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STRONG EQUALITY BETWEEN THE ROMAN DOMINATION AND INDEPENDENT ROMAN DOMINATION NUMBERS IN TREES

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Abstract

A Roman dominating function (RDF) on a graph $G = (V, E)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. An RDF f in a graph G is independent if no two vertices assigned positive values are adjacent. The Roman domination number $\gamma_R(G)$ (respectively, the independent Roman domination number $i_R(G)$) is the minimum weight of an RDF (respectively, independent RDF) on G. We say that $\gamma_R(G)$ strongly equals $i_R(G)$, denoted by $\gamma_R(G) \equiv i_R(G)$, if every RDF on G of minimum weight is independent. In this paper we provide a constructive characterization of trees T with $\gamma_R(T) \equiv i_R(T)$.

Keywords: Roman domination, independent Roman domination, strong equality, trees.

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1. Introduction

We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N_G[v] = N_G(v) \cup \{v\}$. If D is a subset of $V(G)$, then the subgraph induced by D in G is denoted by $G[D]$. The *degree* of v, denoted by $d_G(v)$, is the cardinality of its open neighborhood. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. If v is a support vertex, then L_v will denote the set of the leaves attached at v. A support vertex v is called *strong* if $|L_v| > 1$. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at each support vertex is denoted by $S_{p,q}$. For a vertex v in a rooted tree T, we denote by $D(v)$ the set of all descendants of v. The *maximal subtree* at v is the subtree of T induced by $D(v) \cup \{v\}$, and is denoted by T_v.

For a graph G, let $f : V(G) \to \{0, 1, 2\}$ be a function, and let $(V_0; V_1; V_2)$ be the ordered partition of $V = V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2$. There is a $1-1$ correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partitions $(V_0; V_1; V_2)$ of $V(G)$. So we will write $f = (V_0; V_1; V_2)$.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G if every vertex u of G for which $f(u) = 0$ is adjacent to at least one vertex v of G for which $f(v) = 2$. The weight of a Roman dominating function f on G is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function on G. A function $f = (V_0; V_1; V_2)$ is called a $\gamma_R(G)$-function or γ_R-function for G if it is a Roman dominating function on G and $f(V(G)) = \gamma_R(G)$. Roman domination has been introduced by Cockayne *et al.* [1] and has been further studied for example in [5, 6, 7].

A function $f = (V_0; V_1; V_2)$ is called an *independent Roman dominating function* (IRDF) on G if f is an RDF and no two vertices in $V_1 \cup V_2$ are adjacent. The *independent Roman domination number* $i_R(G)$ is the minimum weight of an independent Roman dominating function of G. A function $f = (V_0; V_1; V_2)$ is called an $i_R(G)$-function or i_R-function for G if it is an IRDF on G and $f(V(G)) = i_R(G)$.

Observe that for every graph G, $\gamma_R(G) \le i_R(G)$. Clearly if G is a graph with $\gamma_R(G) = i_R(G)$, then every $i_R(G)$-function is also a $\gamma_R(G)$-function. However not every $\gamma_R(G)$-function is an $i_R(G)$-function even when $\gamma_R(G) = i_R(G)$. For example the double star $S_{2,3}$ has two $\gamma_R(S_{2,3})$-functions but only one $\gamma_R(S_{2,3})$-function is an $i_R(S_{2,3})$-function. We say that $\gamma_R(G)$ and $i_R(G)$ are *strongly equal*, denoted by $\gamma_R(G) \equiv i_R(G)$, if every $\gamma_R(G)$-function is an $i_R(G)$-function. Note that Haynes and Slater in [4] were the first to introduce strong equality between two parameters. Also in [2] and [3], Haynes, Henning and Slater gave constructive characterizations of trees with strong equality between some domination parameters.

In this paper we present a constructive characterization of trees T with $\gamma_R(T) \equiv i_R(T)$. If f is an RDF on a graph G and H is a subgraph of G, then we denote the restriction of f on H by $f|_{V(H)}$.

2. Trees T with $\gamma_R(T) \equiv i_R(T)$

We begin by the following results that will be useful for the next.

Proposition 1 (Cockayne *et al.* [1]). *Let $f = (V_0; V_1; V_2)$ be any $\gamma_R(G)$-function. Then*

(i) *The subgraph induced by the vertices of V_1 has maximum degree one.*

(ii) *No edge of G joins V_1 to V_2.*

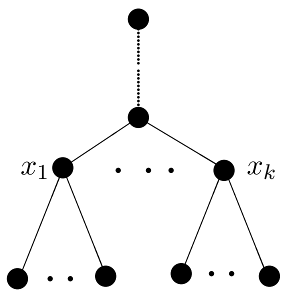

Figure 1. A tree in $\mathcal{T}$.

Proposition 2 (Jafari Rad and Volkmann [7]). *If v is a vertex in a graph G such that $\gamma_R(G - v) > \gamma_R(G)$, then $f(v) = 2$ for every $\gamma_R(G)$-function f.*

Let $\mathcal{T}$ be the family of trees that can be obtained from k ($k \geq 1$) disjoint stars of centers $x_1, x_2, \ldots, x_k$, where each star has order at least three, attached by edges from their center vertices either to a single vertex or to a same leaf of a path P_2. If T is a tree of $\mathcal{T}$, then let us call the vertex adjacent to the centers of stars, the special vertex of T. Note that if T belongs to $\mathcal{T}$, then $\gamma_R(T) \equiv i_R(T)$.

Now we present a constructive characterization of trees T with $\gamma_R(T) \equiv i_R(T)$. For this purpose, we define a family of trees as follows: Let $\mathcal{F}$ be the collection of trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ ($k \geq 1$) of trees, where T_1 is a star $K_{1,t}$ with $t \geq 2$, $T = T_k$, and, if $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the following operations. Also for any tree T_i of $\mathcal{F}$ we let f_i be a $\gamma_R(T_i)$-function.

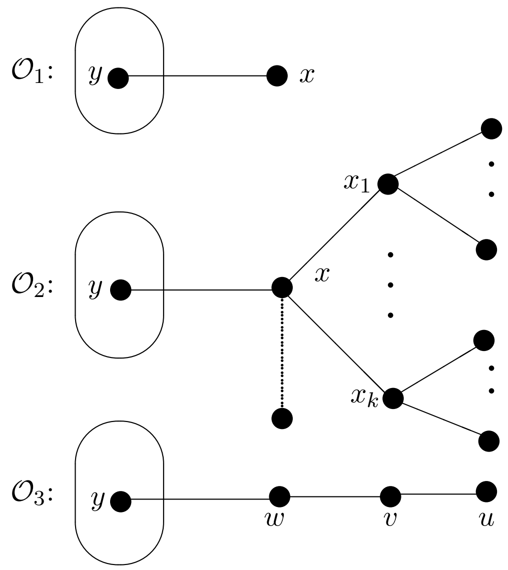

Figure 2. The $\mathcal{O}_i$ Operations.

- Operation $\mathcal{O}_1$: Assume y is a leaf of T_i with $f_i(y) = 0$ and whose support vertex z is either strong or satisfies $\gamma_R(T_i - z) > \gamma_R(T_i)$. Then T_{i+1} is obtained from T_i by adding a new vertex x and adding the edge xy.

- Operation $\mathcal{O}_2$: Assume y is a vertex of T_i. Then T_{i+1} is obtained from T_i by adding a tree $T \in \mathcal{T}$ of special vertex x and adding the edge xy with the condition that if x is a support vertex, then y satisfies $\gamma_R(T_i - y) \geq \gamma_R(T_i)$.

- Operation $\mathcal{O}_3$: Assume y is a vertex of T_i assigned 0 or 1 for every $\gamma_R(T_i)$-function. Then T_{i+1} is obtained from T_i by adding a path $P_3 = u\text{-}v\text{-}w$ and adding the edge wy.

Lemma 3. *If T_i is a tree with $\gamma_R(T_i) \equiv i_R(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_1$, then $\gamma_R(T_{i+1}) \equiv i_R(T_{i+1})$.*

Proof. Since $\gamma_R(T_i) \equiv i_R(T_i)$, it is clear that every $i_R(T_i)$-function with y assigned 0 can be extended to an IRDF for T_{i+1} by assigning 1 to x. Hence $\gamma_R(T_{i+1}) \leq i_R(T_{i+1}) \leq i_R(T_i)+1 = \gamma_R(T_i)+1$. Now let f be a $\gamma_R(T_{i+1})$-function. If $f(y) = 1$, then $f(x) = 1$ and $f|_{V(T_i)}$ is an RDF for T_i. If $f(y) = 0$, then $f(x) = 1$ (else $f(x) = 2$ and we can make a change to obtain $f(x) = 1$ and

$f(y) = 1$) and $f|_{V(T_i)}$ is an RDF for T_i. In both cases, $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 1$ and equality throughout the above chain is obtained. Now assume that $f(y) = 2$. Then $f(x) = 0$ and by Proposition 1 we may assume that $f(z) = 0$. If z has a leaf neighbor, say z', then $f(z') = 1$ and we can change $f(z') = 1$ to $f(z') = 0$, $f(z) = 0$ to $f(z) = 2$, $f(y) = 2$ to $f(y) = 0$ and $f(x) = 0$ to $f(x) = 1$. Clearly we are in the previous situation. Hence we may assume that z is not a support vertex. Then consider the function f' on $V(T_i - z)$ defined by $f'(a) = f(a)$ if $a \in V(T_i) - \{y, z\}$, and $f'(y) = 1$. Then f' is an RDF for $T_i - z$ and so $\gamma_R(T_i - z) \leq f'(V(T_i - z)) - 1 = \gamma_R(T_{i+1}) - 1$. Now since z is a support vertex in T_i but not strong, it satisfies $\gamma_R(T_i - z) > \gamma_R(T_i)$. Then we obtain $\gamma_R(T_i) < \gamma_R(T_i - z) \leq \gamma_R(T_{i+1}) - 1$, implying that $\gamma_R(T_{i+1}) > \gamma_R(T_i) + 1$, which is impossible. Thus for the next we may assume that for any $\gamma_R(T_{i+1})$-function y is not assigned 2.

Next we shall show that $\gamma_R(T_{i+1}) \equiv i_R(T_{i+1})$. Assume to the contrary that $h = (V_0; V_1; V_2)$ is a $\gamma_R(T_{i+1})$-function such that $V_1 \cup V_2$ is not independent. Thus there are two adjacent vertices $u, v \in V_1 \cup V_2$. Recall that $h(y) \in \{0, 1\}$. If $h(y) = 0$, then $h(x) = 1$, and so $h|_{V(T_i)} = (V_0; V_1 - \{x\}; V_2)$ is a $\gamma_R(T_i)$-function. But $h|_{V(T_i)}$ is not independent since u, v belong to $(V_1 - \{x\}) \cup V_2$, contradicting $\gamma_R(T_i) \equiv i_R(T_i)$. If $h(y) = 1$, then $h(x) = 1$. By Proposition 1, $h(z) \neq 2$, and so $h|_{V(T_i - z)}$ is an RDF for $V(T_i - z)$. Observe that z cannot be a support vertex in T_{i+1}. Now by using the fact that z verifies $\gamma_R(T_i - z) > \gamma_R(T_i)$, we obtain $\gamma_R(T_i) < \gamma_R(T_i - z) \leq h(V(T_i - z)) \leq \gamma_R(T_{i+1}) - 1$, which is impossible. Therefore $\gamma_R(T_{i+1}) \equiv i_R(T_{i+1})$. ■

Lemma 4. *If T_i is a tree with $\gamma_R(T_i) \equiv i_R(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_2$, then $\gamma_R(T_{i+1}) \equiv i_R(T_{i+1})$.*

Proof. Let $T \in \mathcal{T}$ be the added tree of special vertex x. Recall that T is obtained from k ($k \geq 1$) disjoint stars of centers $x_1, x_2, \ldots, x_k$, each of order at least three, attached by edges xx_j at x, where x may be a single vertex or belongs to a path $P_2 = x\text{-}x'$.

Clearly every $i_R(T_i)$-function can be extended to an IRDF for T_{i+1} by assigning 2 to every x_j, 1 to x' (if x' exists), and 0 to x and every leaf of T different to x'. Hence $\gamma_R(T_{i+1}) \leq i_R(T_{i+1}) \leq i_R(T_i) + 2k + t = \gamma_R(T_i) + 2k + t$, where $t = 1$ if x' exists and $t = 0$ otherwise. Now let f be a $\gamma_R(T_{i+1})$-function. Without loss of generality we can assume that $f(x_j) = 2$ for every j. Hence every leaf adjacent to some x_i is assigned 0. If $f(x) = 0$ and $f(x') = 1$ (if x' exists), then $f|_{V(T_i)}$ is an RDF for T_i implying that $i_R(T_i) \leq \gamma_R(T_{i+1}) - 2k - t$. Equality throughout the above inequality chain is obtained. Now if either $f(x) = 2$ and $f(x') = 0$ or $f(x) = 0$ and $f(x') = 2$, then we can change by assigning 1 to x' and y, and 0 to x. Clearly we are in the previous situation.

Assume now that $\gamma_R(T_{i+1})$ and $i_R(T_{i+1})$ are not strongly equal and let $h =$

$(V_0; V_1; V_2)$ be a $\gamma_R(T_{i+1})$-function such that $V_1 \cup V_2$ is not independent. Let u and v be any two adjacent vertices in $V_1 \cup V_2$. If $h(x) = 0$, then clearly u, v belong to $V(T_i)$ and $h|_{V(T_i)}$ is a $\gamma_R(T_i)$-function that is not independent, a contradiction with $\gamma_R(T_i) \equiv i_R(T_i)$. If $h(x) = 1$, then $h(x') = 1$ (if x' exists) and so $h|_{V(T_i)}$ is an RDF for T_i with weight $\gamma_R(T_{i+1}) - 2k - t - 1 < \gamma_R(T_i)$, which is impossible. Finally assume that $h(x) = 2$. We may assume that x' exists for otherwise we can decrease the weight of h by assigning 0 to x and 1 to y. Hence $h(x') = 0$ and $h(y) = 0$. Then $h|_{V(T_i - y)}$ is an RDF for $T_i - y$ and so $h(V(T_i - y)) = \gamma_R(T_{i+1}) - 2k - 2$. Now since x is a support vertex in T, y must satisfy $\gamma_R(T_i - y) \geq \gamma_R(T_i)$, implying that $\gamma_R(T_{i+1}) - 2k - 2 = h(V(T_i - y)) \geq \gamma_R(T_i - y) \geq \gamma_R(T_i)$. Therefore we have $\gamma_R(T_{i+1}) \geq \gamma_R(T_i) + 2k + 2$, a contradiction. Consequently $\gamma_R(T_{i+1}) \equiv i_R(T_{i+1})$. ■

Lemma 5. *If T_i is a tree with $\gamma_R(T_i) \equiv i_R(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_3$, then $\gamma_R(T_{i+1}) \equiv i_R(T_{i+1})$.*

Proof. Clearly every $i_R(T_i)$-function can be extended to an IRDF for T_{i+1} by assigning 0 to u, w and 2 to v. Hence $\gamma_R(T_{i+1}) \leq i_R(T_{i+1}) \leq i_R(T_i) + 2 = \gamma_R(T_i) + 2$. Now let f be a $\gamma_R(T_{i+1})$-function. If $f(v) = 2$, then $f(w) = f(u) = 0$ and $f|_{V(T_i)}$ is an RDF for T_i. Hence $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 2$. If $f(v) = 1$, then $f(u) = 1$ and w must be assigned 0. It follows that $f|_{V(T_i)}$ is an RDF for T_i and so $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 2$. Now assume that $f(v) = 0$. Then $f(u) = 2$ and $f(w) \notin \{1, 2\}$. It follows that $f|_{V(T_i)}$ is an RDF for T_i and so $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 2$. For all cases, we obtain $\gamma_R(T_{i+1}) = \gamma_R(T_i) + 2$, implying that $i_R(T_{i+1}) = i_R(T_i) + 2$.

Assume now that $\gamma_R(T_{i+1})$ is not strongly equal to $i_R(T_{i+1})$ and let h be a $\gamma_R(T_{i+1})$-function that is not independent. Thus there are two adjacent vertices a and b assigned positive values. If $h(v) = 2$, then $h(w) = h(u) = 0$ and $h|_{V(T_i)}$ is a $\gamma_R(T_i)$-function, where $a, b \in V(T_i)$, contradicting $\gamma_R(T_i) \equiv i_R(T_i)$. If $h(v) = 1$, then $h(u) = 1$ and $h(w) = 0$. It follows that $h(y) = 2$ and $h|_{V(T_i)}$ is a $\gamma_R(T_i)$-function for which y is assigned 2, a contradiction with the construction. Thus we assume that $h(v) = 0$. Hence $h(u) = 2$. If $h(w) = 1$, then $h|_{V(T_i)}$ is an RDF for T_i of weight $\gamma_R(T_i) - 1$, which is impossible. If $h(w) = 2$, then we change $h(w) = 2$ to $h(w) = 1$ and $h(y) = 0$ to $h(y) = 1$ and we obtain the previous situation. Thus $h(w) = 0$ implying that $h(y) = 2$. But then $h|_{V(T_i)}$ is a $\gamma_R(T_i)$-function for which y is assigned 2, a contradiction with the construction. Therefore $\gamma_R(T_{i+1}) \equiv i_R(T_{i+1})$. ■

We now are ready to establish our main result.

Theorem 6. *Let T be a tree. Then $\gamma_R(T) \equiv i_R(T)$ if and only if $T = K_1$ or $T \in \mathcal{F}$.*

Proof. Obviously, if $T = K_1$, then $\gamma_R(T) \equiv i_R(T)$. Now suppose that $T \in \mathcal{F}$. Then there is a sequence of trees $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ such that T_1 is a star $K_{1,t}$

with $t \geq 2$, $T = T_k$, and, if $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by an operation $\mathcal{O}_1$, $\mathcal{O}_2$ or $\mathcal{O}_3$ for $i = 1, \ldots, (k-1)$. We use an induction on the number of operations performed to construct T. Clearly the property is true if $k = 1$. This establishes the basis case. Assume now that $k \geq 2$ and that the result holds for all trees $T \in \mathcal{F}$ that can be constructed from a sequence of length at most $k-1$, and let $T' = T_{k-1}$. By the induction hypothesis, $\gamma_R(T') \equiv i_R(T')$. By construction T is obtained from T' by using Operation $\mathcal{O}_1$, $\mathcal{O}_2$ or $\mathcal{O}_3$. Hence by Lemmas 3, 4 and 5 it follows that $\gamma_R(T) \equiv i_R(T)$.

Conversely, let T be a tree of order n with $\gamma_R(T) \equiv i_R(T)$. Clearly if $n = 1$, then $T = K_1$. Hence we assume that T has order $n \geq 2$. We use an induction on the order n. Since a path P_2 has a $\gamma_R(P_2)$-function that is not independent, we assume that $n \geq 3$. If $n = 3$, then $T = P_3$ which belongs to $\mathcal{F}$, establishing the base case. Assume that every tree T' of order $2 \leq n' < n$ with $\gamma_R(T') \equiv i_R(T')$ is in $\mathcal{F}$. Let T be a tree of order n with $\gamma_R(T) \equiv i_R(T)$ and let f be a $\gamma_R(T)$-function. Since stars of order at least three belong to $\mathcal{F}$, we may assume that T has diameter at least three. If diam(T)= 3, then T is a double star $S_{1,p}$ with $p \geq 1$ and $T \in \mathcal{F}$ because it is obtained from a star $K_{1,p+1}$ by using Operation $\mathcal{O}_1$. Therefore assume that diam$(T)\geq 4$.

We now root T at a leaf r of a longest path. Let u be a vertex at distance diam$(T) - 1$ from r on a longest path starting at r such that $|L_u|$ is as small as possible. Let v, w be the parents of u and v on this path, respectively. Clearly $f(u) \neq 1$, else u and its leaves belong to V_1, contradicting $\gamma_R(T) \equiv i_R(T)$. We consider the following cases.

Case 1. $f(u) = 2$. Then $f(v) = 0$ and $f(u') = 0$ for every $u' \in L_u$.

Subcase 1.1. v is a support vertex. Then $f(v') = 1$ for every $v' \in L_v$. If v is adjacent to two leaves v' and v'', then we can change $f(v) = 0$ to $f(v) = 2$ and $f(v') = f(v'') = 1$ to $f(v') = f(v'') = 0$. Clearly we obtain a $\gamma_R(T)$-function for which $V_1 \cup V_2$ is not independent. Hence v is adjacent to a unique leaf v'. So $|L_v| = 1$.

Suppose that $|L_u| = 1$ and let u' be the unique leaf neighbor of u. Consider the function h on $V(T)$ defined by $h(x) = f(x)$ if $x \in V(T) - \{u, u', v, v'\}$, $h(u') = 1, h(u) = 0, h(v) = 2$ and $h(v') = 0$. Then h is a $\gamma_R(T)$-function and $h(w) = 0$. Furthermore $d_T(v) = 3$, for otherwise every child y of v different from u is assigned 2, a contradiction. Let T' be the tree obtained from T by removing u'. Note that v is a strong support vertex in T'. Clearly $h|_{V(T')}$ is both an RDF and an IRDF for T' implying that $\gamma_R(T') \leq \gamma_R(T) - 1$ and $i_R(T') \leq i_R(T) - 1$. Since every $\gamma_R(T')$-function can be extended to an RDF for T by assigning 1 to u' we obtain $\gamma_R(T) = \gamma_R(T') + 1$. Also $i_R(T') \leq i_R(T) - 1 = \gamma_R(T) - 1 = \gamma_R(T')$ and so $i_R(T') = \gamma_R(T')$. It follows that $i_R(T) = i_R(T') + 1$ and so $i_R(T') = \gamma_R(T')$. On the other hand, if $\gamma_R(T')$ and $i_R(T')$ are not strongly equal, then every $\gamma_R(T')$-function for which $V_1 \cup V_2$ is not independent can be extended to a $\gamma_R(T)$-function by assigning 1 to u', a contradiction with $\gamma_R(T) \equiv i_R(T)$. Therefore $\gamma_R(T') \equiv i_R(T')$ and by induction on T', we have $T' \in \mathcal{F}$. We conclude that $T \in \mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{O}_1$.

Assume now that $|L_u| \geq 2$. By our choice of u, every child of v which is a support vertex is adjacent to at least two leaves. Hence T_v is a tree of $\mathcal{T}$. Let $u = u_1, u_2, \ldots, u_k$ with $k \geq 1$, denote the support vertices adjacent to v in T_v, and let $T' = T - T_v$. Since $\mathrm{diam}(T) \geq 4$, T' is nontrivial. We observe that $f|_{V(T')}$ is both an RDF and IRDF for T' implying that $\gamma_R(T') \leq \gamma_R(T) - 2k - 1$ and $i_R(T') \leq i_R(T) - 2k - 1$. Equality is obtained by the fact that every $\gamma_R(T')$-function (resp. $i_R(T')$-function) can be extended to an RDF (resp. an IRDF) for T by assigning 2 to every u_i, 0 to v and every leaf in T_v except v', and 1 to v'. On the other hand, observe that if w satisfies $\gamma_R(T' - w) \leq \gamma_R(T') - 1$, then every $\gamma_R(T' - w)$-function can be extended to a $\gamma_R(T)$-function that is not independent by assigning 2 to v and every u_i and 0 to the remaining vertices, a contradiction with $\gamma_R(T) \equiv i_R(T)$. Thus w satisfies $\gamma_R(T' - w) \geq \gamma_R(T')$. If $\gamma_R(T')$ and $i_R(T')$ are not strongly equal, then every $\gamma_R(T')$-function which is not independent can be extended to a $\gamma_R(T)$-function, contradicting $\gamma_R(T) \equiv i_R(T)$. It follows that $\gamma_R(T') \equiv i_R(T')$ and by induction on T' we have $T' \in \mathcal{F}$. Therefore $T \in \mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{O}_2$.

Subcase 1.2. v is not a support vertex. We first assume that $d_T(v) \geq 3$. Then all children of v are support vertices and each one is assigned 2. If some child b of v is adjacent to only one leaf b', then we can change $f(b) = 2$ to $f(b) = 1$ and $f(b') = 0$ to $f(b') = 1$. We then obtain a $\gamma_R(T)$-function that is not independent, a contradiction. Thus every child of v is adjacent to at least two leaves. Let $T' = T - T_v$. Observe that T_v belongs to $\mathcal{T}$. Then $\gamma_R(T) \leq \gamma_R(T') + 2(d_T(v) - 1)$ since every $\gamma_R(T')$-function can be extended to an RDF for T by assigning 2 to every support vertex in T_v. Likewise, $i_R(T) \leq i_R(T') + 2(d_T(v) - 1)$. Both equalities are obtained from the fact that $f|_{V(T')}$ is an RDF and IRDF for T'. It follows that $\gamma_R(T') = i_R(T')$. Now if T' admits a $\gamma_R(T')$-function that is not independent, then such a function can be extended to a $\gamma_R(T)$-function that is not independent, a contradiction with $\gamma_R(T) \equiv i_R(T)$. Thus every $\gamma_R(T')$-function is independent, that is $\gamma_R(T') \equiv i_R(T')$. By induction on T' we have $T' \in \mathcal{F}$ and so $T \in \mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{O}_2$.

Now assume that $d_T(v) = 2$. If $|L_u| \geq 2$, then we consider $T' = T - T_v$. Observe that $T_v \in \mathcal{T}$. It is easy to see that $\gamma_R(T) = \gamma_R(T') + 2$ and $i_R(T) = i_R(T') + 2$, and so $\gamma_R(T') = i_R(T')$. Since every $\gamma_R(T')$-function can be extended to a $\gamma_R(T)$-function, it follows that $\gamma_R(T') \equiv i_R(T')$. By induction on T' we have $T' \in \mathcal{F}$ and so $T \in \mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{O}_2$. Now assume that $|L_u| = 1$, and let u' be the unique leaf adjacent to u. If $f(w) = 2$, then we change $f(u) = 2$ to $f(u) = 1$ and $f(u') = 0$ to $f(u') = 1$. We obtain a $\gamma_R(T)$-function that is not independent, a contradiction. Thus $f(w) \in \{0, 1\}$ for every $\gamma_R(T)$-function f. Let $T' = T - T_v$. Then $\gamma_R(T') \leq \gamma_R(T) - 2$ and $i_R(T') \leq i_R(T) - 2$. Both equalities hold since every $\gamma_R(T')$-function (respectively, $\gamma_R(T')$-function) can be extended to an RDF (respectively, IRDF) for T by assigning 0 to u', v and 2 to u. Hence $i_R(T') = \gamma_R(T')$. Note that since $f(w) \neq 2$, w is assigned 0 or 1 for every $\gamma_R(T')$-function. Now it is clear that $\gamma_R(T') \equiv i_R(T')$ and by induction on T' we have $T' \in \mathcal{F}$. It follows that $T \in \mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{O}_3$.

Case 2. $f(u) = 0$. Then $f(u') > 0$ for every $u' \in L_u$. It follows that $|L_u| \leq 2$, for otherwise we can decrease the weight of f by changing the assignment of u and its leaves. Now if $L_u = \{u', u''\}$, then $f(u') = f(u'') = 1$ and $f(v) = 2$. In this case we change $f(u) = 0$ to $f(u) = 2$ and $f(u') = f(u'') = 1$ to $f(u') = f(u'') = 0$. Clearly we obtain a $\gamma_R(T')$-function that is not independent, a contradiction. Hence $|L_u| = 1$. Let u' be the leaf adjacent to u. If $f(u') = 2$, then we must have $f(v) = 0$ and so we can change $f(u') = 2$ to $f(u') = 0$ and $f(u) = 0$ to $f(u) = 2$. Hence we are in Case 1. Thus we assume that $f(u') = 1$ and so $f(v) = 2$. We consider the following subcases.

Subcase 2.1. v is a support vertex. Then $f(v') = 0$ for every $v' \in L_v$. Let T' be the tree obtained from T by removing u'. As seen in Subcase 1.1 we obtain $\gamma_R(T') \equiv i_R(T')$ and by induction on T', $T' \in \mathcal{F}$. Since T is obtained from T' by using Operation $\mathcal{O}_1$, we have $T \in \mathcal{F}$.

Subcase 2.2. v is not a support vertex but has degree at least three. Thus every child of v is a support vertex with degree two. Also every support vertex in T_v is assigned 0 and every leaf is assigned 1. Now let T' be the tree obtained from T by removing u'. It is easy to see that $\gamma_R(T) = \gamma_R(T')+1$ and $i_R(T) = i_R(T')+1$. Hence $\gamma_R(T') = i_R(T')$. On the other hand suppose that $\gamma_R(T' - v) \leq \gamma_R(T')$ and let f' be any $\gamma_R(T' - v)$-function. Then u is an isolated vertex in $T' - v$ and is assigned 1. Also we may assume, without loss of generality, that every child of v different from u is assigned 2 in $T' - v$. Hence f' can be extended to a $\gamma_R(T)$-function for T by assigning 1 to u'. But then the resulting $\gamma_R(T)$-function is not independent, a contradiction. It follows that v satisfies $\gamma_R(T' - v) > \gamma_R(T')$ and so by Proposition 2, v is assigned 2 for every $\gamma_R(T')$-function. Using this fact and the fact that every $\gamma_R(T')$-function can be extended to a $\gamma_R(T)$-function by assigning 1 to u', we obtain $\gamma_R(T') \equiv i_R(T')$. By induction on T' we have $T' \in \mathcal{F}$ and so $T \in \mathcal{F}$ since it is obtained from T' by using Operation $\mathcal{O}_1$.

Subcase 2.3. $d_T(v) = 2$. Recall that since $f(v) = 2$, we have $f(w) = 0$. Then we can make a change to obtain $f(u') = 0, f(u) = 2, f(v) = 0$ and $f(w) = 1$. Since $\gamma_R(T) \equiv i_R(T)$, no vertex of $N(w) - \{v\}$ is assigned a positive value. Now let $T' = T - T_v$. As seen in Subcase 1.2 (when $d_T(v) = 2$) w is not assigned 2 for every $\gamma_R(T')$-function, $\gamma_R(T) = \gamma_R(T') + 2$, $i_R(T) = i_R(T') + 2$ and $\gamma_R(T') \equiv i_R(T')$.

By induction on T' we have $T' \in \mathcal{F}$ and so $T \in \mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{O}_3$. ■

We close with the following problem.

Problem. *Characterize other classes of graphs (or regular graphs) with strong equality between the Roman domination and the independent Roman domination numbers.*

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