqrt{n^2-4l}$ is an integer, then $\sqrt{n^2-4l} = n-j$ for some $j \in [1,n]$, $4l = j(2n-j)$, hence j is even, $j = 2k$ with $k \in [1, \lfloor \frac{n}{2} \rfloor]$ and $l = k(n-k)$.

If $l = k(n-k)$, where $k \in [1, \lfloor \frac{n}{2} \rfloor]$, then $n - 2k \ge 0$, $n^2 - 4l = (n-2k)^2$ and $\sqrt{n^2-4l} = n-2k$ is an integer. ■

Proposition 9. $w(\mathcal{B},n,m) = n$ *if and only if* $\sqrt{n^2-4m}$ *is an integer.*

Proof. Suppose that $w(\mathcal{B},n,m) = n = w(G)$ for some $G \in \mathcal{B}(n,m)$ with a bipartition $\{X, Y\}$. By Lemma 6 then each edge of G is universal in G and $E(G)$ consists of all edges joining X to Y. Therefore, $G \cong K_{k,n-k}$, where $k = |X|$, $m = |E(G)| = k(n-k)$ and $k^2 - nk + m = 0$. Thus k, as a root of the quadratic equation $x^2 - nx + m = 0$, is either $\frac{1}{2}(n - \sqrt{n^2-4m})$ or $\frac{1}{2}(n + \sqrt{n^2-4m})$, from which it follows that $\sqrt{n^2-4m}$ is an integer.

If $\sqrt{n^2-4m}$ is an integer, then, by Lemma 8, $m = k(n-k)$ with $k \in [1, \lfloor \frac{n}{2} \rfloor]$, $K_{k,n-k} \in \mathcal{B}(n,m)$ and, since $w(K_{k,n-k}) = n$, using Corollary 7 we obtain $w(\mathcal{B},n,m) = n$. ■

Proposition 10. *The following two statements are equivalent:*

(1) $w(\mathcal{B},n,m) = n-1$.

(2) *The number* $\sqrt{n^2-4m}$ *is not an integer, while* (*exactly*) *one of the numbers* $\sqrt{(n-1)^2-4m}$ *and* $\sqrt{n^2-4m-4}$ *is.*

Proof. (1) $\Rightarrow$ (2): The fact that $\sqrt{n^2-4m}$ is not an integer follows from Proposition 9.

To prove the rest consider a pair (n,m) with $w(\mathcal{B},n,m) = n-1 = w(G)$, where $G \in \mathcal{B}(n,m)$ has a bipartition $\{X,Y\}$. Without loss of generality we may suppose that X does not contain isolated vertices of G. Let $d := \min\{\deg_G(x) : x \in X\}$ and pick $x \in X$ so that $\deg_G(x) = d$. Clearly, $d < |Y|$, because $d = |Y|$ means that G is a complete bipartite graph with $w(G) = n$. On the other hand, $d > |Y|-2$, since $d = |Y|-i$ with $i \geq 2$ yields $w_G(xy) \leq |Y|-i+|X| = n-i < n-1$ for any edge $xy \in E(G)$. Thus, $d = |Y|-1$.

Now let y be the unique vertex of Y with $xy \notin E(G)$. If y is isolated in G, then $G-y \in \mathcal{B}(n-1,m)$ and $w(G-y) = w(G) = n-1$ so that, by Proposition 9, $\sqrt{(n-1)^2-4m}$ is an integer.

If y is not isolated in G, then $\deg_G(y) = |X|-1$, since from $\deg_G(y) = |X|-j$ with $j \geq 2$ we obtain $w_G(x'y) \leq |Y|+|X|-j = n-j < n-1$ for any edge $x'y \in E(G)$. Further, if $x_1y_1 \neq xy$, $x_1 \in X$ and $y_1 \in Y$, then $x_1y_1 \in E(G)$. Indeed, if $x_1y_1 \notin E(G)$, then $y_1 \neq y$ and $w_G(xy_1) \leq |Y|-1+|X|-1 = n-2$. So with $k := |X|$ we have $G = K_{k,n-k} - e$, $m = k(n-k)-1$, $k^2-nk+m+1 = 0$ and $\sqrt{n^2-4m-4}$ is an integer.

(2) $\Rightarrow$ (1): As a consequence of Proposition 9 and Corollary 7 we obtain $w(\mathcal{B},n,m) \leq n-1$.

If $\sqrt{(n-1)^2-4m}$ is an integer, then, by Lemma 8, $m = k(n-1-k)$ for some $k \in [1, \lfloor \frac{n-1}{2} \rfloor]$, hence $K_{k,n-1-k} \cup K_1 \in \mathcal{B}(n,m)$ and $w(\mathcal{B},n,m) \geq w(K_{k,n-1-k} \cup K_1) = n-1$.

If $\sqrt{n^2-4m-4}$ is an integer, then, again by Lemma 8, $m+1 = k(n-k)$, where $k \in [1, \lfloor \frac{n}{2} \rfloor]$, $K_{k,n-k} - e \in \mathcal{B}(n,m)$ and $w(\mathcal{B},n,m) \geq w(K_{k,n-k}-e) = n-1$. ■

If $G \in \mathcal{B}(n,m)$, there are $i_1 \in [1, \lfloor \frac{n}{2} \rfloor]$ and $i_2 \in [i_1, n-i_1]$ such that $G \subseteq K_{i_1,i_2} \cup (n-i_1-i_2)K_1$. In general, the pair (i_1,i_2) is not necessarily unique; it is said to be *standard for G* if it is lexicographically minimal from among all such pairs. Clearly, if (i_1,i_2) is standard for G, then no vertex of G belonging to K_{i_1,i_2} is isolated.

Let us define some numbers that will be important in our analysis:

$$i_{\min} := \left\lceil \frac{n-\sqrt{n^2-4m}}{2} \right\rceil,\ i_{\text{mid}} := \lceil \sqrt{m}\, \rceil,\ i_{\max} := \left\lfloor \frac{n+\sqrt{n^2-4m}}{2} \right\rfloor;$$

it is easily seen that $i_{\min} \leq \lfloor \frac{n}{2} \rfloor$ and $i_{\min} \leq i_{\max}$. Further, for $i \in [1, n-1]$ let

$$a_i := i, \quad b_i := \lceil m/a_i \rceil, \quad s_i := a_i b_i - m, \; p_i := \min\{s_i, 2\}, \; w_i := a_i + b_i - p_i,$$
$$a^* := i_{\min}, \; b^* := \lceil m/a^* \rceil, \; s^* := a^* b^* - m, \; p^* := \min\{s^*, 2\}, \; w^* := a^* + b^* - p^*.$$

Clearly, $w^* = w^*(n, m)$ is an integer depending on n and m.

Proposition 11. *If $G \in \mathcal{B}(n, m)$ and (i_1, i_2) is the standard pair for G, then $i_{\min} \leq i_1 \leq i_2 \leq i_{\max}$.*

Proof. For both $l = 1, 2$, the graph G is a subgraph of the graph $K_{i_l, n-i_l}$. Therefore, $m = |E(G)| \leq i_l(n - i_l)$, $i_l^2 - n i_l + m \leq 0$, and so $i_l \in [\lceil x_1 \rceil, \lfloor x_2 \rfloor] = [i_{\min}, i_{\max}]$, where $x_{1,2} := \frac{n \mp \sqrt{n^2 - 4m}}{2}$ are solutions of the quadratic equation $x^2 - nx + m = 0$. ■

Proposition 12. *For every $i \in [1, n-1]$ the following hold:*

1. *$i + b_i \leq n$ if and only if $i \in [i_{\min}, i_{\max}]$.*
2. *If $i + b_i \leq n$ and $i \leq b_i + 1$, then $w(\mathcal{B}, n, m) \geq w_i$.*

Proof. 1. If $i + b_i \leq n$, then $i + \frac{m}{i} \leq n$, $i^2 - ni + m \leq 0$ and (as in the proof of Proposition 11) $i \in [\lceil x_1 \rceil, \lfloor x_2 \rfloor]$. To show that $i \in [\lceil x_1 \rceil, \lfloor x_2 \rfloor]$ implies $i + b_i \leq n$ we prove an equivalent assertion $i + b_i > n \Rightarrow (i < \lceil x_1 \rceil \vee i > \lfloor x_2 \rfloor)$. For that purpose notice that $i + \frac{m}{i} + 1 > i + b_i \geq n + 1$, $i^2 - ni + m > 0$, and then either $i < \lceil x_1 \rceil$ or $i > \lfloor x_2 \rfloor$, as required.

2. We have $0 \leq s_i = i\lceil \frac{m}{i} \rceil - m \leq i\frac{m+i-1}{i} - m = i - 1$. If $i - 1 \leq b_i$, then the graph $K_{i,b_i} \cup (n - i - b_i)K_1$ has a matching of size s_i, and so $G_i := (K_{i,b_i} - s_i K_2) \cup (n - i - b_i)K_1$ is a bipartite graph of order n and size $ib_i - s_i = m$. If $p_i = 0$, then $s_i = 0$ and all edges of G_i are of weight $i + b_i = w_i$. If $p_i = 1$, then $s_i = 1$ and the weight of G_i is attained on any edge sharing a vertex with the unique non-edge of G_i so that $w(G_i) = i + b_i - 1 = w_i$. Finally, $p_i = 2$ implies $s_i \geq 2$ and the weight of G_i is attained on any edge joining a vertex of a non-edge of G_i to a vertex of another non-edge of G_i, which yields $w(G_i) = i + b_i - 2 = w_i$. Thus $w(\mathcal{B}, n, m) \geq w(G_i) = w_i$. ■

Lemma 13. *The following statements are equivalent:*

(1) $a^* = k$.

(2) $(k-1)(n-k+1) + 1 \leq m \leq k(n-k)$.

(3) $\lceil \frac{m}{k} \rceil + k \leq n \leq \left\lfloor \frac{m+k(k-2)}{k-1} \right\rfloor$.

Proof. The equivalence of (1) and (2) follows from the defining inequalities for $a^* = \left\lceil \frac{n-\sqrt{n^2-4m}}{2} \right\rceil$, i.e., $\frac{n-\sqrt{n^2-4m}}{2} \leq a^* < \frac{n-\sqrt{n^2-4m}}{2} + 1$, and from the fact that m is an integer.

The equivalence of (2) and (3) is an obvious consequence of the fact that n is an integer. (For $k = 1$ the righthand side of (3) can be formally set to ∞ indicating that n is not bounded from above.) ■

Corollary 14. *If $a^* = k$, then $m \geq k^2$.*

Proof. The assumption $a^* = k$ by Lemma 13 means that $\frac{m}{k} + k \leq \lceil \frac{m}{k} \rceil + k \leq n \leq \left\lfloor \frac{m+k(k-2)}{k-1} \right\rfloor \leq \frac{m+k(k-2)}{k-1}$. Standard manipulations applied to the inequality $\frac{m}{k} + k \leq \frac{m+k(k-2)}{k-1}$ yield the desired result. ■

Theorem 15. $w(\mathcal{B}, n, m) = \max\{w_i : i \in [i_{\min}, i_{\mathrm{mid}}]\}$.

Proof. Let us first show that i_{mid} (in the role of i) satisfies the assumptions of Proposition 12.2. We have $i_{\mathrm{mid}} \leq \left\lceil \sqrt{n^2/4} \right\rceil \leq \frac{n+1}{2}$, and so $i_{\mathrm{mid}} = \frac{n+k}{2}$ with $k \in [2-n, 1]$ and $k \equiv n \pmod 2$. From $\left(\frac{n+k-2}{2}\right)^2 < m \leq \left(\frac{n+k}{2}\right)^2$ it follows that $\left\lceil \frac{m}{i_{\mathrm{mid}}} \right\rceil \leq \lceil (\frac{n+k}{2})^2/(\frac{n+k}{2}) \rceil = \frac{n+k}{2}$ and $i_{\mathrm{mid}} + \left\lceil \frac{m}{i_{\mathrm{mid}}} \right\rceil \leq n + k$. If $k \leq 0$, then $i_{\mathrm{mid}} + \left\lceil \frac{m}{i_{\mathrm{mid}}} \right\rceil \leq n + k \leq n$. On the other hand, the assumption $i_{\mathrm{mid}} = \frac{n+1}{2}$ yields $n \equiv 1 \pmod 2$, $m \leq \frac{n^2-1}{4}$, $\left\lceil \frac{m}{i_{\mathrm{mid}}} \right\rceil \leq \left\lceil (\frac{n^2-1}{4})/(\frac{n+1}{2}) \right\rceil = \frac{n-1}{2}$ and $i_{\mathrm{mid}} + \left\lceil \frac{m}{i_{\mathrm{mid}}} \right\rceil \leq n$. Thus, by Proposition 12.1, $i_{\mathrm{mid}} \in [i_{\min}, i_{\max}]$, and $i + b_i \leq n$ for any $i \in [i_{\min}, i_{\mathrm{mid}}]$.

Moreover, $\frac{m}{i_{\mathrm{mid}}} > \left(\frac{n+k-2}{2}\right)^2 / \left(\frac{n+k}{2}\right) > \frac{n+k-4}{2}$, and hence $\left\lceil \frac{m}{i_{\mathrm{mid}}} \right\rceil \geq \frac{n+k-2}{2} = i_{\mathrm{mid}} - 1$. Let us prove by descending induction that $\lceil \frac{m}{i} \rceil \geq i - 1$ for every $i \in [i_{\min}, i_{\mathrm{mid}}]$. The first step has been performed above. So, suppose that $i \in [i_{\min} + 1, i_{\mathrm{mid}}]$ and $\lceil \frac{m}{i} \rceil \geq i - 1$. If the inequality $\left\lceil \frac{m}{i-1} \right\rceil \geq i - 2$ is not true, then $\frac{m}{i-1} \leq i - 3$, $m \leq (i-1)(i-3) < (i-2)^2$, $i > \sqrt{m} + 2$ and $i \geq \lceil \sqrt{m} \rceil + 2 > i_{\mathrm{mid}}$, a contradiction. By Proposition 12.1 we know that $i + b_i \leq n$ for any $i \in [i_{\min}, i_{\mathrm{mid}}]$. Therefore, with help of Proposition 12.2, we see that $w(\mathcal{B}, n, m) \geq M := \max\{w_i : i \in [i_{\min}, i_{\mathrm{mid}}]\}$.

To prove the inequality $w(\mathcal{B}, n, m) \leq M$ consider an arbitrary graph $G \in \mathcal{B}(n, m)$. Let (i_1, i_2) be the standard pair for G and let U_1, U_2 be partite sets of the graph K_{i_1,i_2} with $E(K_{i_1,i_2}) \supseteq E(G)$ satisfying $|U_l| = i_l$, $l = 1, 2$. Then $m = |E(G)| \leq i_1 i_2$, $i_2 \geq \left\lceil \frac{m}{i_1} \right\rceil$, $i_1 + \left\lceil \frac{m}{i_1} \right\rceil \leq i_1 + i_2 \leq n$, and so, by Proposition 12.1, $i_1 \geq i_{\min}$.

If $i_1 \leq i_{\mathrm{mid}}$, we can show that $w(G) \leq w_{i_1}$. Suppose first that there is a vertex $u_2 \in U_2$ such that $\deg_G(u_2) \in [1, i_1 - 1]$, say $\deg_G(u_2) = i_1 - t$ for some $t \in [1, i_1 - 1]$. If $w(G) \geq w_{i_1} + 1 = i_1 + b_{i_1} - p_{i_1} + 1$, it follows that $\deg_G(u_1) \geq b_{i_1} + t + 1 - p_{i_1}$ for all vertices $u_1 \in N_G(u_2) \subseteq U_1$. Further, $\deg_G(u_1) \geq b_{i_1} + 1 - p_{i_1}$ for all vertices $u_1 \in U_1 - N_G(u_2)$. Since $\min\{x(i_1 - x) : x \in \langle 1, i_1 - 1 \rangle\} = i_1 - 1$

and $i_1(2-p_{i_1}) > 1-p_{i_1}$ (which is a consequence of $p_{i_1} \in [0,2]$), we have

$$\begin{aligned} m = |E(G)| &\geq t(b_{i_1}+1-p_{i_1}) + (i_1-t)(b_{i_1}+t+1-p_{i_1}) \\ &= i_1(b_{i_1}+1-p_{i_1}) + t(i_1-t) \geq i_1(b_{i_1}+1-p_{i_1}) + i_1 - 1 \\ &= i_1 b_{i_1} - 1 + i_1(2-p_{i_1}) > i_1 b_{i_1} - p_{i_1} \geq i_1 b_{i_1} - s_{i_1} = m, \end{aligned}$$

a contradiction.

Now we may assume that $\deg_G(u_2) = i_1$ for every $u_2 \in U_2$. In such a case $m = i_1 i_2$, $i_2 = \frac{m}{i_1} = b_{i_1}$, $p_{i_1} = 0$, $G = K_{i_1,i_2} \cup (n-i_1-i_2)K_1$ and $w(G) = i_1+i_2 = i_1 + b_{i_1} - p_{i_1} = w_{i_1}$.

In the remaining part of the proof we suppose that $i_1 \geq i_{\mathrm{mid}}+1 \geq \sqrt{m}+1$. We have $\lceil\sqrt{m}\,\rceil(\lceil\sqrt{m}\,\rceil-2) < (\sqrt{m}+1)(\sqrt{m}-1) < m$, hence $m/\lceil\sqrt{m}\,\rceil > \lceil\sqrt{m}\,\rceil - 2$ and $\lceil m/\lceil\sqrt{m}\,\rceil\rceil \geq \lceil\sqrt{m}\,\rceil - 1$; on the other hand, $m/\lceil\sqrt{m}\,\rceil \leq m/\sqrt{m} = \sqrt{m}$, which implies $\lceil m/\lceil\sqrt{m}\,\rceil\rceil \leq \lceil\sqrt{m}\,\rceil$. So,

$$\lceil\sqrt{m}\,\rceil - 1 \leq b_{i_{\mathrm{mid}}} = \left\lceil \frac{m}{\lceil\sqrt{m}\,\rceil} \right\rceil \leq \lceil\sqrt{m}\,\rceil. \tag{1}$$

Choose $u_l \in U_l$ so as to satisfy $\deg_G(u_l) = \min\{\deg_G(u) : u \in U_l\}$, choose $v_{3-l} \in N_G(u_l) \subseteq U_{3-l}$ and put $d_l := \deg_G(u_l)$. Let us prove the inequality

$$d_l \leq \lfloor\sqrt{m}\rfloor - 1, \ \ l = 1,2. \tag{2}$$

First, a weaker (in general) inequality $d_l < \sqrt{m}$ is evident, since with $d_l \geq \sqrt{m}$ we would obtain $m \geq i_l d_l \geq (\sqrt{m}+1)\sqrt{m} > m$, a contradiction.

To show (2), admit that $d_l \geq \lfloor\sqrt{m}\rfloor$ for some $l \in [1,2]$. From the above weaker inequality we see that then $\sqrt{m} \notin \mathbb{Z}$ and

$$\begin{aligned} m = \sum\nolimits_{u \in U_l} \deg_G(u) &\geq i_l d_l \geq (\lceil\sqrt{m}\,\rceil+1)(\lceil\sqrt{m}\,\rceil-1) \\ &= \lceil\sqrt{m}\,\rceil^2 - 1 \geq m+1-1 = m, \end{aligned}$$

hence

$$m = (\lceil\sqrt{m}\,\rceil+1)(\lceil\sqrt{m}\,\rceil-1), \tag{3}$$

every vertex in U_l is of degree $\lceil\sqrt{m}\,\rceil - 1$ and

$$w(G) = \lceil\sqrt{m}\,\rceil - 1 + d_{3-l} \leq \lceil\sqrt{m}\,\rceil - 1 + \lfloor\sqrt{m}\rfloor = 2\lceil\sqrt{m}\,\rceil - 2. \tag{4}$$

Because of (1), there are two cases to be considered.

If $b_{i_{\mathrm{mid}}} = \lceil\sqrt{m}\,\rceil$, then, by (4), $M \geq w_{i_{\mathrm{mid}}} = 2\lceil\sqrt{m}\,\rceil - p_{\lceil\sqrt{m}\,\rceil} \geq 2\lceil\sqrt{m}\,\rceil - 2 \geq w(G)$, which contradicts our assumption.

If, however, $\lceil\sqrt{m}\,\rceil - 1 = b_{i_{\rm mid}} = \lceil m/\lceil\sqrt{m}\,\rceil\rceil$, then $m/\lceil\sqrt{m}\,\rceil \le \lceil\sqrt{m}\,\rceil - 1$, so that (3) yields $\lceil\sqrt{m}\,\rceil - 1 = m/(\lceil\sqrt{m}\,\rceil + 1) < m/\lceil\sqrt{m}\,\rceil \le \lceil\sqrt{m}\,\rceil - 1$, a contradiction.

Let us prove by the way of contradiction that $w(G) \le M$. So, suppose that $a^* = k$ and

$$(5) \qquad e \in E(G) \Rightarrow w_G(e) \ge M + 1 \ge \max\{w^* + 1, w_{i_{\rm mid}} + 1\}.$$

If $k = 1$, then $b^* = m$, $w^* = m + 1$ and, by (2), $\deg_G(v_{3-l}) \ge w^* + 1 - d_l \ge m + 3 - \sqrt{m}$, $l = 1, 2$. We have $d_1 = d_2 = 1$, since $d_l \ge 2$ for some $l \in [1, 2]$ yields $m = \sum_{u\in U_{3-l}} \deg_G(u) \ge \sum_{u\in N_G(u_l)} \deg_G(u) \ge 2(m + 3 - \sqrt{m}) > m$, a contradiction. Thus, $w_G(u_1v_2) \le 1 + m = w^*$ in contradiction to (5).

If $k = 2$, then $b^* = \lceil \frac{m}{2} \rceil$, $s^* = 2\lceil \frac{m}{2} \rceil - m \le 1$, $p^* = s^*$ and $w^* = 2 + \lceil \frac{m}{2} \rceil - p^* \ge \frac{m+2}{2}$. Further, Corollary 14 yields $m \ge 4$, hence $i_2 \ge i_1 \ge \lceil\sqrt{m}\,\rceil + 1 \ge 3$. If $l \in [1, 2]$, then, by (5) and (2), $w_G(u_lv_{3-l}) \ge w^* + 1 \ge \frac{m+4}{2}$ and $\deg_G(u) \ge \frac{m+4}{2} - d_l \ge \frac{m+6}{2} - \sqrt{m}$. Now $d_l \le 2$, for otherwise

$$m \ge \sum\nolimits_{u\in N_G(u_l)} \deg_G(u) \ge 3\left(\frac{m+6}{2} - \sqrt{m}\right) > m,$$

a contradiction. Therefore, $\deg_G(v_{3-l}) \ge \frac{m+4}{2} - 2 = \frac{m}{2}$. In the case $d_l = 2$ we obtain (having in mind that $i_{3-l} \ge 3 > d_l$) $m = \sum_{u\in U_{3-l}} \deg_G(u) > \sum_{u\in N_G(u_l)} \deg_G(u) \ge 2 \cdot \frac{m}{2} = m$, a contradiction. If $d_1 = d_2 = 1$, then $\deg_G(v_{3-l}) \ge \frac{m+4}{2} - d_l = \frac{m+2}{2}$, $l = 1, 2$, and $m \ge \deg_G(v_1) + \deg_G(v_2) - 1 \ge 2 \cdot \frac{m+2}{2} - 1 > m$, a contradiction.

Henceforth we may suppose that $k \ge 3$, and, consequently, by Corollary 14, $m \ge k^2 \ge 9$.

If $k = 3$, then $b^* = \lceil \frac{m}{3} \rceil$, $s^* = 3\lceil \frac{m}{3} \rceil - m \le 2$, $p^* = s^*$ and $w^* = m + 3 - 2\lceil \frac{m}{3} \rceil \ge \frac{m+5}{3}$. If $l \in [1, 2]$ and $u \in N_G(u_l)$, then, by (5), $w_G(u_lu) \ge w^* + 1 \ge \frac{m+8}{3}$ and $\deg_G(u) \ge \frac{m+8}{3} - d_l$. Since $v_{3-l} \in N_G(u_l)$, $l = 1, 2$, the assumption $d_1 + d_2 \le 5$ leads to

$$\begin{aligned} n &\ge \sum\nolimits_{l=1}^{2} i_l \ge \sum\nolimits_{l=1}^{2} \deg_G(v_{3-l}) \ge \sum\nolimits_{l=1}^{2} \left(\frac{m+8}{3} - d_l\right) \\ &= \frac{2m+16}{3} - (d_1 + d_2) \ge \frac{2m+1}{3} > \left\lfloor \frac{m+3}{2} \right\rfloor, \end{aligned}$$

which contradicts Lemma 13. The above assumption is fulfilled if $9 \le m \le 15$, because then, by (2), $d_l \le 2$, $l = 1, 2$.

So we may assume that $d_1 + d_2 \ge 6$ and $m \ge 16$. Pick $l \in [1, 2]$. Since

$$m \ge \sum\nolimits_{u\in N_G(u_l)} \deg_G(u) \ge d_l \left(\frac{m+8}{3} - d_l\right),$$

the inequality $d_l(\frac{m+8}{3} - d_l) > m$ equivalent to $3d_l^2 - (m+8)d_l + 3m < 0$ suffices for obtaining a contradiction. The discriminant of the quadratic equation

$$3x^2 - (m+8)x + 3m = 0$$

is $D_1(m) = m^2 - 20m + 64 \geq 0$ and $\sqrt{D_1(m)} \geq m - 16$. Thus, a contradiction will appear as soon as there is $l \in [1,2]$ with

$$d_l \in \left(\frac{m+8-\sqrt{D_1(m)}}{6}, \frac{m+8+\sqrt{D_1(m)}}{6}\right) \supseteq \left(4, \frac{m-4}{3}\right).$$

Therefore, for the rest of our analysis of the case $k = 3$ we may suppose that either $d_l \leq 4$ or $d_l \geq \frac{m-4}{3}$ for both $l = 1, 2$. However, the latter possibility does not apply at all, for otherwise, by (2), we would obtain $\frac{m-4}{3} \leq d_l \leq \lfloor\sqrt{m}\rfloor - 1 \leq \sqrt{m} - 1$, which yields $m \leq 10$, a contradiction; thus, $3 \leq \max\{d_1, d_2\} \leq 4$.

If there is $l \in [1,2]$ with $d_l = 4$, then $4 = d_l \leq \sqrt{m} - 1$, $m \geq 25$, $\deg_G(u) \geq \frac{m-4}{3}$ for each $u \in N_G(u_l)$ and $m \geq 4 \cdot \frac{m-4}{3} \geq m + 3$, a contradiction.

Finally, if $d_1 = d_2 = 3$, then

$$\sum\nolimits_{u \in N_G(u_1)} \deg_G(u) \geq 3\left(\frac{m+8}{3} - 3\right) = m - 1,$$

hence $i_2 = 3$ (as a consequence of $d_2 = 3$). Thus, in U_2 there are two vertices of degree $\frac{m-1}{3}$ and one vertex of degree $\frac{m+2}{3}$, so that $3 = d_1 = \frac{m-1}{3}$ and $m = 10$, a contradiction.

From now on suppose $k \geq 4$, so that $n \geq 2k \geq 8$, and, by Lemma 13, $m \geq 3n - 8 \geq 16$. Putting

$$j_l := \lfloor\sqrt{m}\rfloor - d_l$$

we see from (2) that $j_l \in [1, \lfloor\sqrt{m}\rfloor - 1]$, $l = 1, 2$.

The following assertion will be important for the rest of the proof of our theorem.

Claim. *If* $l \in [1,2]$, *then*

(i) $\deg_G(u) \geq \lceil\sqrt{m}\,\rceil$ *for every* $u \in N_G(u_l)$,

(ii) $N_G(u_l) \subsetneq U_{3-l}$,

(iii) $j_l + j_{3-l} \geq \frac{\sqrt{m}}{2}$.

Proof. Consider the distance

$$\alpha := \lceil\sqrt{m}\,\rceil - \sqrt{m} \in \langle 0, 1)$$

between $\sqrt{m}$ and $\lceil\sqrt{m}\,\rceil$. First notice that Claim (ii) is a direct consequence of Claim (i); indeed, if Claim (i) is true, then the assumption $N_G(u_l) = U_{3-l}$ would mean

$$m = \sum\nolimits_{u \in U_{3-l}} \deg_G(u) \geq i_{3-l}\lceil\sqrt{m}\,\rceil \geq (\lceil\sqrt{m}\,\rceil + 1)\lceil\sqrt{m}\,\rceil > m,$$

a contradiction.

Let $u \in N_G(u_l)$. Using (5) we have $w_G(u_l u) \geq w_{i_{\rm mid}} + 1$ and

$$\deg_G(u) \geq \lceil \sqrt{m}\,\rceil + \left\lceil \frac{m}{\lceil \sqrt{m}\,\rceil} \right\rceil - p_{\lceil \sqrt{m}\,\rceil} + 1 - \lfloor \sqrt{m} \rfloor + j_l. \tag{6}$$

Suppose first that $\sqrt{m} \notin \mathbb{Z}$ (which implies $\alpha > 0$ and $\lceil \sqrt{m}\,\rceil = \lfloor \sqrt{m} \rfloor + 1$). By (1) there are two cases to be considered.

If $\lceil m/\lceil \sqrt{m}\,\rceil\rceil = \lceil \sqrt{m}\,\rceil$, then (6) is transformed into

$$\deg_G(u) \geq (\lceil \sqrt{m}\,\rceil - \lfloor \sqrt{m} \rfloor + 1 - p_{\lceil \sqrt{m}\,\rceil}) + \lceil \sqrt{m}\,\rceil + j_l \geq \lceil \sqrt{m}\,\rceil + j_l,$$

so that

$$\begin{aligned} \sum\nolimits_{u \in N_G(u_l)} \deg_G(u) &\geq (\lfloor \sqrt{m} \rfloor - j_l)(\lceil \sqrt{m}\,\rceil + j_l) = (\lfloor \sqrt{m} \rfloor - j_l)(\lfloor \sqrt{m} \rfloor + j_l + 1) \\ &= \lfloor \sqrt{m} \rfloor^2 - j_l^2 + \lfloor \sqrt{m} \rfloor - j_l \end{aligned} \tag{7}$$

and $N_G(u_l) \subsetneqq U_{3-l}$. Therefore, (7) yields

$$\begin{aligned} \lfloor \sqrt{m} \rfloor - j_{3-l} = d_{3-l} &\leq \frac{\sum_{u \in U_{3-l} - N_G(u_l)} \deg_G(u)}{|U_{3-l} - N_G(u_l)|} = \frac{m - \sum_{u \in N_G(u_l)} \deg_G(u)}{i_{3-l} - (\lfloor \sqrt{m} \rfloor - j_l)} \\ &\leq \frac{m - \lfloor \sqrt{m} \rfloor^2 + j_l^2 + j_l - \lfloor \sqrt{m} \rfloor}{j_l + (i_{3-l} - \lfloor \sqrt{m} \rfloor)}. \end{aligned} \tag{8}$$

Since $i_{3-l} - \lfloor \sqrt{m} \rfloor \geq \lceil \sqrt{m}\,\rceil + 1 - \lfloor \sqrt{m} \rfloor = 2$ and $\frac{j_l^2 + j_l}{j_l + 2} \leq j_l - \frac{1}{3}$ (as a consequence of $j_l \geq 1$), from (8) it follows

$$\lfloor \sqrt{m} \rfloor - j_{3-l} \leq \frac{m - \lfloor \sqrt{m} \rfloor^2 - \lfloor \sqrt{m} \rfloor}{3} + \frac{j_l^2 + j_l}{j_l + 2} \leq \frac{m - \lfloor \sqrt{m} \rfloor^2 - \lfloor \sqrt{m} \rfloor - 1}{3} + j_l,$$

and

$$\begin{aligned} j_l + j_{3-l} &\geq \frac{4\lfloor \sqrt{m} \rfloor + \lfloor \sqrt{m} \rfloor^2 - m + 1}{3} \\ &= \frac{4(\sqrt{m} + \alpha - 1) + (\sqrt{m} + \alpha - 1)^2 - m + 1}{3} \\ &= \frac{\sqrt{m}(2 + 2\alpha) + \alpha^2 + 2\alpha - 2}{3} > \frac{2\sqrt{m} - 2}{3} \geq \frac{\sqrt{m}}{2} \end{aligned}$$

(where the last inequality comes from $m \geq 16$).

If $\lceil m/\lceil \sqrt{m}\,\rceil\rceil = \lceil \sqrt{m}\,\rceil - 1$, then $m/(\sqrt{m} + \alpha) = m/\lceil \sqrt{m}\,\rceil \leq \lceil \sqrt{m}\,\rceil - 1 = \sqrt{m} + \alpha - 1$ and $m \leq m + \sqrt{m}(2\alpha - 1) + \alpha(\alpha - 1)$, so that necessarily $\alpha > \frac{1}{2}$. From (6) we have

$$\begin{aligned} \deg_G(u) &\geq (\lceil \sqrt{m}\,\rceil - \lfloor \sqrt{m} \rfloor + 1 - p_{\lceil \sqrt{m}\,\rceil}) + \lceil \sqrt{m}\,\rceil - 1 + j_l \\ &\geq \lceil \sqrt{m}\,\rceil - 1 + j_l = \lfloor \sqrt{m} \rfloor + j_l, \end{aligned}$$

so that

$$\sum_{u\in N_G(u_l)} \deg_G(u) \geq (\lfloor\sqrt{m}\rfloor - j_l)(\lfloor\sqrt{m}\rfloor + j_l) = \lfloor\sqrt{m}\rfloor^2 - j_l^2$$

and $N_G(u_l) \subsetneqq U_{3-l}$. Since $\frac{j_l^2}{j_l+2} \leq j_l - \frac{2}{3}$, similarly as above we obtain

$$\lfloor\sqrt{m}\rfloor - j_{3-l} = d_{3-l} \leq \frac{m - \lfloor\sqrt{m}\rfloor^2 + j_l^2}{j_l+2} \leq \frac{m - \lfloor\sqrt{m}\rfloor^2 - 2}{3} + j_l,$$

$$j_l + j_{3-l} \geq \frac{3\lfloor\sqrt{m}\rfloor + \lfloor\sqrt{m}\rfloor^2 - m + 2}{3} = \frac{\sqrt{m}(2\alpha+1) + \alpha(\alpha+1)}{3} > \frac{2\sqrt{m}}{3} > \frac{\sqrt{m}}{2}.$$

Finally, suppose that $\sqrt{m} \in \mathbb{Z}$, which yields $w_{i_{\text{mid}}} = 2\sqrt{m}$. Then (6) reads as $\deg_G(u) \geq \lceil\sqrt{m}\,\rceil + j_l + 1 \geq \lceil\sqrt{m}\,\rceil + 2$, hence

$$\sum_{u\in N_G(u_l)} \deg_G(u) \geq (\sqrt{m} - j_l)(\sqrt{m} + j_l + 1) = m + \sqrt{m} - j_l^2 - j_l$$

and $N_G(u_l) \subsetneqq U_{3-l}$. As $|U_{3-l} - N_G(u_l)| = j_l + (i_{3-l} - \sqrt{m}) \geq j_l + 1$, proceeding analogously as above we obtain $\sqrt{m} - j_{3-l} = d_{3-l} \leq \frac{j_l^2 + j_l - \sqrt{m}}{j_l+1} \leq j_l - \frac{\sqrt{m}}{2}$ and $j_l + j_{3-l} \geq \frac{\sqrt{m}}{2}$. □

Since $v_{3-l} \in N_G(u_l)$, $l = 1, 2$, using (5) and Claim (iii) we get

$$\begin{aligned}(9)\qquad n &\geq \sum_{l=1}^{2} i_l \geq \sum_{l=1}^{2} |N_G(v_{3-l})| \geq \sum_{l=1}^{2} (w^* + 1 - d_l)\\ &= \sum_{l=1}^{2} (a^* + b^* - p^* + 1 - \lfloor\sqrt{m}\rfloor + j_l)\\ &= 2\left(k + \left\lceil\frac{m}{k}\right\rceil - p^* + 1 - \lfloor\sqrt{m}\rfloor\right) + (j_1 + j_2)\\ &\geq 2\left(k + \frac{m}{k} - 1 - \sqrt{m}\right) + \frac{\sqrt{m}}{2} = 2\left(k + \frac{m}{k} - 1 - \frac{3\sqrt{m}}{4}\right).\end{aligned}$$

From (9) it is clear that to obtain a contradiction it suffices to show that $k + \frac{m}{k} - 1 - \frac{3\sqrt{m}}{4} > \frac{n}{2}$. The function $f_1(x) = k + \frac{x}{k} - 1 - \frac{3\sqrt{x}}{4}$ is nondecreasing in the interval $\langle\frac{9k^2}{64}, \infty)$. If $a^* = k$, then, by Lemma 13, $m \geq (k-1)(n-k+1) + 1$. We have $[(k-1)(n-k+1)+1, \infty) \subseteq \langle\frac{9k^2}{64}, \infty)$; indeed, from $k = i_{\min} \leq \frac{n}{2}$ it follows that $(k-1)(n-k+1) \geq (k-1)(2k-k+1) + 1 = k^2 > \frac{9k^2}{64}$. Therefore, in order to obtain a contradiction mentioned above, it is sufficient to check that

$$\frac{n}{2} < f_1((k-1)(n-k+1)+1) = n + 1 - \frac{n}{k} - \frac{3\sqrt{(k-1)(n-k+1)+1}}{4},$$

or, equivalently,

$$(10)\qquad n(2k-4) + 4k > 3k\sqrt{(k-1)(n-k+1)+1},$$

or either (after squaring both sides of (10))

$$n^2(2k-4)^2 + n(-9k^3 + 25k^2 - 32k) + 7k^2 + 9k^2(k-1)^2 > 0. \quad (11)$$

The discriminant of the quadratic equation

$$x^2(2k-4)^2 + x(-9k^3 + 25k^2 - 32k) + 7k^2 + 9k^2(k-1)^2 = 0$$

is $D_2(k) = k^3 D_3(k)$ with $D_3(k) := -63k^3 + 414k^2 - 783k + 576$. The function $D_3(x)$ is nonincreasing in the interval $\langle 3, \infty)$. Since $D_3(5) = -864$, it is clear that $D_2(k) < 0$ for every $k \in [5, \lfloor \frac{n}{2} \rfloor]$, which confirms the validity of (11) yielding a contradiction.

If $k = 4$, then (11) is equivalent to $n^2 - 19n + 88 > 0$. The last inequality is true whenever $n \geq 12$. On the other hand, the assumption $n \in [8, 11]$ (recall that we have $n \geq 8$) together with the inequality $m \geq 2n - 8$ (Lemma 13) lead to $n \geq i_1 + i_2 \geq 2(\lceil \sqrt{m} \rceil + 1) \geq 2(\sqrt{m} + 1) \geq 2(\sqrt{3n-8} + 1) > n$, a final contradiction. ∎

Lemma 16. *If $i \in [i_{\mathrm{min}}, i_{\mathrm{mid}} - 1]$, then the following hold:*

1. $w_{i+1} \leq w_i + 1$.
2. *If* $w_{i+1} = w_i + 1$, *then* $b_{i+1} = b_i - 2$, $s_i \geq 2$ *and* $s_{i+1} = 0$.
3. *If* $w_{i+1} = w_i$ *and* $s_{i+1} \geq 2$, *then* $b_{i+1} = b_i - 1$.
4. *If* $s_i \leq 1$ *and* $i \leq i_{\mathrm{mid}} - 2$, *then* $w_{i+1} \leq w_i - 1$.

Proof. We have $b_{i+1} = \lceil \frac{m}{i+1} \rceil \leq \lceil \frac{m}{i} \rceil = b_i$. Let us prove that $b_{i+1} < b_i$. If $i \leq i_{\mathrm{mid}} - 2$, then $i(i+1) \leq (\lceil \sqrt{m} \rceil - 2)(\lceil \sqrt{m} \rceil - 1) < (\sqrt{m} - 1)\sqrt{m} < m$, $\frac{m}{i} - \frac{m}{i+1} = \frac{m}{i(i+1)} > 1$ and the desired inequality follows. It remains to be shown that $b_{\lceil \sqrt{m} \rceil - 1} \neq b_{\lceil \sqrt{m} \rceil}$. Since $m/\lceil \sqrt{m} \rceil \leq m/\sqrt{m} = \sqrt{m} < m/(\lceil \sqrt{m} \rceil - 1)$, we see that $\lceil m/\lceil \sqrt{m} \rceil \rceil$ can be equal to $\lceil m/(\lceil \sqrt{m} \rceil - 1) \rceil$ only if each of those two numbers is $\lceil \sqrt{m} \rceil$. In such a case, however, both $m/(\lceil \sqrt{m} \rceil - 1)$ and $m/\lceil \sqrt{m} \rceil$ are in the interval $(\lceil \sqrt{m} \rceil - 1, \lceil \sqrt{m} \rceil \rangle$, and then $(\lceil \sqrt{m} \rceil - 1)\lceil \sqrt{m} \rceil < m \leq (\lceil \sqrt{m} \rceil - 1)\lceil \sqrt{m} \rceil$, a contradiction.

If $b_{i+1} \leq b_i - 4$, then $w_{i+1} \leq i + 1 + b_i - 4 - p_{i+1} \leq i + b_i - 3 < i + b_i - p_i = w_i$.

If $b_{i+1} = b_i - 3$, then $w_{i+1} = i + 1 + b_i - 3 - p_{i+1} \leq i + b_i - 2 \leq i + b_i - p_i = w_i$ and $w_{i+1} = w_i$ implies $p_i = 2$ and $p_{i+1} = 0$, hence $s_i \geq 2$ and $s_{i+1} = 0$.

If $b_{i+1} = b_i - 2$, then $w_{i+1} = i + 1 + b_i - 2 - p_{i+1} \leq i + b_i - 1 \leq i + b_i - p_i + 1 = w_i + 1$. Moreover, $w_{i+1} = w_i + 1$ yields $p_i = 2$ and $p_{i+1} = 0$ (and, consequently, $s_i \geq 2$ and $s_{i+1} = 0$), while $w_{i+1} = w_i$ implies either $p_i = 1$ and $p_{i+1} = 0$ ($s_i = 1$ and $s_{i+1} = 0$) or $p_i = 2$ and $p_{i+1} = 1$ ($s_i \geq 2$ and $s_{i+1} = 1$).

Finally, if $b_{i+1} = b_i - 1$, then $m = ib_i - s_i = (i+1)(b_i - 1) - s_{i+1}$. From $b_i - (i+1) \geq \lceil m/(\lceil \sqrt{m} \rceil - 1) \rceil - \lceil \sqrt{m} \rceil \geq \lceil m/\sqrt{m} \rceil - \lceil \sqrt{m} \rceil = 0$ and $(i+1)(b_i - 1) = ib_i + b_i - (i+1) \geq ib_i$ it follows that $s_{i+1} \geq s_i$, $p_{i+1} \geq p_i$ and

$w_{i+1} = i+1+b_i-1-p_{i+1} \le i+b_i-p_i = w_i$. Besides that, from the assumption $i \le i_{\text{mid}}-2$ we obtain $b_i-(i+1) \ge \lceil m/(\lceil\sqrt{m}\,\rceil-2)\rceil - \lceil\sqrt{m}\,\rceil+1 \ge \lceil m/(\sqrt{m}-1)\rceil - \lceil\sqrt{m}\,\rceil+1 \ge \lceil\sqrt{m}+1\rceil - \lceil\sqrt{m}\,\rceil+1 = 2$, $s_{i+1} \ge s_i+2$, and then w_{i+1} can be equal to w_i only if $s_i \ge 2 = p_i = p_{i+1}$.

The statements of lemma follow by inspecting the above assertions. ■

Lemma 17. *If* $i \in [i_{\text{min}}, i_{\text{mid}}-1]$ *and* $j \in [i+1, i_{\text{mid}}]$, *then* $w_j \le w_i+1$.

Proof. If there is $l \in [i+1, i_{\text{mid}}]$ with $w_l \ge w_i+1$, then, by Lemma 16.1, $J := \{j \in [i+1, i_{\text{mid}}] : w_j = w_{j-1}+1\} \ne \emptyset$. Moreover, $s_{j-1} \ge 2$ and $s_j = 0$ for every $j \in J$ (Lemma 16.2) and $w_{j+1} \le w_j - 1$ for every $j \in J - \{i_{\text{mid}}-1, i_{\text{mid}}\}$ (Lemma 16.4). Let $r := |J|$ and let $J = \{j_k : k \in [1,r]\}$, where the sequence $(j_1, \dots, j_r)$ is increasing. (Notice that $j_{k+1} \ge j_k+2$ for every $k \in [1, r-1]$.) Then $w_j \le w_i$ for every $j \in [i+1, j_1-1]$ and $w_{j_1} \le w_i+1$. Further, if $k \in [1, r-1]$, then (by induction one can prove) $w_j \le w_{j_k}-1 \le w_i$ for every $j \in [j_k+1, j_{k+1}-1]$ and $w_{j_{k+1}} \le w_{j_k} \le w_i+1$. Finally, if $j_r = i_{\text{mid}}$, then $w_j \le w_{j_r}-1 \le w_i$ for every $j \in [j_r+1, i_{\text{mid}}]$. If $j_r = i_{\text{mid}}-1$, then $w_{i_{\text{mid}}} \le w_{j_r} \le w_i+1$ (the first inequality follows from the fact that $i_{\text{mid}} \notin J$). ■

Theorem 18. $w(\mathcal{B}, n, m)$ *is either* w^* *or* w^*+1 *and in the latter case there is a positive integer* l *such that* $a^*+l \le i_{\text{mid}}$, $m = (a^*+l)(b^*-l-1)$, $b^* \le 2a^*$, $s^* \ge 2$ *and* $p^* = 2$.

Proof. By Theorem 15 and by Lemma 17 with $i = i_{\text{min}} = a^*$ we have $w^* = w_{i_{\text{min}}} \le w(\mathcal{B}, n, m) \le w^*+1$.

If $w(\mathcal{B}, n, m) = w^*+1$, by Theorem 15 there is $j \in [1, i_{\text{mid}}-i_{\text{min}}]$ such that $w_{a^*+j} = w^*+1$. With $l := \min\{j \in [1, i_{\text{mid}}-i_{\text{min}}] : w_{a^*+j} = w^*+1\}$ Lemma 17 yields $w_{a^*+j} = w^*$ for every $j \in [1, l-1]$ ($w_{a^*+j} \le w^*-1$ for some $j \in [1, l-1]$ would imply $w_{a^*+l} \le w_{a^*+j}+1 \le w^*$, a contradiction).

Then, by Lemma 16.2, $s_{a^*+l} = 0$ and $s_{a^*+l-1} \ge 2$. If $s_{a^*+j} \le 1$ for some $j \in [0, l-2]$, then by taking j to be maximum, we have $s_{a^*+j+1} \ge 2$. Since $a^*+j \le a^*+l-2 \le i_{\text{mid}}-2$, by using Lemma 16.4, we have $w_{a^*+j+1} \le w_{a^*+j}-1$, a contradiction. Thus $s_{a^*+j} \ge 2$ for every $j \in [0, l-1]$, in particular $s^* \ge 2$ and $p^* = 2$. Moreover, by Lemma 16.3, $b_{a^*+j} = b_{a^*}-j = b^*-j$ for each $j \in [0, l-1]$, and by Lemma 16.2, $b_{a^*+l} = b_{a^*+l-1}-2 = b^*-l-1$ and $s_{a^*+l} = 0 = p_{a^*+l}$. Consequently,

$$m = (a^*+l)b_{a^*+l} - p_{a^*+l} = (a^*+l)(b^*-l-1), \tag{12}$$

where $a^*+l \le a^*+i_{\text{mid}}-i_{\text{min}} = i_{\text{mid}}$.
Let us show that $b^* \le 2a^*$. Since $a^*+1 \le a^*+l \le i_{\text{mid}}$,

$$m = a^*b^* - s^* = (a^*+1)b_{a^*+1} - s_{a^*+1}. \tag{13}$$

If $l = 1$, then $b_{a^*+1} = b^* - 2$ and $s_{a^*+1} = 0$. Thus, by (12) and (13), $2a^* - b^* = s^* - 2 \geq 0$ as required. If $l \geq 2$, then $b_{a^*+1} = b^* - 1$. Since $s_{a^*+1} \leq a^*$, from (12) and (13) we obtain $2a^* - b^* = a^* - s_{a^*+1} + s^* - 1 > 0$ and the proof follows. ■

Theorem 19. *If $w(\mathcal{B}, n, m) = w^* + 1$, then $a^* + b^* = n$ and $w(\mathcal{B}, n, m) = n - 1$.*

Proof. The assumption $w(\mathcal{B}, n, m) = w^* + 1$ gives us $a^* \geq 2$, because $a^* = 1$ yields $b^* = m$ and $s^* = 0 = p^*$ so that, by Theorem 18, $w(\mathcal{B}, n, m) = w^*$.

From Theorem 18 we know that $2a^* \geq b^*$, $p^* = 2$ and $s^* \geq 2$, hence, by Proposition 12.1, $w(\mathcal{B}, n, m) = a^* + b^* - p^* + 1 = a^* + b^* - 1 = a_{i_{\min}} + b_{i_{\min}} - 1 \leq n - 1$, $a^* + b^* \leq n$ and $a^* + b^* = n - r$ with $r \geq 0$. Suppose that $r \geq 1$. The complete bipartite graph K_{a^*-1, b^*+1+r} is of order $a^* + b^* + r = n$ and (as $m = a^*b^* - s^*$) of size $(a^* - 1)(b^* + 1 + r) = m + (s^* - 2) + (r - 1)(a^* - 1) + (2a^* - b^*) \geq m$. Consider an arbitrary subgraph G of K_{a^*-1, b^*+1+r} belonging to $\mathcal{B}(n, m)$. Then the standard pair (i_1, i_2) for G satisfies $i_1 \leq a^* - 1 = i_{\min} - 1$ in contradiction to Proposition 11. Therefore, $r = 0$, $a^* + b^* = n$ and $w(\mathcal{B}, n, m) = n - 1$. ■

Theorem 20. *Suppose that $r_0 = \sqrt{n^2 - 4m}$, $r_1 = \sqrt{(n-1)^2 - 4m}$ and $r_1' = \sqrt{n^2 - 4m - 4}$.*

1. *If r_0 is an integer, then $w(\mathcal{B}, n, m) = n$.*
2. *If r_0 is not an integer and (exactly) one of r_1, r_1' is, then $w(\mathcal{B}, n, m) = n - 1$.*
3. *If r_0, r_1, r_1' are not integers, then $w(\mathcal{B}, n, m) = w^*$.*

Proof. The theorem is a direct consequence of Propositions 9 and 10, and of Theorems 18 and 19. ■

The rest of the paper is devoted to showing that there are parameters n, m such that $w(\mathcal{B}, n, m) = w^* + 1$.

Lemma 21. *Suppose that $w(\mathcal{B}, n, m) = w^* + 1$.*

1. *If $n \equiv 0 \pmod 2$, then $a^* \leq \frac{n-4}{2}$.*
2. *If $n \equiv 1 \pmod 2$, then $a^* \leq \frac{n-3}{2}$.*

Proof. The lemma will be proved by the way of contradiction with the help of Theorem 18. Namely, we shall show that if the inequalities for a^* are invalid, then $w(\mathcal{B}, n, m) = w^*$. This will be done mostly by exhibiting that $s^* \in [0, 1]$.

1. Assume that n is even and $a^* \geq \frac{n-2}{2}$. Then $\frac{n - \sqrt{n^2-4m}}{2} > \frac{n-4}{2}$, $n^2 - 4m < 16$ and $m \in \{\frac{n^2-4i}{2} : i \in [0, 3]\}$. If $m = \frac{n^2}{4}$, then $a^* = \frac{n}{2} = b^*$ and $s^* = a^*b^* - m = 0$. Let $m = \frac{n^2-4i}{4}$, $i \in [1, 3]$, so that $n \geq 4$ and $a^* = \left\lceil \frac{n-\sqrt{4i}}{2} \right\rceil = \frac{n-2}{2}$. By Theorem 19, $b^* = n - a^* = \frac{n+2}{2}$ and $s^* = \frac{n^2-4}{4} - \frac{n^2-4i}{4} = i - 1$ so that with $i \in [1, 2]$ the mentioned contradiction follows. If $i = 3$, then $s^* = 2$, $w^* = n - 2$, $i_{\text{mid}} =$

$\left\lceil \sqrt{(n^2-12)/4} \right\rceil \le \frac{n}{2}$, $\frac{n-2}{2} < \frac{n^2-12}{4} \le \frac{n^2}{4}$, hence $b_{\frac{n}{2}} = \frac{n}{2}$, $s_{\frac{n}{2}} = \frac{n^2}{4} - \frac{n^2-12}{4} = 3$, $p_{\frac{n}{2}} = 2$ and $w_{\frac{n}{2}} = n-2 = w^*$ so that, by Theorem 15, $w(\mathcal{B}, n, m) = w^*$.

2. Provided that n is odd and $a^* \ge \frac{n-1}{2}$, we have $\frac{n-\sqrt{n^2-4m}}{2} > \frac{n-3}{2}$, $n^2-4m < 9$ and $m = \frac{n^2-1-4i}{4}$ with $i \in [0,1]$ and $a^* = \left\lceil \frac{n-\sqrt{4i+1}}{2} \right\rceil = \frac{n-1}{2}$. By Theorem 19, $b^* = n - a^* = \frac{n+1}{2}$ and $s^* = \frac{n^2-1}{4} - \frac{n^2-1-4i}{4} = i \in [0,1]$. ■

Theorem 22. *If $w(\mathcal{B}, n, m) = w^*+1$, then $m \le \left\lfloor \frac{n^2-2n+1}{4} \right\rfloor$ and there is $i \in [0,\infty)$ such that one of the following three series of conditions is satisfied:*
$n \equiv 0 \pmod 3$, $n \ge 9$ and $m = \left(\frac{n+3}{3}+i\right)\left(\frac{2n-6}{3}-i\right) \ge \frac{n+3}{3}\cdot\frac{2n-6}{3} = \frac{2n^2-18}{9}$;
$n \equiv 2 \pmod 3$, $n \ge 11$ and $m = \left(\frac{n+4}{3}+i\right)\left(\frac{2n-7}{3}-i\right) \ge \frac{n+4}{3}\cdot\frac{2n-7}{3} = \frac{2n^2+n-28}{9}$;
$n \equiv 1 \pmod 3$, $n \ge 16$ and $m = \left(\frac{n+5}{3}+i\right)\left(\frac{2n-8}{3}-i\right) \ge \frac{n+5}{3}\cdot\frac{2n-8}{3} = \frac{2n^2+2n-40}{9}$.

Proof. Let us first show that with $w(\mathcal{B}, n, m) = w^*+1$ we cannot have $n \le 8$ or $n \in \{10, 13\}$.

If $n \le 8$, then, by Lemma 21, $a^* \le \frac{n-3}{2} < 3$, $a^* \le 2$, $s^* \le 1$ and so, by Theorem 18, $w(\mathcal{B}, n, m) = w^*$.

Suppose $n = 10$ and $w(\mathcal{B}, n, m) = w^*+1$. By Theorem 18 and Lemma 21 then $2 \le s^* \le a^*-1 \le 2$, $s^* = 2$ and $a^* = 3$ so that Theorem 19 yields $b^* = 10 - a^* = 7$, which contradicts the inequality $b^* \le 2a^*$ of Theorem 18.

Suppose $n = 13$ and $w(\mathcal{B}, n, m) = w^*+1$. By Lemma 21, $a^* \le \frac{n-3}{2} = 5$, while Theorems 18 and 19 imply $b^* = 13 - a^* \le 2a^*$, which yields $a^* > 4$. Thus $a^* = 5$ and $b^* = 8$. By Theorem 18, $s^* \ge 2$, and then $m = a^*b^* - s^* = 40 - s^* \le 38$. Since $\left\lceil \frac{13-\sqrt{169-4m}}{2} \right\rceil = a^* = 5$, we have $m > 36$, thus $m \in [37, 38]$. Then, however, m cannot be expressed as $(a^*+l)(b^*-l-1)$, where l is a positive integer with $a^*+l \le i_{\mathrm{mid}} = \lceil \sqrt{m}\, \rceil = 7$, a contradiction to Theorem 18.

So, in the sequel we suppose that $w(\mathcal{B}, n, m) = w^*+1$, $n \ge 9$ and $n \notin \{10, 13\}$. By Theorem 19 and Theorem 18 then $n-1 = w(\mathcal{B}, n, m) = w^*+1 = a^*+b^*-1$ and $n = a^*+b^* \le 3a^*$ so that $a^* \ge \lceil \frac{n}{3} \rceil$. Therefore, $a^* \ge \frac{n+c(n)}{3}$, where $c(n) \in [0,2]$ is such that $n + c(n) \equiv 0 \pmod 3$. As a consequence, $a^* = \frac{n+c(n)}{3} + j$ and $b^* = \frac{2n-c(n)}{3} - j$ for some nonnegative integer j. By Theorem 18 there is a positive integer l such that

$$a^* + l \le i_{\mathrm{mid}} = \lceil \sqrt{m}\, \rceil \tag{14}$$

and

$$m = \left(\frac{n+c(n)}{3} + j + l\right)\left(\frac{2n-c(n)}{3} - j - 1 - l\right)$$
$$= \left(\frac{n+c(n)+3}{3} + i\right)\left(\frac{2n-c(n)-6}{3} - i\right) =: f_4(i)$$

with $i := j+l-1 \in [0, \infty)$. Thus we know that $m = k_1 k_2 = k_1(n-1-k_1) \leq \left(\frac{n-1}{2}\right)^2$ and $m \leq \left\lfloor \frac{n^2-2n+1}{4} \right\rfloor$. Moreover, it is easy to check that $f_4(x) = f_4(\frac{n-2c(n)-9}{3} - x)$ and that

$$\text{(15)} \qquad \min\left\{ f_4(x) : x \in \left\langle 0, \frac{n-2c(n)-9}{3} \right\rangle \right\} = f_4(0) = f_4\left(\frac{n-2c(n)-9}{3}\right).$$

If n is even, then $i_{\text{mid}} = \lceil \sqrt{m} \rceil \leq \left\lceil \sqrt{n^2}/4 \right\rceil = \frac{n}{2}$, hence, by (14), $\frac{n+c(n)+3}{3} + i = a^* + l \leq \frac{n}{2}$, and

$$0 \leq i \leq \frac{n-2c(n)-6}{6} \leq \frac{n-2c(n)-9}{3}$$

(where the last inequality immediately follows from our assumptions on n).

If n is odd, then $i_{\text{mid}} \leq \frac{n-1}{2}$, $\frac{n+c(n)+3}{3} + i \leq \frac{n-1}{2}$, and

$$0 \leq i \leq \frac{n-2c(n)-9}{6} \leq \frac{n-2c(n)-9}{3}.$$

Thus, independently from the parity of n, because of (15) we have $m = f_4(i) \geq f_4(0)$. So, the statement of our theorem follows from the fact that $f_4(0) = \frac{n+c(n)+3}{3} \cdot \frac{2n-c(n)-6}{3}$ is exactly the claimed lower bound for m depending on the congruence class modulo 3 containing n. ∎

Let us prove now the tightness of the bounds for m in Theorem 22. Recall that $c(n) \in [0, 2]$ is such that $n + c(n) \equiv 0 \pmod 3$.

Proposition 23. 1. *If $n \geq 9$, $n \notin \{10, 13\}$ and $m = \frac{n+c(n)+3}{3} \cdot \frac{2n-c(n)-6}{3}$, then $w(\mathcal{B}, n, m) = w^* + 1$.*

2. *If $n = 2^{2q+1} + 1$ with $q \in \mathbb{Z}^+$ and $m = \left\lfloor \frac{n^2-2n+1}{4} \right\rfloor$, then $w(\mathcal{B}, n, m) = w^* + 1$.*

Proof. 1. If $m = \frac{n+c(n)+3}{3} \cdot \frac{2n-c(n)-6}{3}$, then $n^2 - 4m = \frac{1}{9}[n^2 - 4nc(n) + 4(c(n) + 3)(c(n) + 6)]$ and $a^* = \left\lceil \frac{1}{2}(n - \sqrt{n^2 - 4m}) \right\rceil = \frac{n+c(n)}{3}$, because a necessary and sufficient pair of inequalities is

$$\frac{n+c(n)-3}{3} < \frac{1}{2}\left[n - \frac{1}{3}\sqrt{n^2 - 4nc(n) + 4(c(n)+3)(c(n)+6)}\right] \leq \frac{n+c(n)}{3};$$

the first inequality is equivalent to $5c(n) + 3 < n$ and the second one is obvious.

Therefore, we have

$$\begin{aligned} b^* &= \left\lceil \frac{m}{a^*} \right\rceil = \left\lceil \frac{n+c(n)+3}{3} \cdot \frac{2n-c(n)-6}{3} \cdot \frac{3}{n+c(n)} \right\rceil = \frac{2n-c(n)}{3}, \\ s^* &= \frac{n+c(n)}{3} \cdot \frac{2n-c(n)}{3} - \frac{n+c(n)+3}{3} \cdot \frac{2n-c(n)-6}{3} = c(n)+2 \\ &\geq 2 = p^*, \\ w^* &= \frac{n+c(n)}{3} + \frac{2n-c(n)}{3} - 2 = n-2. \end{aligned}$$

On the other hand, $a^*+1 = \frac{n+c(n)+3}{3} \leq \frac{2n-c(n)-6}{3}$, hence $(a^*+1)^2 \leq \frac{n+c(n)+3}{3} \cdot \frac{2n-c(n)-6}{3} = m$ and $i_{\min} \leq a^*+1 \leq \sqrt{m} \leq i_{\text{mid}}$. By Theorems 15 and 18 then $w^*+1 \geq w(\mathcal{B},n,m) \geq w_{a^*+1} = \frac{n+c(n)+3}{3} + \frac{2n-c(n)-6}{3} - 0 = n-1 = w^*+1$ and $w(\mathcal{B},n,m) = w^*+1$.

2. If $n = 2^{2q+1}+1$ and $m = \left\lfloor \frac{n^2-2n+1}{4} \right\rfloor = 2^{4q}$, then

$$\begin{aligned} a^* &= \left\lceil (2^{2q+1}+1-\sqrt{2^{2q+2}+1})/2 \right\rceil = 2^{2q}-2^q+1, \\ b^* &= \lceil 2^{4q}/(2^{2q}-2^q+1) \rceil = 2^{2q}+2^q, \\ s^* &= (2^{2q}-2^q+1)(2^{2q}+2^q) - 2^{4q} = 2^q \geq 2 = p^*, \\ w^* &= (2^{2q}-2^q+1) + (2^{2q}+2^q) - 2 = 2^{2q+1}-1 = n-2. \end{aligned}$$

Besides that, $i_{\min} = a^* \leq 2^{2q} = \sqrt{m} = i_{\text{mid}}$, and, since $w_{2^{2q}} = 2^{2q}+2^{2q}-0 = 2^{2q+1} = w^*+1$, as above we obtain $w(\mathcal{B},n,m) = w^*+1$. ■

Note that there are n's such that the maximum m satisfying $w(\mathcal{B},n,m) = w^*+1$ is smaller than $\left\lfloor \frac{n^2-2n+1}{4} \right\rfloor$. Indeed, if $n = 2q^2$, $q \in \mathbb{Z}^+$, then with $m = \left\lfloor \frac{n^2-2n+1}{4} \right\rfloor = q^2(q^2-1)$ we have $a^* = q(q-1)$, $b^* = q(q+1)$, $s^* = 0 = p^*$ and $w^* = q(q-1) + q(q+1) = n$ so that $w(\mathcal{B},n,m) = w^*$.

Acknowledgement

The authors are extremely grateful to one of two anonymous referees for his/her useful comments and especially for having found a gap in the original proof of Theorem 15.

References

[1] O.V. Borodin, *Computing light edges in planar graphs*, in: Topics in Combinatorics and Graph Theory, R. Bodendiek and R. Henn (Ed(s)), (Physica-Verlag, Heidelberg, 1990) 137–144.

[2] H. Enomoto and K. Ota, *Connected subgraphs with small degree sum in* 3*-connected planar graphs*, J. Graph Theory **30** (1999) 191–203.
doi:10.1002/(SICI)1097-0118(199903)30:3⟨191::AID-JGT4⟩3.0.CO;2-X

[3] I. Fabrici and S. Jendroľ, *Subgraphs with restricted degrees of their vertices in planar* 3*-connected graphs*, Graphs Combin. **13** (1997) 245–250.

[4] B. Grünbaum, *Acyclic colorings of planar graphs*, Israel J. Math. **14** (1973) 390–408.
doi:10.1007/BF02764716

[5] J. Ivančo, *The weight of a graph*, in: Fourth Czechoslovakian Symposium on Combinatorics, Graphs and Complexity, J. Nešetřil and M. Fiedler (Ed(s)), (North-Holland, Amsterdam, 1992) 113–116.

[6] J. Ivančo and S. Jendroľ, *On extremal problems concerning weights of edges of graphs*, in: Sets, Graphs and Numbers, G. Halász, L. Lovász, D. Miklós and T. Szőnyi (Ed(s)), (North-Holland, Amsterdam, 1992) 399–410.

[7] E. Jucovič, *Strengthening of a theorem about* 3*-polytopes*, Geom. Dedicata **13** (1974) 233–237.
doi:10.1007/BF00183214

[8] S. Jendroľ and I. Schiermeyer, *On a max-min problem concerning weights of edges*, Combinatorica **21** (2001) 351–359.
doi:10.1007/s004930100001

[9] S. Jendroľ, M. Tuhársky and H.-J. Voss, *A Kotzig type theorem for large maps on surfaces*, Tatra Mt. Math. Publ. **27** (2003) 153–162.

[10] A. Kotzig, *Contribution to the theory of Eulerian polyhedra*, Mat.-Fyz. Časopis. Slovensk. Akad. Vied **5** (1955) 111–113.

[11] J. Zaks, *Extending Kotzig's theorem*, Israel J. Math. **45** (1983) 281–296.
doi:10.1007/BF02804013

Received 2 February 2012
Revised 8 January 2012
Accepted 9 January 2012