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ON THE DOMINATION OF CARTESIAN PRODUCT OF DIRECTED CYCLES: RESULTS FOR CERTAIN EQUIVALENCE CLASSES OF LENGTHS

Michel Mollard

CNRS Université Joseph Fourier
Institut Fourier
100, rue des Maths
38402 St Martin d'Hères Cedex France

e-mail: michel.mollard@ujf-grenoble.fr

Abstract

Let $\gamma(\overrightarrow{C_m}\Box\overrightarrow{C_n})$ be the domination number of the Cartesian product of directed cycles $\overrightarrow{C_m}$ and $\overrightarrow{C_n}$ for $m, n \geq 2$. Shaheen [13] and Liu *et al.* ([11], [12]) determined the value of $\gamma(\overrightarrow{C_m}\Box\overrightarrow{C_n})$ when $m \leq 6$ and [12] when both m and $n \equiv 0(\bmod 3)$. In this article we give, in general, the value of $\gamma(\overrightarrow{C_m}\Box\overrightarrow{C_n})$ when $m \equiv 2(\bmod 3)$ and improve the known lower bounds for most of the remaining cases. We also disprove the conjectured formula for the case $m \equiv 0(\bmod\ 3)$ appearing in [12].

Keywords: directed graph, Cartesian product, domination number, directed cycle.

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1. Introduction and Definitions

Let $D = (V, A)$ be a finite directed graph (digraph for short) without loops or multiple arcs.

A vertex u *dominates* a vertex v if $u = v$ or $uv \in A$. A set $W \subseteq V$ is a *dominating* set of D if any vertex of V is dominated by at least one vertex of W. The *domination number* of D, denoted by $\gamma(D)$ is the minimum cardinality of a dominating set. The set V is a dominating set thus $\gamma(D)$ is finite. These definitions extend to digraphs the classical domination notion for undirected graphs.

The determination of the domination number of a directed or undirected graph is, in general, a difficult question in graph theory. Furthermore this problem has connections with information theory. For example the domination number of hypercubes is linked to error-correcting codes. Among the lot of related works, Haynes *et al.* ([7], [8]) mention the special cases of the domination of Cartesian products of undirected paths, cycles or more general graphs ([1] to [6], [9], [10]).

For two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ the *Cartesian product* $D_1 \square D_2$ is the digraph with vertex set $V_1 \times V_2$ and $(x_1, x_2)(y_1, y_2) \in A(D_1 \square D_2)$ if and only if $x_1y_1 \in A_1$ and $x_2 = y_2$ or $x_2y_2 \in A_2$ and $x_1 = y_1$. Note that $D_2 \square D_1$ is isomorphic to $D_1 \square D_2$. In [13] Shaheen determined the domination number of $\overrightarrow{C_m} \square \overrightarrow{C_n}$ for $m \leq 6$ and arbitrary n. In two articles [11], [12] Liu *et al.* considered independently the domination number of the Cartesian product of two directed cycles. They gave also the value of $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n})$ when $m \leq 6$ and when both m and $n \equiv 0(\bmod 3)$ [12]. Furthermore they proposed lower and upper bounds for the general case.

In this paper we are able to give, in general, the value of $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n})$ when $m \equiv 2(\bmod 3)$ and we improve the lower bounds for most of the still unknown cases. We also disprove the conjectured formula appearing in [12] for the case $m \equiv 0(\bmod 3)$.

We denote the vertices of a directed cycle $\overrightarrow{C_n}$ by $C_n = \{0, 1, \ldots, n-1\}$, the integers considered modulo n. Thus, when used for vertex labeling, $a+b$ and $a-b$ will denote the vertices $a+b$ and $(a-b)(\bmod n)$. Notice that there exists an arc xy from x to y in $\overrightarrow{C_n}$ if and only if $y \equiv x+1(\bmod n)$, thus with our convention, if and only if $y = x+1$. For any i in $\{0, 1, \ldots, n-1\}$ we will denote by $\overrightarrow{C_m^i}$ the subgraph of $\overrightarrow{C_m} \square \overrightarrow{C_n}$ induced by the vertices $\{(k, i) \mid k \in \{0, 1, \ldots, m-1\}\}$. Note that $\overrightarrow{C_m^i}$ is isomorphic to $\overrightarrow{C_m}$. We will denote by C_m^i the set of vertices of $\overrightarrow{C_m^i}$.

2. General Bounds and the Case $m \equiv 2(\bmod 3)$

We start this section by developing a general upper bound for $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n})$. Then we will construct minimum dominating sets for $m \equiv 2(\bmod 3)$. These optimal sets will be obtained from integer solutions of a system of equations.

Proposition 1. *Let W be a dominating set of $\overrightarrow{C_m} \square \overrightarrow{C_n}$. Then for all i in $\{0, 1, \ldots, n-1\}$ considered modulo n we have $\left|W \cap C_m^{i-1}\right| + 2\left|W \cap C_m^i\right| \geq m$.*

Proof. The m vertices of C_m^i can only be dominated by vertices of $W \cap C_m^i$ and $W \cap C_m^{i-1}$. Each of the vertices of $W \cap C_m^i$ dominates two vertices in C_m^i. Similarly, each of the vertices of $W \cap C_m^{i-1}$ dominates one vertex in C_m^i. The result follows. ■

Theorem 2. *Let $m, n \geq 2$ and $k_1 = \left\lfloor \frac{m}{3} \right\rfloor$. Then*

(i) *if* $m \equiv 0(\bmod 3)$, *then* $\gamma(\overrightarrow{C_m}\Box\overrightarrow{C_n}) \geq nk_1$, *or*

(ii) *if* $m \equiv 1(\bmod 3)$, *then* $\gamma(\overrightarrow{C_m}\Box\overrightarrow{C_n}) \geq nk_1 + \frac{n}{2}$, *or*

(iii) *if* $m \equiv 2(\bmod 3)$, *then* $\gamma(\overrightarrow{C_m}\Box\overrightarrow{C_n}) \geq nk_1 + n$.

Proof. Let W be a dominating set of $\overrightarrow{C_m}\Box\overrightarrow{C_n}$ and for any i in $\{0, 1, \ldots, n-1\}$ let $a_i = |W \cap C_m^i|$. Notice first, as noticed by Liu *et al.* [12], that each of the vertices of W dominates three vertices of $\overrightarrow{C_m}\Box\overrightarrow{C_n}$ and thus $|W| \geq \frac{mn}{3}$. This general bound give the announced result for $m = 3k_1$, $\gamma(\overrightarrow{C_m}\Box\overrightarrow{C_n}) \geq nk_1 + \frac{n}{3}$ for $m = 3k_1 + 1$ and $\gamma(\overrightarrow{C_m}\Box\overrightarrow{C_n}) \geq nk_1 + 2\frac{n}{3}$ for $m = 3k_1 + 2$. We will improve these two last results to verify parts (ii) and (iii) of the theorem.

Assume first $m = 3k_1 + 1$. Let J be the set of $j \in \{0, 1, \ldots, n-1\}$ such that $a_j \leq k_1$. If $J = \emptyset$, then $|W| \geq n(k_1 + 1) \geq nk_1 + \frac{n}{2}$ and we are done. Otherwise let $J' = \{j \mid j + 1 \,(\bmod n) \in J\}$. By Proposition 1, for any i in $\{0, 1, \ldots, n-1\}$ considered modulo n, we have $a_{i-1} + 2a_i \geq 3k_1 + 1$. Then if i belongs to J, $a_{i-1} + a_i \geq 2k_1 + 1$. A first consequence is that there are no consecutive indices, taken modulo n, in J. Indeed, if $j - 1$ and j are in J then, by definition of J, $a_{j-1} + a_j \leq 2k_1$ in contradiction with the previous inequality. By definition of J' we have thus $J \cap J' = \emptyset$.

Now let $K = \{j \in \{0, 1, \ldots, n-1\},\ j \notin J \cup J'\}$. We can write $\{0, 1, \ldots, n-1\} = J \cup J' \cup K$ where J, J' and K are disjoint sets. Notice that $\theta : j \mapsto j - 1(\bmod n)$ induces a one to one mapping between J and J'.

The cardinality of W is $|W| = \sum_{i\in\{0,1,\ldots,n-1\}} a_i = \sum_{i\in J} a_i + \sum_{i\in J'} a_i + \sum_{i\in K} a_i$. We can use θ for grouping 2 by 2 the elements of $J \cup J'$ and write $\sum_{i\in J} a_i + \sum_{i\in J'} a_i = \sum_{i\in J} a_i + \sum_{i\in J} a_{\theta(i)} = \sum_{i\in J}(a_i + a_{i-1})$. Using $a_{i-1} + a_i \geq 2k_1 + 1$, because $i \in J$, we obtain $\sum_{i\in J} a_i + \sum_{i\in J'} a_i \geq |J|\,(2k_1 + 1)$.

If $i \in K$ then $i \notin J$ and $a_i \geq k_1 + 1$. Since $|K| = n - 2|J|$ we have $\sum_{i\in K} a_i \geq (n - 2|J|)(k_1 + 1)$. Then $|W| = \sum_{i\in\{0,1,\ldots,n-1\}} a_i \geq |J|(2k_1 + 1) + (n - 2|J|)(k_1 + 1) = nk_1 + n - |J|$. Since $|J| = |J'|$ and $J \cap J' = \emptyset$, $n - |J| \geq \frac{n}{2}$ and the conclusion for (ii) follows.

The case $m = 3k_1 + 2$ is similar. Let J be the set of $j \in \{0, 1, \ldots, n-1\}$ such that $a_j \leq k_1$. If $J = \emptyset$ then we are done. Otherwise let $J' = \{j \mid j + 1(\bmod n) \in J\}$. If $i \in J$ we have $a_{i-1} + 2a_i \geq 3k_1 + 2$ thus $a_{i-1} + a_i \geq 2k_1 + 2$. Then $J \cap J' = \emptyset$ and $\sum_{i\in J\cup J'} a_i \geq |J|\,(2k_1 + 2)$. Therefore $\sum_{i\in\{0,1,\ldots,n-1\}} a_i \geq |J|(2k_1 + 2) + (n - 2|J|)(k_1 + 1) \geq n(k_1 + 1)$. ■

Let us now study in detail the case $m \equiv 2(\bmod 3)$. Assume $m = 3k_1 + 2$. Let A be the set of $k_1 + 1$ vertices of $\overrightarrow{C_m}$ defined by $A = \{0\} \cup \{2 + 3p \mid p = 0, 1, \ldots, k_1 - 1\} = \{0\} \cup \{2, 5, \ldots, m-6, m-3\}$. For any i in $\{0, 1, \ldots, m-1\}$ let us call $A_i = \{j \mid j - i\,(\bmod m) \in A\}$ the *translate*, considered modulo m, of A by i. We have thus $A_i = \{i\} \cup \{i+2, i+5, \ldots, i-6, i-3\}$ (see Figure 1).

We will call a set S of vertices of $\overrightarrow{C_m}\Box\overrightarrow{C_n}$ an A-set if for any j in $\{0, 1, \ldots, n-1\}$ we have $S \cap C_m^j = A_i$ for some i in $\{0, 1, \ldots, n-1\}$. It will be convenient to denote this index i, function of j, as i_j. If S is a A-set then $|S| = n(k_1 + 1)$; thus if a set is both a A-set and a dominating set, by Theorem 2, it is minimum and we have $\gamma(\overrightarrow{C_m}\Box\overrightarrow{C_n}) = n(k_1 + 1)$.

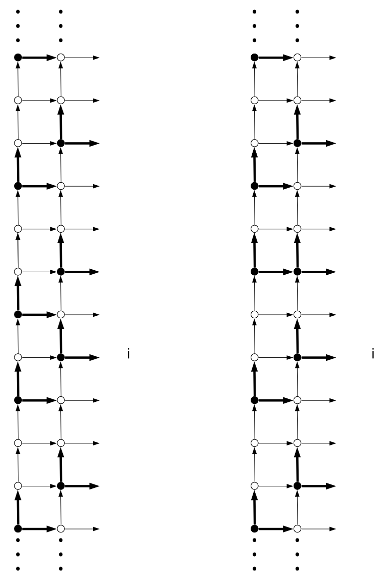

Figure 1. A_{i-1}, A_i and A_{i+2}, A_i.

Lemma 3. *Let $m = 3k_1 + 2$. Let S be a A-set and for any j in $\{0, 1, \ldots, n-1\}$ define i_j as the index such that $S \cap C_m^j = A_{i_j}$. Assume that*

(i) *for any $j \in \{1, \ldots, n-1\}$ $i_j \equiv i_{j-1} + 1 \pmod{m}$ or $i_j \equiv i_{j-1} - 2 \pmod{m}$ and*

(ii) *$i_0 \equiv i_{n-1} + 1 \pmod{m}$ or $i_0 \equiv i_{n-1} - 2 \pmod{m}$.*

Then S is a dominating set of $\overrightarrow{C_m}\Box\overrightarrow{C_n}$.

Proof. Note first that for any i in $\{0, 1, \ldots, m-1\}$ the set of non dominated vertices of C_m by A_i is $T = \{i + 4, i + 7, \ldots, i - 4, i - 1\}$. Note also that $A_{i+2} = \{i+2\} \cup \{i+4, i+7, \ldots, i-4, i-1\}$ and $A_{i-1} = \{i-1\} \cup \{i+1, i+4, \ldots, i-7, i-4\}$. Thus $T \subset A_{i+2}$ and $T \subset A_{i-1}$.

Let j in $\{1, \ldots, n-1\}$. Let us prove that the vertices of C_m^j are dominated. Indeed, by the previous remark and the lemma hypothesis, the vertices non dominated by $S \cap C_m^j$ are dominated by $S \cap C_m^{j-1}$ (see Figure 1). For the same reasons the vertices of C_m^0 are dominated by those of $S \cap C_m^0$ and $S \cap C_m^{n-1}$. ■

We will prove next that the existence of solutions to some system of equations over integers implies the existence of an A-set satisfying the hypothesis of Lemma 3.

Lemma 4. *Let $m = 3k_1 + 2$. If there exist integers $a, b \geq 0$ such that*
(i) $a + b = n - 1$ *and*
(ii) $a - 2b \equiv 2 \pmod m$ *or* $a - 2b \equiv m - 1 \pmod m$.
Then $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) = n(k_1 + 1)$.

Proof. Consider a word $w = w_1 \dots w_{n-1}$ on the alphabet $\{1, -2\}$ with a occurrences of 1 and b of -2. Such a word exists, for example $w = 1^a(-2)^b$. We can associate with w a set S of vertices of $\overrightarrow{C_m} \square \overrightarrow{C_n}$ using the following algorithm:

$S \cap C_m^0 = A_0$
For $i = 1$ to $n - 1$ do
begin
Let k such that $S \cap C_m^{i-1} = A_k$
If $w_i = 1$ let $k' \equiv k + 1 \pmod m$ else $k' \equiv k - 2 \pmod m$
$S \cap C_m^i := A_{k'}$
end

By construction S is an A-set. Notice that we have $S \cap C_m^{n-1} := A_{i_{n-1}}$ where $i_{n-1} \equiv \sum_{k=1}^{n-1} w_k \equiv a - 2b \pmod m$. Thus $i_{n-1} \equiv 2 \pmod m$ or $i_{n-1} \equiv m - 1 \pmod m$. By Lemma 3, S is a dominating set. Furthermore, because S is a A-set, $|S| = n(k_1 + 1)$, thus by Theorem 2 it is minimum and we have $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) = n(k_1 + 1)$. ■

With the exception of one subcase we can find solutions (a, b) of the system and thus obtain minimum dominating sets for $m \equiv 2 \pmod 3$.

Theorem 5. *Let $m, n \geq 2$ and $m \equiv 2 \pmod 3$. Let $k_1 = \lfloor \frac{m}{3} \rfloor$ and $k_2 = \lfloor \frac{n}{3} \rfloor$.*
(i) *If $n = 3k_2$, then $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) = n(k_1 + 1)$, and*
(ii) *if $n = 3k_2 + 1$ and $2k_2 \geq k_1$, then $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) = n(k_1 + 1)$, and*
(iii) *if $n = 3k_2 + 1$ and $2k_2 < k_1$, then $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) > n(k_1 + 1)$, and*
(iv) *if $n = 3k_2 + 2$ and $n \geq m$, then $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) = n(k_1 + 1)$, and*
(v) *if $n = 3k_2 + 2$ and $n \leq m$, then $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) = m(k_2 + 1)$.*

Proof. We will use Lemma 4 considering the following integer solutions of

$$\begin{cases} a, b \geq 0 \\ a + b = n - 1 \\ a - 2b \equiv 2 \pmod m \text{ or } a - 2b \equiv m - 1 \pmod m \end{cases}.$$

(i) If $n = 3k_2$, then $k_2 \geq 1$. Take $a = 2k_2$ and $b = k_2 - 1$.

(ii) If $n = 3k_2 + 1$ and $2k_2 \geq k_1$, then take $a = 2k_2 - k_1$ and $b = k_2 + k_1$.

(iii) If $n = 3k_2 + 1$ and $2k_2 < k_1$, then $\gamma(\overrightarrow{C_m} \Box \overrightarrow{C_n}) = \gamma(\overrightarrow{C_n} \Box \overrightarrow{C_m}) \geq \frac{(2k_2+1)m}{2}$ by Theorem 2. Furthermore, $\frac{(2k_2+1)m}{2} - n(k_1 + 1) = \frac{k_1}{2} - k_2 > 0$.

(iv) If $n = 3k_2 + 2$ and $k_2 \geq k_1$, then take $a = 2k_2 - 2k_1$ and $b = k_2 + 2k_1 + 1$.

(v) If $n = 3k_2 + 2$ and $k_2 \leq k_1$, then use $\gamma(\overrightarrow{C_m} \Box \overrightarrow{C_n}) = \gamma(\overrightarrow{C_n} \Box \overrightarrow{C_m})$. ∎

3. THE CASE $m \equiv 0 (\mod 3)$

In [12] Liu *et al.* conjectured the following formula:

Conjecture 6. *Let $k \geq 2$. Then $\gamma(\overrightarrow{C_{3k}} \Box \overrightarrow{C_n}) = k(n + 1)$ for $n \not\equiv 0 (\mod 3)$.*

Our Theorem 5 confirms the conjecture when $n \equiv 2 (\mod 3)$. Unfortunately, the formula is not always valid when $n \equiv 1 (\mod 3)$.

Indeed, consider $C_{3k} \Box C_4$. In [11] the following result is proved:

Theorem 7. *Let $n \geq 2$. Then $\gamma(\overrightarrow{C_4} \Box \overrightarrow{C_n}) = \frac{3n}{2}$ if $n \equiv 0 (\mod 8)$ and $\gamma(\overrightarrow{C_4} \Box \overrightarrow{C_n}) = n + \left\lceil \frac{n+1}{2} \right\rceil$ otherwise.*

We have thus $\gamma(\overrightarrow{C_{3k}} \Box \overrightarrow{C_4}) = \gamma(\overrightarrow{C_4} \Box \overrightarrow{C_{3k}}) = 3k + \left\lceil \frac{3k+1}{2} \right\rceil$ when $k \not\equiv 0 (\mod 8)$. Alternately, Conjecture 6 proposes the value $\gamma(\overrightarrow{C_{3k}} \Box \overrightarrow{C_4}) = 5k$. These two numbers are different when $k \geq 3$.

4. CONCLUSION

Consider the possible remainder of m, n modulo 3. For some of the nine possibilities, we have found exact values for $\gamma(\overrightarrow{C_m} \Box \overrightarrow{C_n})$. The remaining cases are:

a) $m \equiv 0 (\mod 3)$ and $n \equiv 1 (\mod 3)$

b) The symmetrical case $m \equiv 1 (\mod 3)$ and $n \equiv 0 (\mod 3)$.

c) m and $n \equiv 1 (\mod 3)$.

d) The case m or $n \equiv 2 (\mod 3)$ is not completely solved by Theorem 5. The following subcases are still open

   i) $m \equiv 2 (\mod 3)$ and $n \equiv 1 (\mod 3)$ with $m > 2n + 1$

   ii) the symmetrical case $m \equiv 1 (\mod 3)$ and $n \equiv 2 (\mod 3)$ with $n > 2m + 1$.

For these values of m, n there does not always exist a dominating set reaching the bound stated if Theorem 2 and thus the determination of $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n})$ seems to be a more difficult problem.

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