*Dedicated to Mieczysław Borowiecki on his 70th birthday*

# UNIVERSALITY FOR AND IN INDUCED-HEREDITARY GRAPH PROPERTIES

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## Abstract

The well-known Rado graph $R$ is universal in the set of all countable graphs $\mathcal{I}$, since every countable graph is an induced subgraph of $R$. We study universality in $\mathcal{I}$ and, using $R$, show the existence of $2^{\aleph_0}$ pairwise non-isomorphic graphs which are universal in $\mathcal{I}$ and denumerably many other universal graphs in $\mathcal{I}$ with prescribed attributes. Then we contrast *universality for* and *universality in* induced-hereditary properties of graphs and show that the overwhelming majority of induced-hereditary properties contain no universal graphs. This is made precise by showing that there are $2^{(2^{\aleph_0})}$ properties in the lattice $\mathbb{K}_{\leq}$ of induced-hereditary properties of which only at most $2^{\aleph_0}$ contain universal graphs.

In a final section we discuss the outlook on future work; in particular the question of characterizing those induced-hereditary properties for which there is a universal graph in the property.

**Keywords:** countable graph, universal graph, induced-hereditary property.

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## 1.   Introduction and Motivation

In this article a graph shall (with one illustrative exception) be simple, undirected, unlabelled, with a countable (i.e., finite or denumerably infinite) vertex set. For graph theoretical notions undefined here, we generally follow [14].

A *property* (of graphs) is a class of countable graphs, closed under isomorphisms. If two graphs are isomorphic, we refer to any one of them as a *clone* of the other. The symbol $\mathcal{I}$ denotes the class of all countable graphs. To avoid potential conceptual problems with proper classes or large numbers of clones, we may select a particular subset $\mathrm{Skel}(\mathcal{I})$ of the class $\mathcal{I}$, with elements one specific graph chosen from each isomorphism class in $\mathcal{I}$. (Since clones share all their graph properties, and since we have for many purposes, in a property, no reason to distinguish between clones, this move is unproblematic.) Similarly, an arbitrary property $\mathcal{P}$ may for most purposes be thought of "concretely" or "extensionally" as its skeleton, the set $\mathrm{Skel}(\mathcal{P}) := \{G \in \mathrm{Skel}(\mathcal{I}) \mid G \in \mathcal{P}\}$. Corresponding to any graph $G$ we have the property $\leq G := \{H \mid H \leq G\}$ of all induced subgraphs of $G$.

A property $\mathcal{P}$ is an *induced-hereditary property of graphs*, or an *induced-hereditary* (i-h) property for short, if, whenever $G \in \mathcal{P}$ and $H \leq G$, then $H \in \mathcal{P}$ too. As an easy example, note that every $\leq G$ is an i-h property, since $\leq$ is transitive. For i-h properties, we generally follow the notation of [1]. In particular, we use the notation $\mathbb{K}_{\leq}$ to denote the lattice of all i-h properties (although the graphs we consider are generally countable, whereas in [1] they are assumed to be finite).

Let $\mathcal{P}$ be a set of countable graphs. Following [14], we define a graph $U$ to be a *universal graph for* $\mathcal{P}$ if every graph in $\mathcal{P}$ is an induced subgraph of $U$; it is a *universal graph in* $\mathcal{P}$ if $U \in \mathcal{P}$ too. Since a universal graph $U$ for $\mathcal{P}$ is allowed to be outside $\mathcal{P}$ and hence, presumably, to be uncountable, the existence of at least one such $U$ becomes trivial: take $U$ to be the disjoint union of one clone from each isomorphism class in $\mathcal{P}$. The fact that this $U$ is in general uncountable follows from Lemma 1 of [6]; a countable universal graph for any i-h property is constructed in that paper too—see Theorem 3 of [6]. Another such construction, again depending on the specific property, occurs in Theorem 3 and Corollary 3.1 of [5].

For given natural numbers $n$ and $k \geq 2$, consider the sequence $(n_0, n_1, n_2, \ldots)$ with $0 \leq n_i < k$ for each $i$ such that $n = \sum_{i=0}^{\infty} n_i k^i$. We shall refer to $n_{i-1}$ as the entry in the $i$'th position of the $k$-sequence associated with $n$. When $k = 2$, this is of course the *binary expansion* of $n$. Rado [21] constructed a (simple) denumerable graph $R$ with the positive integers as vertex set with the following edges: For given $m$ and $n$ with $m < n$, $m$ is adjacent to $n$ if $n$ has a 1 in the $m$'th position of its binary expansion. It is well known that $R$ is a universal graph in

the set $\mathcal{I}$ of all countable graphs and that $R$ is a connected, self-complementary graph.

The following notation is used for cardinalities: for any graph $G$, $|G| := card(V(G))$; for any i-h property $\mathcal{P}$: $|\mathcal{P}| := card(\mathrm{Skel}(\mathcal{P}))$; and for the set $\mathbb{K}_\leq$ of all (skeletons of) i-h properties, $|\mathbb{K}_\leq| := card\{\mathrm{Skel}(\mathcal{P}) \mid \mathcal{P} \in \mathbb{K}_\leq\}$.

Note that our choice to restrict ourselves to countable graphs means that $|G| \leq \aleph_0$ for every graph $G$. Upper bounds for the other cardinalities mentioned above are contained in our first result.

**Lemma 1.**  (i)  *For any i-h property $\mathcal{P}$, $|\mathcal{P}| \leq 2^{\aleph_0}$, and $|\mathcal{I}| = 2^{\aleph_0}$.*

(ii)  $|\mathbb{K}_\leq| \leq 2^{(2^{\aleph_0})}$.

**Proof.** (i) $\mathrm{Skel}(\mathcal{P}) \subseteq \mathrm{Skel}(\mathcal{I})$ and $|\mathcal{I}| = 2^{\aleph_0}$, since $|\mathcal{I}| \geq 2^{\aleph_0}$ by Lemma 1 of [6] and $|\mathcal{I}| \leq 2^{\aleph_0}$, since each graph in $\mathcal{I}$ is an induced subgraph of the Rado graph $R$ and hence is determined by a subset of the denumerable vertex set of $R$.

(ii) $|\mathbb{K}_\leq| \leq 2^{(2^{\aleph_0})}$ since each i-h property is a subset of $\mathcal{I}$, which has cardinality $2^{\aleph_0}$ by (i). ∎

We note that we shall later (in Theorem 8) show that also $|\mathbb{K}_\leq| \geq 2^{(2^{\aleph_0})}$, so that $|\mathbb{K}_\leq| = 2^{(2^{\aleph_0})}$ (Corollary 3).

In Section 2 we study universality in $\mathcal{I}$ and, using $R$, show the existence of $2^{\aleph_0}$ pairwise non-isomorphic graphs which are universal in $\mathcal{I}$ and denumerably many other universal graphs in $\mathcal{I}$ with prescribed attributes. Then we contrast in Sections 3 and 4 *universality for* and *universality in* induced-hereditary properties of graphs and show that the overwhelming majority of induced-hereditary properties contain no universal graphs. This is made precise in Section 5 by showing that there are $2^{(2^{\aleph_0})}$ properties in the lattice $\mathbb{K}_\leq$ of induced-hereditary properties of which only at most $2^{\aleph_0}$ contain universal graphs. In Section 6 we conclude by highlighting the problem of characterizing those i-h properties with universal members.

## 2.    Universality in $\mathcal{I}$

We note that $\mathcal{I}$ is the only property $\mathcal{P}$ for which universality *for* $\mathcal{P}$ is equivalent to universality *in* $\mathcal{P}$. The Rado graph $R$ is of course the archetypal universal graph in $\mathcal{I}$. Many different constructions for clones of $R$ are known, some of which can be found in [2] and [4]. We do not here engage with the view of $R$ as "the random graph" [8]. But is there universality in $\mathcal{I}$ beyond $R$ and its clones? *Yes*, as we demonstrate in this section.

In our first result it is convenient to let $\mathcal{F}$ denote the set of $2^{\aleph_0}$ pairwise non-isomorphic linear forests which are shown to exists in Lemma 1 of [6].

**Lemma 2.** *There exist exactly $2^{\aleph_0}$ pairwise non-isomorphic (denumerable) graphs which are universal in $\mathcal{I}$.*

**Proof.** Since (by Lemma 1(i)) $|\mathcal{I}| = 2^{\aleph_0}$, there cannot be more that $2^{\aleph_0}$ such graphs. The set $\{R \sqcup F \mid F \in \mathcal{F}\}$ (where $\sqcup$ denotes disjoint union) of graphs universal in $\mathcal{I}$ has $2^{\aleph_0}$ pairwise non-isomorphic graphs. This follows from the observation that every vertex of $R$ is of denumerable degree while every vertex of every linear forest $F \in \mathcal{F}$ is of degree one or two. Hence if the isomorphism $\alpha : R \sqcup F_1 \cong R \sqcup F_2$, then $\alpha$ has to map every vertex of $R$ to a vertex of $R$ and correspondingly for vertices outside $R$, so that $F_1 \cong F_2$.                               ∎

If we write $\mathcal{U}(\mathcal{P})$ for the class of graphs which are universal in $\mathcal{P}$, and $|\mathcal{U}(\mathcal{P})|$ for the cardinality of a skeleton of $\mathcal{U}(\mathcal{P})$, then what this lemma says is (more concisely) that $|\mathcal{U}(\mathcal{I})| = 2^{\aleph_0}$.

Given such an abundance of graphs universal in $\mathcal{I}$, one may be interested in sets $\mathcal{S}$ of graphs such that every $G \in \mathcal{S}$ is universal in $\mathcal{I}$, but moreover has certain specified attributes (like being connected and self-complementary—which $R \sqcup F$ above is not) or relations like being induced subgraphs of some special type of other graphs in $\mathcal{S}$. As an example of such a denumerable set $\mathcal{S}$ we present the next theorem. In order to formulate it, we define a sequence of graphs based on the following (new) graph operation: For a graph $G$ and any countable non-zero cardinal $\kappa$, we define the graph $G^{(\kappa)}$ by taking $\kappa$ pairwise disjoint copies of $G$ on pairwise disjoint vertex sets and additionally joining each vertex of each copy to the neighbours of the corresponding vertex in each other copy too.

The required sequence $\mathcal{S}$ of graphs, indexed by $I = \{1, 2, \ldots, \aleph_0\}$, is now defined by specifying that for each non-zero countable cardinal $i \in I$ the graph $G_i$ is $R^{(i)}$ (where $G_1$ is of course the Rado graph $R$ itself). Note that $R$ is an induced subgraph of each $G_i$, which is a denumerable graph, so that each $G_i$ is indeed a universal graph in the property $\mathcal{I}$ of all countable graphs.

**Theorem 1.** *The denumerable set $\{G_i \mid i \in I\}$ of these denumerable graphs $G_i$, indexed by $I = \{1, 2, \ldots, \aleph_0\}$, has the following properties:*
 (i) *for every $i \in I$, $\leq G_i = \mathcal{I}$, so every $G_i$ is universal in the property of all countable graphs;*
 (ii) *for every $i \in I$, $G_i$ is connected;*
 (iii) *for every $i \in I$, $G_i$ is self-complementary;*
 (iv) *for every vertex $v \in V(G_i)$ there are exactly $i - 1$ vertices of $G_i$ different from $v$ with the same neighbourhood as $v$ (where $\aleph_0 - 1 = \aleph_0$);*
 (v) *the distance beween any two vertices of $G_i$ is either $1$ or $2$, so $G_i$ has diameter $2$;*
 (vi) *the $G_i$'s are pairwise non-isomorphic: if $i, j \in I, i \neq j$, then $G_i \not\cong G_j$;*

(vii) *for all $i, j \in I, i \neq j$, both $G_i < G_j$ and $G_j < G_i$; and*

(viii) *for any permutation $p : I \to I, G_{p(1)} < G_{p(2)} < \cdots < G_{p(\aleph_0)}$.*

**Proof.** (i) is the remark before the theorem and (ii) is trivial.

To prove (iii) we remark that, for any index $i$, any isomorphism from $R$ onto its complement $\overline{R}$ can be applied to each copy of $R$ in $G_i$ to obtain an isomorphism from $G_i$ to $\overline{G_i}$.

(iv) follows immediately from the construction of the $G_i$'s, (v) follows from the corresponding property of $R$, while (vi) is a consequence of (iv).

To see that (vii) is true: if $i < j$, then $G_i < G_j$ by the construction of the $G_i$'s, while $G_j < G_i$ since $G_j$ is a denumerable graph while $G_i$ is universal in the property of all countable graphs.

Finally, (viii) is a consequence of (vii). ∎

## 3.   Universality for i-h Properties

For any property $\mathcal{P} \subseteq \mathcal{I}$ we use, in this and in the next section, the notations $\mathcal{P}_f = \{G \in \mathcal{P} \mid |G| \text{ is finite}\}$ and $\mathcal{P}_d = \{G \in \mathcal{P} \mid |G| \text{ is denumerable}\}$. Let $\mathcal{P}$ be any i-h property. Since $\mathcal{P} \subseteq \mathcal{I}$, any graph universal in $\mathcal{I}$ is universal for $\mathcal{P}$. Hence, by Lemma 2, we know that there are exactly $2^{\aleph_0}$ pairwise non-isomorphic graphs which are universal for $\mathcal{P}$. If $\mathcal{P} \subset \mathcal{I}$, then many of these, for some $\mathcal{P}$ even all of them, may be outside $\mathcal{P}$.

Examples of $\mathcal{P} \subset \mathcal{I}$ and $G$ universal for $\mathcal{P}$ where $G \notin \mathcal{P}$, or it is not known whether $G \in \mathcal{P}$ or not—excluding the usual suspects like $G \cong R$ or the $G_i$ of Theorem 1—are known. The universal graphs $U$ B-constructed for any $\mathcal{P}$ in [6] Theorem 3 sometimes may contingently happen to be in $\mathcal{P}$, but at least employed $\mathcal{P}$ explicitly in their construction. The same holds for the universal graphs $X(\mathcal{P})$ (with $\mathcal{P}$-extensibility—see Section 4) for any $\mathcal{P}$ in [5], Theorem 3 and Corollary 3.1.

Examples elsewhere in the literature, especially those $\mathcal{P}$ for which it is claimed that $\mathcal{U}(\mathcal{P}) = \emptyset$ (so that the many graphs universal for $\mathcal{P}$ are all outside $\mathcal{P}$), are summarised in the table below. In this table, $\mathcal{S}$ denotes a finite set of cycles and $\mathcal{S}_k$, for $k$ a positive integer, denotes the set of odd cycles $\{C_3, C_5, \ldots, C_{2k+1}\}$. Furthermore, for a given countable set of connected finite graphs $\mathcal{T}$, the (additive and) i-h graph property $-\mathcal{T}$ is defined by

$-\mathcal{T} = \{G \in \mathcal{I}_f \mid \text{for each } T \in \mathcal{T}, T \text{ is not an induced subgraph of } G\}$.

| Property | Description | $U \in \mathcal{P}_d$? | Reference |
|----------|-------------|------------------------|-----------|
| $\mathcal{P}_{fin}$ | Graphs with all vertices of finite degree | Does not exist in $\mathcal{P}_{fin}$ | [21] (accredited to N.G. de Bruijn) |
| $-\{K_{m,n}\}$ | $K_{m,n}$-free graphs | Exists if and only if $m = 1$ and $n \leq 3$ | [18] |
| $-\{C_4\}$ | $C_4$-free graphs | Does not exist in $-\{C_4\}$ | [16] |
| $-\{C_n\}$, $n \geq 5$ | $C_n$-free graphs | Does not exist in $-\{C_n\}$ | [9] |
| $-\mathcal{S}$ | Limited cycle-free graphs | Exists in $-\mathcal{S}$ if and only if $\mathcal{S} = \mathcal{S}_k$ | [11] |

## 4.   Universality in Some i-h Properties: When $\mathcal{U}(\mathcal{P}) \neq \emptyset$

In this section we are interested in i-h properties $\mathcal{P} \subset \mathcal{I}$, with $\mathcal{U}(\mathcal{P}) \neq \emptyset$, i.e., with at least one graph universal in $\mathcal{P}$—of which we have already seen two examples in the table above. (The property $\mathcal{I}$ has been treated extensively in Section 2.)

We start by listing some results in the next theorem of which the proofs are easy.

**Theorem 2.** *Let $G$ be any graph.*
 (i)  *$\leq G$ is an i-h graph property;*
 (ii)  *$G$ is universal in $\leq G$ (so any graph is universal in at least one i-h property); and*
 (iii)  *If $G$ is finite, then $G$ is (up to isomorphism) the unique universal graph in $\leq G$.*

It is of interest to observe that for i-h properties with (even uncountably) many universal members as in Lemma 2 (and Lemma 3 below), there may be exactly one of those members which has the "extension property". We say that a graph $G$ has the (classical) *extension property* when the following holds: For every two finite disjoint sets $U$ and $V$ of vertices of $G$ there is a vertex not in $U \cup V$ which is adjacent to every vertex in $U$ and to no vertex in $V$. Now, among the uncountably many graphs universal in $\mathcal{I}$ (Lemma 2) there is, up to isomorphism, exactly one—namely the Rado graph $R$—with the extension property. Not only in $\mathcal{U}(\mathcal{I})$, but even in the whole of $\mathcal{I}$, $R$ is the unique graph with the extension property, which hence characterises $R$ among all countable graphs.

In [5] the extension property is generalized from $\mathcal{I}$ to any i-h property $\mathcal{P}$ as the "$\mathcal{P}$-extension property". There Theorem 6 then proves that $\mathcal{P}$ has *at most* one member with the $\mathcal{P}$-extension property. So, by imposing an apt strict extra attribute—like some form of extensibility, $\mathcal{E}$—one may cull down $|\mathcal{U}(\mathcal{P}) \cap \mathcal{E}|$ to

zero or one. In [4], Theorem 2, $\mathcal{U}(\to H)$ is culled down by the attribute $\mathcal{E} =$ "$H$-extensibility" to a singleton: $\mathcal{U}(\to H) \cap \mathcal{E} = \{U(H)\}$.

In [15] and [17] the graph $G_k \in \mathcal{U}(-\{K_{k+2}\})$ is the unique universal one having an adapted extension property. A similar result holds for $L_k \in \mathcal{U}(\mathcal{L}_k)$ in [3].

The concepts of extensibility and homogeneity used in the characterisation of the Rado graph, as relativised in [5] to $\mathcal{P}$-extensibility and $\mathcal{P}$-homogeneity respectively, yield in that paper the following result:

**Theorem 3.** *If $G$ is universal in the i-h property $\mathcal{P}$ and $G$ has $\mathcal{P}$-extensibility or $\mathcal{P}$-homogeneity, then $G$ is (up to isomorphism) the unique universal graph in $\mathcal{P}$ with $\mathcal{P}$-extensibility or $\mathcal{P}$-homogeneity—and is in fact both $\mathcal{P}$-extensible and $\mathcal{P}$-homogeneous.*

The next result is also of interest although its proof is easy.

**Theorem 4.** *Let $G$ be any graph and $\mathcal{P}$ any i-h property. Then $G$ is universal in $\mathcal{P}$ if and only if $\mathcal{P} = \leq G$.*

**Corollary 1.** *If there exists at least one graph which is universal in both the i-h properties $\mathcal{P}$ and $\mathcal{Q}$, then $\mathcal{P} = \mathcal{Q}$.*

In the following table we summarise a few instances from the literature where $\mathcal{U}(\mathcal{P}) \neq \emptyset$, i.e., where there is at least one graph universal in the i-h property $\mathcal{P}$. In this table we also indicate, where it is known, a brief description of a characterisation of a (the) universal graph in $\mathcal{P}$. For a given finite (or countable) graph $H$ the induced-hereditary graph property $\to H$ is defined by $\to H = \{G \in \mathcal{I}_f :$ there is a homomorphism from $G$ into $H\}$ (using $G \in \mathcal{I}$ if $H$ is countable, respectively). Throughout this table, $k$, $m$ and $n$ are positive integers.

In Theorem 2(iii) we have seen examples where $|\mathcal{U}(\mathcal{P})| = 1$, namely $\mathcal{P} = \leq G$ for a *finite* $G$, with $G$ (up to isomorphism) as the unique element of $\mathcal{U}(\leq G)$. We end this section by showing that many i-h properties $\mathcal{P}$ indeed have $2^{\aleph_0}$ pairwise non-isomorphic graphs which are universal in $\mathcal{P}$, just like $\mathcal{I}$ (Lemma 2).

**Lemma 3.** *Let $\mathcal{P}$ be an additive i-h property containing the set of all linear forests $\mathcal{F}$, and for which there is a universal graph $G$ in $\mathcal{P}$ of which every vertex is of degree at least three. Then there exist exactly $2^{\aleph_0}$ pairwise non-isomorphic graphs which are universal in $\mathcal{P}$.*

**Proof.** Almost the same as the proof of Lemma 2, with $G$ playing the role $R$ plays there. ∎

| Property | Description | $U \in \mathcal{P}_d$? | Characteri-sation of $U$? | Refere-nce(s) |
|---|---|---|---|---|
| $\mathcal{I}$ $= -\emptyset$ | All graphs | Yes, the Rado graph $R \in \mathcal{I}$ | $C \cong R$ iff $C$ has the extension property | [21] |
| $-\{K_{k+2}\}$ | $K_{k+2}$-free graphs | Yes, the graph $G_k \in -\{K_{k+2}\}$ | $C \cong G_k$ iff $C$ has an adapted extension property | [15], [17] |
| $-\{K_{m,n}\}$ | $K_{m,n}$-free graphs | Exists if and only if $m = 1$ and $n \leq 3$ | | [18] |
| $-\{C_3\}$ | $C_3$-free graphs | Yes, the graph $G_1 \in -\{C_3\}$ | Same as $K_3$-free graphs above | |
| $-\mathcal{S}$ | Limited cycle-free graphs | Exists in $-\mathcal{S}$ if and only if $\mathcal{S} = \mathcal{S}_k$ | | [11] |
| $\rightarrow H$ | Hom-property for finite $H$ | Known to exist in $\rightarrow H$ | | [20] |
| $\rightarrow H$ | Hom-property for countable $H$ | Known to exist in $\rightarrow H$ | Two characteri-sations known | [4] |
| $\mathcal{C}_k$ | Colouring number at most $k + 1$ | Known to exist in $\mathcal{C}_k$ | | [7] |
| $\mathcal{L}_k$ | Directed label-led graphs | Yes, the graph $L_k \in \mathcal{L}_k$ | $C \cong L_k$ iff $C$ has the $k$-extension property | [3] |

## 5.   For Most Properties We Have $\mathcal{U}(\mathcal{P}) = \emptyset$

We immediately have, by Theorem 4, that $\mathcal{U}(\mathcal{P}) = \emptyset$ if and only if, for all $G \in \mathcal{I}$, $\leq G \neq \mathcal{P}$. The number of properties $\leq G$ is at most equal to the number of graphs $G \in \mathcal{I}$, namely $|\mathcal{I}| = 2^{\aleph_0}$. But the number of i-h properties is $|\mathbb{K}_{\leq}| = 2^{(2^{\aleph_0})}$ by Corollary 3 below. Hence, for the overwhelmimg majority of i-h properties we have that $\mathcal{U}(\mathcal{P}) = \emptyset$.

**Theorem 5.** *For any i-h property $\mathcal{P}$, $\mathcal{P} = \bigcup\{\leq G \mid G \in \mathcal{P}\}$.*

**Proof.** Immediate.                                                                      ∎

We remark that Theorem 5 demonstrates that any i-h property (even one in the large majority of those without a universal member) is the union of (maybe uncountably many) properties with a universal member, those having the special attributes mentioned in Theorem 2. This theorem also illustrates that a graph (say universal for $\mathcal{P}$) may be universal for even an uncountable number of different properties (like the $\leq G$ for $G$ in $\mathcal{P}$). Lastly, this theorem demonstrates that in the lattice $\mathbb{K}_{\leq}$, even though this lattice is not closed under taking unions of the i-h properties in it [1], each of its elements is the union of those i-h properties

below it which are "generated" by single elements; a result reminding one of the distinguishing feature of an algebraic lattice in which the compact elements play this role [13].

By Corollary 1 we have the following pleasing situation: If $\mathcal{P}_1$ and $\mathcal{P}_2$ are i-h properties and $G_1$ and $G_2$ are countable graphs with $G_i$ universal in $\mathcal{P}_i$, $i = 1, 2$, then

$$\mathcal{P}_1 \neq \mathcal{P}_2 \text{ implies } G_1 \not\cong G_2.$$

Hence the cardinality of the set of i-h properties of countable graphs with universal graphs in them is at most that of the set of countable graphs. This latter set has cardinality at most $2^{\aleph_0}$ since each such graph is an induced subgraph of the Rado graph $R$, a graph on $\aleph_0$ vertices, Lemma 1(i).

Next we consider i-h properties of graphs determined by forbidding sets: If $\mathcal{X}$ is any set of graphs we denote the set of graphs defined by forbidding members of $\mathcal{X}$ (similar to the definition of $-\mathcal{T}$ in Section 3, but now not restricted to finite graphs) as induced subgraphs by $-\mathcal{X}$, i.e.,

$$-\mathcal{X} = \{G \in \mathcal{I} \mid X \not\leq G \text{ for all } X \in \mathcal{X}\}.$$

Note that if every element of $\mathcal{X}$ is a finite graph, then $-\mathcal{X}$ is a property of finite character. Also, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are sets of cycles, then it is easy to see that they satisfy the implication

$$\mathcal{S}_1 \neq \mathcal{S}_2 \text{ implies } -\mathcal{S}_1 \neq -\mathcal{S}_2.$$

One way of proving this implication is as follows: Show that the set $\mathcal{S}$ of all cycles satisfies the implication

$$X, Y \in \mathcal{S} \text{ and } X \leq Y \text{ imply } X \cong Y.$$

This then shows that there are at least as many i-h properties of the form $-\mathcal{S}_1$ with $\mathcal{S}_1 \subseteq \mathcal{S}$ as there are subsets of the set $\mathcal{S}$ of cycles, i.e., at least $2^{\aleph_0}$.

Using the same type of argumentation as above, we now work towards showing that there are indeed at least $2^{(2^{\aleph_0})}$ distinct i-h properties. This is accomplished by constructing a set $\mathcal{G}$ of graphs satisfying the same implications as above satisfied by the set of all cycles; this set, however, will be of cardinality at least $2^{\aleph_0}$, with hence at least $2^{(2^{\aleph_0})}$ subsets.

Consider any sequence $\mathbf{k} = k_1, k_2, \ldots$ of positive integers with each $k_i \geq 3$ and construct, for each such sequence, a graph $G_{\mathbf{k}}$ as follows: $G_{\mathbf{k}}$ has denumerably many components, one for each positive integer $i$, consisting of a cycle $C_{i+3}$ with $k_i$ additional vertices which induce a path $P_{k_i}$. Furthermore, there is a designated vertex $v_i$ for each cycle $C_{i+3}$ and all the vertices of degree two of the path $P_{k_i}$ are adjacent to the vertex $v_i$ on the cycle $C_{i+3}$—see the diagram of a typical

component of $G_{\mathbf{k}}$ below.
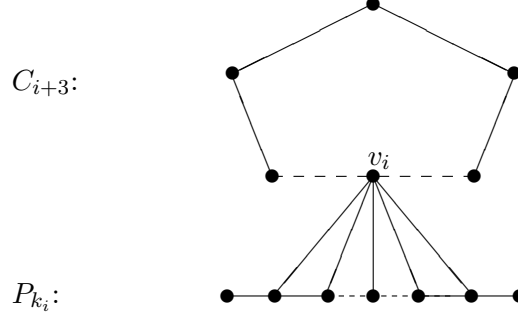


A typical component of $G_{\mathbf{k}}$

Now let $\mathbf{k} = k_1, k_2, \ldots$ and $\mathbf{m} = m_1, m_2, \ldots$ be any two permissible such sequences. Note that each $G_{\mathbf{k}}$ has exactly one induced subgraph $C_n$ for each $n \geq 4$; hence any possible isomorphism from $G_{\mathbf{k}}$ to a $G_{\mathbf{m}}$ should take the vertices of such a $C_n$ in $G_{\mathbf{k}}$ to the vertices of the corresponding $C_n$ in $G_{\mathbf{m}}$. This fact is now used in

**Lemma 4.** *If $\mathbf{k} \neq \mathbf{m}$, then $G_{\mathbf{k}} \ncong G_{\mathbf{m}}$.*

**Proof.** If $\mathbf{k} \neq \mathbf{m}$, then $k_i \neq m_i$ for some $i$. But then the lengths of the paths attached to the cycle $C_{i+3}$ are different and it is impossible to find an isomorphism between $G_{\mathbf{k}}$ and $G_{\mathbf{m}}$ since such a function should take the copy of $C_{i+3}$ in $G_{\mathbf{k}}$ with the path attached to it to the copy of $C_{i+3}$ in $G_{\mathbf{m}}$ with the path attached to it. ∎

The idea of the proof of this lemma is taken further in Theorem 6 below; the proof of this theorem is a refinement of the above proof.

Let $\mathcal{G}$ be the set $\mathcal{G} = \{G_{\mathbf{k}} \mid$ there exist integers $k_i \geq 3$ such that $\mathbf{k} = k_1, k_2, \ldots\}$. First we note

**Corollary 2.** *There are $2^{\aleph_0}$ graphs in the set $\mathcal{G}$.*

**Proof.** There are $2^{\aleph_0}$ such sequences of positive integers. ∎

**Theorem 6.** *If $G_{\mathbf{k}}, G_{\mathbf{m}} \in \mathcal{G}$ and $G_{\mathbf{k}} \leq G_{\mathbf{m}}$, then $G_{\mathbf{k}} \cong G_{\mathbf{m}}$.*

**Proof.** Suppose $G_{\mathbf{k}}, G_{\mathbf{m}} \in \mathcal{G}$ and $G_{\mathbf{k}} \leq G_{\mathbf{m}}$ with $\psi$ an embedding from $G_k$ into $G_m$. Then each cycle $C_{i+3}$ of $G_{\mathbf{k}}$ gets mapped by $\psi$ onto the corresponding cycle $C_{i+3}$ of $G_{\mathbf{m}}$. Consequently, for each $i$ the vertex $v_i$ of the cycle $C_{i+3}$ of $G_{\mathbf{k}}$, being the only one of degree greater than two in this $C_{i+3}$, gets mapped onto the corresponding vertex of the corresponding cycle $C_{i+3}$ of $G_{\mathbf{m}}$. But then the induced

path attached to this vertex in $G_{\mathbf{k}}$ has to be mapped onto the corresponding path attached to the cycle $C_{i+3}$ of $G_{\mathbf{m}}$; if this part of the mapping is not onto, then the $i$'th component of $G_{\mathbf{k}}$ is not an induced subgraph of $G_{\mathbf{m}}$ as assumed, given the restriction on the adjacencies of $v_i$ to the vertices of its associated paths $P_{k_i}$ and $P_{m_i}$. Hence $k_i = m_i$ for each $i$, i.e., $\mathbf{k} = \mathbf{m}$ and it follows that $G_{\mathbf{k}} \cong G_{\mathbf{m}}$.     ∎

We are now ready to work towards finding the cardinality of $\mathbb{K}_\leq$.

**Theorem 7.** *If $\mathcal{S}$ and $\mathcal{T}$ are subsets of $\mathcal{G}$ and $\mathcal{S} \neq \mathcal{T}$, then $-\mathcal{S} \neq -\mathcal{T}$.*

**Proof.** If $\mathcal{S} \neq \mathcal{T}$, then there is a graph $G_{\mathbf{m}}$ which is in exactly one of them. Suppose without loss of generality that $G_{\mathbf{m}}$ is in $\mathcal{S}$ but $G_{\mathbf{m}}$ is not in $\mathcal{T}$. Then $G_{\mathbf{m}}$ is not in $-\mathcal{S}$ since it is a forbidden graph for it. It is in $-\mathcal{T}$ since if not, then some graph $G_{\mathbf{k}} \in \mathcal{T}$ is an induced subgraph of it. But then $G_{\mathbf{k}} \cong G_{\mathbf{m}}$ by the above theorem and hence $G_{\mathbf{m}} \in \mathcal{T}$, contrary to our assumption.     ∎

The next result can now be proven as an easy consequence; it reminds one of our motivating example of sets of cycles.

**Theorem 8.** *There are $2^{(2^{\aleph_0})}$ distinct i-h properties of the form $-\mathcal{S}$ with $\mathcal{S} \subseteq \mathcal{G}$.*

**Proof.** There are as many distinct i-h properties of the form $-\mathcal{S}$ with $\mathcal{S} \subseteq \mathcal{G}$ as there are subsets of the set $\mathcal{G}$ by Theorem 7, i.e., $2^{(2^{\aleph_0})}$.     ∎

**Corollary 3.** $|\mathbb{K}_\leq| = 2^{(2^{\aleph_0})}$.

**Proof.** This follows from Lemma 1(ii) and Theorem 8.     ∎

## 6.   Recapitulation and Outlook

We can conclude that the $2^{(2^{\aleph_0})}$ many i-h properties have considerably fewer—yes, at most $2^{\aleph_0}$—graphs available as candidates to serve as universal graphs in some of them. Even though we introduce no probabilities into this world of infinities, this means that if you would pick an arbitrary i-h property, it seems highly unlikely that you would have one with a universal graph in it. An ultimate challenge in this context would be to find an elegant characterisation of all and only those $\mathcal{P}$ for which $\mathcal{U}(\mathcal{P})$ is not empty—a task beyond the horizon of our present outlook, maybe a mission impossible. Let us elaborate somewhat on this issue.

We define $\mathcal{U}_\leq := \{\mathcal{P} \in \mathbb{K}_\leq \mid \mathcal{U}(\mathcal{P}) \neq \emptyset\}$. By the grace of Theorem 4 and Corollary 1 it is clear that $\mathcal{U}_\leq = \{\leq G \mid G \in \mathrm{Skel}(\mathcal{I})\}$; so, seen from the perspective of $\mathcal{I}$ (so to speak) it is trivial to identify $\mathcal{U}_\leq$. But from the perspective of the big $\mathbb{K}_\leq$ it is extremely difficult to distinguish its small subset $\mathcal{U}_\leq$, to decide from

some mathematical description of a $\mathcal{P} \in \mathbb{K}_{\leq}$ whether that $\mathcal{P}$ is in $\mathcal{U}_{\leq}$ or not. The challenge is aggravated when the same $\mathcal{P}$ has seemingly disparate mathematical descriptions. Here comes a famous example. Let $\mathcal{P} := \{G \in \mathcal{I} \mid \chi(H) = \omega(H)$ for all $H \leq G\}$, the i-h property of all graphs for which every induced subgraph has equal chromatic and clique number. Seemingly quite incommensurable is the *Berge* property $\mathcal{Q} := \{G \in \mathcal{I} \mid G$ contains no odd hole or antihole$\}$. And yet, a famous mathematical result, the Strong Perfect Graph Theorem [12], demonstrates that $\mathcal{P} = \mathcal{Q} = \mathcal{P}erf$, the i-h property of perfect graphs. The "positive" $\mathcal{P}$ (equating graph parameters) turns out to be equivalent to the "negative" $\mathcal{Q}$ (forbidding certain induced subgraphs).

When we muster the i-h properties $\mathcal{P}$ which are mentioned explicitly in this paper as having some information known on the cardinality of $\mathcal{U}(\mathcal{P})$, they represent mainly two types: properties *with forbidding*, and properties *with assignment*. This suggests two plausible strategies for investigating universality in these two types of property. Both involve the *form* of how the property is formulated in mathematical language. The i-h properties *with forbidding* that we mentioned are the following, all of the form $-\mathcal{T}$, prohibiting a graph in the property from having any induced subgraph from the countable set $\mathcal{T}$ of connected, finite graphs:

$-\{K_{m,n}\}$, forbidding some complete bipartite graph, with $\mathcal{U}(\mathcal{P})$ empty except for the three cases forbidding stars with $min(m,n) = 1$ and $max(m,n) \leq 3$;

$-\mathcal{S}$, forbidding some finite set of cycles, with $\mathcal{U}(\mathcal{P})$ empty except for the cases when forbidden $\mathcal{S}$ is a set $\mathcal{S}_k = \{C_3, C_5, \ldots, C_{2k+1}\}$ of odd cycles;

$-\{C_n\}$, with $n \geq 4$, all $n$-cycle-free graphs, with $\mathcal{U}(\mathcal{P})$ empty;

$-\{K_{k+2}\}$, $K_{k+2}$-free graphs (including the case of $C_3$-free graphs), with $\mathcal{U}(\mathcal{P})$ non-empty.

Substantial work has already been done on universality in such properties which forbid specific induced substructures, for graphs and also for more general relational structures, in e.g. [10] and [19]. That work links universality to aspects of logic, set theory, model theory, constraint satisfaction problems, and complexity theory, amply demonstrating the inherent difficulty of the relevant questions. Here a broad approach is indicated.

The notion of i-h graph properties *with assignment* was introduced in [7], Section 4, by the following definition: We say that $\mathcal{P}$ is a property *with assignment* when (a part of) the definition of $\mathcal{P}$ stipulates an instance of the following schema: "For a graph $G$ to be in $\mathcal{P}$ it is necessary that there exists a finite, non-empty set $A = \{f_1, f_2, \ldots, g_1, g_2, \ldots\}$, where each $f_i$ is a function defined on $V(G)$ and each $g_j$ is a function defined on $E(G)$, and these functions satisfy ...". The definition of such a property $\mathcal{P}$ of graph $G$ then involves not just the internal graph-theoretical structure of $G$, but also links extraneous mathematical structure to $G$ globally. The general graph-theoretic notions of induced subgraph, i-h property, and homomorphism (including isomorphism)—which co-determine

universality in an i-h property $\mathcal{P}$—evoke kindred but slightly strengthened notions when $\mathcal{P}$ has assignment. This happens because now these notions (unlike anything in the case of forbidding) have to *respect the assignment*. It is rather striking that for all three the i-h properties with assignment at the bottom of the table in Section 4—namely $\rightarrow H$, $\mathcal{C}_k$ and $\mathcal{L}_k$—explicit constructions utilizing $\mathcal{P}$ of an element of $\mathcal{U}(\mathcal{P})$ are known. It is almost as if in these cases the assignment embodied in $\mathcal{P}$ forces, or at least guides, the construction of a universal member in $\mathcal{P}$. Here is the nature of the assignments in these three cases:

(i) $G \in \rightarrow H$ if and only if there exists a function $f_1 : V(G) \rightarrow V(H)$ which is a graph homomorphism $f_1 : G \rightarrow H$ (and we then say that $G$ is $H$-*colourable*).

(ii) $G \in \mathcal{C}_k$ if and only if there exists a bijection $f_1 : V(G) \rightarrow B$, where $B$ is some well-ordered set, with specified properties involving $k$ and degree restrictions on the vertices of $G$ as stipulated in Definition 5 of [7], (and we then say that $G$ *has colouring number at most* $k + 1$).

(iii) Pick a positive integer $k$ and a set $L_v$ of $k$ vertex-labels and a set $L_e$ of $k$ edge-labels. $G \in \mathcal{L}_k$ if and only if $G$ has an assignment $A = \{f_1, g_1, g_2\}$, where $f_1 : V(G) \rightarrow L_v$; $g_1 : E(G) \rightarrow V(G) \times V(G)$, assigning an orientation to each edge; and $g_2 : E(G) \rightarrow L_e$ (and we then say that $G$ is a *directed labelled graph*).

It would be interesting to identify more i-h properties $\mathcal{P}$ with assignment and investigate whether the nature of the assignment perhaps directs us towards the construction of an element in $\mathcal{U}(\mathcal{P})$.

Finally, we note that the mother of all i-h properties, $\mathcal{I}$, belongs in trivial ways to both the type of properties with forbidding and the type of properties with assignment: $\mathcal{I} = -\emptyset$; and for any $G \in \mathcal{I}$, its vertex set $V(G)$ can always be indexed or labelled by some set of positive integers.

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