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Dedicated to Mietek Borowiecki on the occasion of his 70th birthday and to his wife Wanda who together with Mietek create a necassary pair in each stable marriage matching.

A NOTE ON THE UNIQUENESS OF STABLE MARRIAGE MATCHING[1]

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Abstract

In this note we present some sufficient conditions for the uniqueness of a stable matching in the Gale-Shapley marriage classical model of even size. We also state the result on the existence of exactly two stable matchings in the marriage problem of odd size with the same conditions.

Keywords: stable matching, Gale-Shapley model, stable perfect matching.

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1. Introduction

The stable marriage problem was introduced in the seminal paper of Gale and Shapley [4]. Its many variants have been widely studied in the literature [7] mainly because of some important practical applications, such as National Resident Matching Program [8] and similar large-scale matching schemes. In the classical form of the problem we consider two disjoint n-element sets, the set of women $W = \{W_1, \ldots, W_n\}$ and the set of men $M = \{M_1, \ldots, M_n\}$. All women and men have preferences over the opposite sex represented by the linearly ordered lists. The notation $(W_i : M_{i_1}, \ldots, M_{i_n})$ means that for W_i the man M_{i_j}

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is better than M_{i_k} for all k greater than j. In the case when we only know, or we are only interested in some parts of the lists, the symbol "|" substitutes their possible elements. We write $(W_i : |M_{i_1}, M_{i_2}|M_{i_3}|)$ when W_i prefers M_{i_1} to M_{i_2}; M_{i_1} to M_{i_3}; and M_{i_2} to M_{i_3}. Moreover, it is possible that there are other men at the $W_i's$ list of preferences who are better than M_{i_1}, there are possibly some men between M_{i_2} and M_{i_3} and there is no man between M_{i_1} and M_{i_2}. It has to be noted that we can construct many $W_i's$ lists of preferences given by the description $(W_i : |M_{i_1}, M_{i_2}|M_{i_3}|)$.

Let $\mathbf{W}$ and $\mathbf{M}$ be two tables, each of n^2 elements (of size n), representing preferences of the sexes. The ith row of the table $\mathbf{W}$ shows preferences of W_i and has the form $M_{i_1}, \ldots, M_{i_n}$, when $(W_i : M_{i_1}, \ldots M_{i_n})$ $(M = \{M_{i_1}, \ldots M_{i_n}\})$. The table $\mathbf{M}$ is constructed analogously. If $\mathbf{W}$, $\mathbf{M}$ are of size n, then a pair $(\mathbf{W}, \mathbf{M})$ is called a *marriage problem* of size n.

A matching σ is an arbitrary bijection of W onto M. For simplicity, we write $\sigma(i) = j$ instead of $\sigma(W_i) = M_j$. A matching σ is *unstable* for given tables of preferences, if there is a woman W_i and a man M_j such that $\sigma(i) \neq j$ but each of W_i, M_j prefers the other to her/his partner in σ $((W_i : |M_j|M_{\sigma(i)}|)$, $(M_j : |W_i|W_{\sigma^{-1}(j)}|))$. Such a pair W_i, M_j is said to be a *blocking pair* for σ. A matching for which there is no blocking pair is called *stable*.

In 1962 Gale and Shapley, using a model of college admission, showed [4, 3] that for any tables of preferences a stable matching always exists, but it does not have to be unique.

The problem of determining the maximum number of stable matchings among all the marriage problems of size n was posed by Knuth [6] and still remains an open question. In [1] it was shown that this number is not greater than three quarters of all $n!$ possible matchings. On the other hand Knuth established that this number exceeds $2^{\frac{n}{2}}$ for $n \geq 2$ and Gusfield and Irving showed that for n being a power of 2 it is at least 2^{n-1}, which can be improved to $(2.28)^n/(1+\sqrt{3})$ based on the construction given by Irving and Leather [5].

Another interesting question concerning the number of stable matchings is how many of all the marriage problems of size n have exactly one stable matching. To solve this problem there must be recognized some necessary and sufficient conditions, which guarantee the existance of exactly one stable matching (recall that there always exists one stable matching). Let $[n] = \{1, \ldots, n\}$. In [2] Eeckhout derived the sufficient conditions on the tables $\mathbf{W}$, $\mathbf{M}$ of size n that yield the uniqueness of the stable matching $\sigma(i) = i$, $i \in [n]$. These conditions are: for each $i \in [n]$ the woman W_i prefers M_i to M_j for all $j > i$, and the man M_i prefers W_i to W_j for all $j > i$. Using this result and taking into account that for fixed $i \in [n]$, the number of $W_i's$ lists of preferences, that satisfy the condition: "W_i prefers M_i to M_j for all $j > i$" is $\frac{n!}{(n-i+1)}$, we can produce m problems ($\mathbf{W}$, $\mathbf{M}$) of size n for which there exists exactly one stable matching $\sigma(i) = i$, $i \in [n]$,

where

$$m = \left(\prod_{i=1}^{n} \frac{n!}{(n-i+1)}\right)^2 = \left(\frac{(n!)^n}{n\cdot(n-1)\cdots 1}\right)^2 = \left((n!)^{n-1}\right)^2 = \left(n!\right)^{2n-2}.$$

In this note we give $\frac{\left((n-2)!\right)^{2n}}{4}$ new marriage problems of even size n (not included in the set of problems that satisfy Eeckhout's conditions), for which there exists the unique stable matching $\sigma(i) = i$, $i \in [n]$, (Theorem 1). Our result generalizes the example given in [2] by Ahmed Alkan for $n = 4$. Along the way, we observe that the conditions described in this note as sufficient for the uniqueness, in the above mentioned case (even n), create pairs $\mathbf{W}$, $\mathbf{M}$ with exactly two stable matchings for an odd n (Theorem 2). It seems to be an interesting fact too.

All the results presented herein can be expressed in the graph theory language as results on the number of stable perfect matchings in a complete balanced bipartite graph with $2n$ vertices. In this model, each vertex v has assigned a list consisting of all the vertices of the opposite bipartition set, whose elements are linearly ordered by $\overset{v}{\prec}$. A perfect matching N of such a graph is not stable if there are verices x, y from different bipartition sets, such that $xy \notin N$ and for a, b satisfying $xa \in N$, $yb \in N$ there hold $a \overset{x}{\prec} y$ and $b \overset{y}{\prec} x$. In our paper we do not use the graph theory model and describe our results in language of two-sided matching problem, widely known in economics and computer science oriented papers.

2. Main Results

We shall say that tables of preferences $\mathbf{W}$, $\mathbf{M}$ for women $\{W_1, \ldots, W_n\}$ and men $\{M_1, \ldots, M_n\}$ satisfy conditions $(*)$ if the following hold:

1. $(W_1 : M_2, M_1|)$, and
2. $(W_{n-1} : M_{n-2}, M_{n-1}|M_n|M_1|)$, and
3. $(W_i : M_{i-1}, M_i|)$ for $i \in [n] \setminus \{1, n-1\}$, and
4. $(M_1 : W_{n-1}, W_1|)$, and
5. $(M_2 : W_n, W_2, |W_3|W_1|)$, and
6. $(M_n : W_{n-1}, W_n|)$, and
7. $(M_i : W_{i-2}, W_i|)$ for $i \in [n] \setminus \{1, 2, n\}$.

In the remaining part of this section we shall prove the two following main results of this note.

Theorem 1. *Let $n \geq 4$ and let $(\mathbf{W}, \mathbf{M})$ be a marriage problem of even size n that satisfies the conditions $(*)$. There exists a unique stable matching for $\mathbf{W}$, $\mathbf{M}$.*

Theorem 2. *Let $n \geq 5$ and let $(\mathbf{W}, \mathbf{M})$ be a marriage problem of odd size n that satisfies the conditions $(*)$. There exist exactly two different stable matchings σ_1, σ_2 for $\mathbf{W}$, $\mathbf{M}$ such that:*

(1) *$\sigma_1(i) = i$ for all $i \in [n]$, and*

(2) *$\sigma_2(i) = i$ for all odd $i \in [n-2]$ and $\sigma_2(n) = 2$, $\sigma_2(n-1) = n$, $\sigma_2(n-i) = n-i+2$ for all odd $i \in \{3, \ldots, n-2\}$.*

To make the proofs of Theorems 1, 2 more readable, we divide them into lemmas.

Let tables $\mathbf{W}$, $\mathbf{M}$ of preferences for women $\{W_1, \ldots, W_n\}$ and men $\{M_1, \ldots, M_n\}$ be given. We define, for $i, j \in [n]$, a set $A_{i,j}(\mathbf{W}, \mathbf{M})$ as the set of all the cuples (s, l) such that W_i, M_j is a blocking pair for each matching σ satisfying $\sigma(i) = s$ and $\sigma(l) = j$. We can easily see that

$$A_{i,j}(\mathbf{W}, \mathbf{M}) = \{(s, l) : \ (W_i : |M_j|M_s|) \text{ and } (M_j : |W_i|W_l|)\}.$$

Lemma 3. *Let $n \geq 4$ and let $(\mathbf{W}, \mathbf{M})$ be a marriage problem of size n that satisfies the conditions $(*)$. If σ is a stable matching for $\mathbf{W}$, $\mathbf{M}$, then $\sigma(1) = 1$.*

Proof. To the contrary, let us first assume that $\sigma(1) = 2$. It implies that W_3, M_2 is a blocking pair for σ, which is impossible because σ is stable. Next, assume that $\sigma(1) = s$, for some $s \geq 3$. Because for each $k \in [n] \setminus \{1, n-1\}$ we have $(s, k) \in A_{1,1}(\mathbf{W}, \mathbf{M})$ and because $\sigma(1) \neq 1$ it follows that $\sigma(n-1) = 1$. Hence, consequently, W_{n-1}, M_n is a blocking pair for σ, a contradiction. ■

Lemma 4. *Let $n \geq 4$ and let $(\mathbf{W}, \mathbf{M})$ be a marriage problem of size n that satisfies the conditions $(*)$. Next assume that σ is a stable matching for $\mathbf{W}$, $\mathbf{M}$.*

(1) *If n is even, then $\sigma(2) = 2$.*

(2) *If n is odd and $\sigma(2) \neq 2$, then $\sigma(n) = 2$, $\sigma(n-1) = n$ and for each odd $i \in \{3, \ldots, n-2\}$ there holds $\sigma(n-i) = n-i+2$.*

Proof. From Lemma 3 we have $\sigma(1) = 1$, which implies $\sigma(2) \neq 1$. Assume that n is arbitrary (even or odd) and $\sigma(2) \neq 2$. Then $\sigma(2) = s$, where $s \in \{3, \ldots, n\}$. Because $(s, l) \in A_{2,2}$ for all $l \in \{3, \ldots, n-1\} \cup \{1\}$ we deduce that $\sigma(n) = 2$. Similarly, the argument $(p, q) \in A_{n,n}$ for all $p, q \in \{1, \ldots, n-2\}$ yields $\sigma(n-1) = n$.

Now we focus on the fact that $(n, l) \in A_{n-1,n-1}$ for all $l \in [n] \setminus \{n-1, n-3\}$. Hence $\sigma(n-3) = n-1$. Next, step by step, assuming that $\sigma(1) = 1$, $\sigma(2) = s$, where $s \in \{3, \ldots, n\}$, $\sigma(n) = 2$, $\sigma(n-1) = n$ and $\sigma(n-j) = n-j+2$ for all odd $j \in \{3, \ldots, i\}$ with odd i $(i \leq n-3)$ we shall note that $\sigma(n-(i+2)) = n-(i+$

$2)+2 = n-i$. Indeed, $(n-(i+2), l) \in A_{n-i,n-i}$ for all $l \in [n] \setminus \{n-i, n-(i+2)\}$, which shows the assertion.

Thus we obtained that if n is even, then $1 = \sigma(1) = \sigma(n-(n-1)) = 3$, which is impossible. If n is odd we have $p = \sigma(2) = \sigma(n-(n-2)) = 4$, which gives $\sigma(1) = 1$, $\sigma(n) = 2$, $\sigma(n-1) = n$ and $\sigma(n-i) = n-i+2$ for odd $i \in \{3, \ldots, n-2\}$. Resuming, the assumptions $(*)$ and stability of σ imply that if n is even the only possibility for $\sigma(2)$ is to be equal 2 and if n is odd the assertion of the lemma describing this case holds. ■

Lemma 5. *Let $n \geq 4$ and let $(\mathbf{W}, \mathbf{M})$ be a marriage problem of size n that satisfies the conditions $(*)$. If σ is a stable matching for $\mathbf{W}$, $\mathbf{M}$ and $\sigma(1) = 1$ and $\sigma(2) = 2$, then $\sigma(i) = i$ for all $i \in [n]$.*

Proof. First we shall prove that if $\sigma(i) = i$ for $i \in \{1, \ldots, j-1\}$, where $3 \leq j \leq n-1$, then $\sigma(j) = j$. Suppose, the above assumption is satisfied. The assertion immediately follows from two observations:

- $(s, l) \in A_{j,j}$ for all $s \in [n] \setminus \{j-1, j\}$ and $l \in [n] \setminus \{j-2, j\}$, and
- $\sigma(j-1) = j-1$ and $\sigma(j-2) = j-2$.

The last step $(\sigma(n) = n)$ is implied by the just obtained statement $\sigma(i) = i$ for all $i \in [n-1]$. ■

Lemma 6. *Let $n \geq 5$ and let $(\mathbf{W}, \mathbf{M})$ be a marriage problem of size n that satisfies the conditions $(*)$. If σ is a stable matching for $\mathbf{W}$, $\mathbf{M}$ and $\sigma(1) = 1$, $\sigma(n) = 2$, $\sigma(n-1) = n$ and $\sigma(n-i) = n-i+2$ for all odd $i \in \{3, \ldots, n-2\}$, then $\sigma(i) = i$ for all odd $i \in [n-2]$.*

Proof. To observe the statement, as in the previous two proofs, we shall show that the assumptions $\sigma(n) = 2$, $\sigma(n-1) = n$, $\sigma(n-i) = n-i+2$ for all odd $i \in \{3, \ldots, n-2\}$ and the assumption $\sigma(j) = j$ for all odd j satisfying $j \leq i$ and $i \in [n-4]$, lead to $\sigma(i+2) = i+2$. The key argument is that $(s, l) \in A_{i+2,i+2}$ for $s \in [n] \setminus \{i+1, i+2\}$ and $l \in [n] \setminus \{i, i+2\}$. ■

***Proof of Theorem* 1.** It follows from Lemmas 3, 4, 5 and the fact that for any tables $\mathbf{W}$, $\mathbf{M}$ there always exists at least one stable matching. ■

***Proof of Theorem* 2.** From Lemmas 3, 4, 6 we know that for $\mathbf{W}$, $\mathbf{M}$ there is no stable matching, which is different from σ_1 and different from σ_2. Thus it is enough to observe that both σ_1, σ_2 are stable for $\mathbf{W}$, $\mathbf{M}$. By the way of contradiction, assume that σ_1 is unstable. It means one can find a blocking pair W_{i_1}, M_{j_1} for σ_1. Next, by the construction of σ_1 we have $i_1 \neq j_1$. If $i_1 < j_1$, then the condition $(W_{i_1} : |M_{j_1}|M_{i_1}|)$, which must be satisfied for the blocking pair,

and the conditions $(*)$ imply $j_1 = 2$ and $i_1 = 1$. But $(M_2 : |W_2|W_1|)$ yields W_1, M_2 could not be a blocking pair for σ_1. If $i_1 > j_1$, then the fact $(M_{j_1} : |W_{i_1}|W_{j_1}|)$ and the conditions $(*)$ imply two possibilities: either $i_1 = n-1$ and $j_1 = 1$ or $i_1 = n$ and $j_1 = 2$. Both of them are forbidden because $(W_n : |M_n|M_2|)$ and $(W_{n-1} : |M_{n-1}|M_1|)$. It leads to the assertion that σ_1 is stable. Now let us assume that σ_2 is unstable and there is a blocking pair W_{i_2}, M_{j_2} for σ_2. Because W_i is the first element at the list of $M_{\sigma_2(i)}$ for $\sigma_2(i) = n$ and for all even numbers $\sigma_2(i)$, we deduce that j_2 has to be odd and less than $n-1$. Consequently, because for the odd $i \in [n]$ the first element at $W_i's$ list is M_k with k odd and the second one is $M_{\sigma_2(i)}$, we obtain that i_2 must be even.

Next, we can see that M_1 could create a blocking pair only with W_{n-1} (W_{n-1} is the first element at the list of M_1 and $W_1 = W_{\sigma_2^{-1}(1)}$ is the second one). But W_{n-1} prefers her partner in σ_2 ($M_{\sigma_2(n-1)} = M_n$) to M_1, which excludes the possibility $j_2 = 1$. Actually, by the above consideration we have that j_2 has to be odd, different from 1 and different from n. For each such j_2, the conditions $\sigma_2^{-1}(j_2) = j_2$ and $(M_{j_2} : W_{j_2-2}, W_{j_2}|)$ are fulfilled. It finally gives $i_2 = j_2 - 2$, which means that i_2 must be odd, contrary to the previous claim. ■

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