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THE BALANCED DECOMPOSITION NUMBER OF TK_4 AND SERIES-PARALLEL GRAPHS

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Abstract

A *balanced colouring* of a graph G is a colouring of some of the vertices of G with two colours, say red and blue, such that there is the same number of vertices in each colour. The *balanced decomposition number* $f(G)$ of G is the minimum integer s with the following property: For any balanced colouring of G, there is a partition $V(G) = V_1 \dot\cup \cdots \dot\cup V_r$ such that, for every i, V_i induces a connected subgraph of order at most s, and contains the same number of red and blue vertices. The function $f(G)$ was introduced by Fujita and Nakamigawa in 2008. They conjectured that $f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$ if G is a 2-connected graph on n vertices. In this paper, we shall prove two partial results, in the cases when G is a subdivided K_4, and a 2-connected series-parallel graph.

Keywords: graph decomposition, vertex colouring, k-connected.

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1. INTRODUCTION

In this paper, all graphs will be simple and finite. For such a graph G, let $V(G)$ be its vertex set and $E(G)$ be its edge set. For $X \subset V(G)$, let $X^c = V(G) \setminus X$; let $G[X]$ be the subgraph of G induced by X; and let $N(X) = \{v \in X^c : vx \in E(G)$ for some $x \in X\}$ be the open neighbourhood of X in G. For a subgraph $H \subset G$, the graph $H - X$ is the subgraph of H induced by $V(H) \setminus X$. We write $H - u$ for $H - \{u\}$. For $k \in \mathbb{N}$, G is a *k-connected* graph if $|V(G)| \geq k+1$, and $G - X$ is connected for every $X \subset V(G)$ with $|X| \leq k-1$. For $u, v \in V(G)$, the graph distance from u to v in G is denoted by $d_G(u, v)$. If P is a path with end-vertices u and v, then $\operatorname{int} P$ is the path $P - \{u, v\}$ (this is vacuous if $|V(P)| \leq 2$).

We refer the reader to [1] for any undefined graph theoretic terms.

In 2008, Fujita and Nakamigawa [6] introduced the *balanced decomposition number* of a graph. For a graph G, a *balanced colouring* of G is a pair (R, B), where $R, B \subset V(G)$, $R \cap B = \emptyset$, and $|R| = |B|$. We refer the vertices of R (resp. B) as the *red* (resp. *blue*) *vertices*, and those of $V(G) \setminus (R \cup B)$ the *uncoloured vertices*. A set $X \subset V(G)$ is a *balanced set* if $|X \cap R| = |X \cap B|$, and $G[X]$ is connected. A *balanced decomposition* of G is a partition $V(G) = V_1 \,\dot\cup \cdots \dot\cup\, V_r$ (for some $r \geq 1$), such that each V_i is a balanced set. We may also write the balanced decomposition as $\mathcal{P} = \{V_1, \ldots, V_r\}$. The *size* of $\mathcal{P}$ is the maximum of $|V_1|, \ldots, |V_r|$.

If G is a disconnected graph, then any balanced colouring of G with one red vertex and one blue vertex, in different components, has no possible balanced decomposition. Hence, we will only consider balanced decompositions for connected graphs.

If G is a connected graph of order n, and $k \in \mathbb{Z}$, $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, define

$$\begin{aligned} f(k, G) \;=\; & \min\{s \in \mathbb{N} : \text{ every balanced colouring } (R, B) \text{ of } G \text{ with} \\ & |R| = |B| = k \text{ has a balanced decomposition of size} \leq s\}. \end{aligned}$$

Note that $f(k, G) \leq n$, so that $f(k, G)$ is well-defined. The *balanced decomposition number* of G is then defined as

$$f(G) = \max\left\{ f(k, G) : 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

Fujita and Nakamigawa [6] made the following conjecture.

Conjecture 1 [6]**.** *If G is a 2-connected graph of order n, then $f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

The partial result when $G = C_n$, the cycle of order n, was solved [6].

Theorem 2 [6]**.** *If $n \geq 3$, then $f(C_n) = \lfloor \frac{n}{2} \rfloor + 1$.*

Also, the partial result when G is a generalised Θ-graph was solved [4]. A *generalised Θ-graph* (*with t paths*) is a graph G which is the union of $t \geq 2$ paths,

$Q_1, \ldots, Q_t$ say, with each having the same two end-vertices, x and y say, such that $V(Q_i) \cap V(Q_j) = \{x, y\}$ for any $i \neq j$. Note that the Q_i are pairwise internally vertex-disjoint paths. In addition, all but at most one of the Q_i have order at least 3. The vertices x and y are the *source* and *sink* of G. We also write $G = \Theta(Q_1, \ldots, Q_t)$.

In the proof of Theorem 3 [4], the following assertion, which contains a structural statement about balanced decompositions, was in fact proved.

Theorem 3 [4]**.** *Let $G = \Theta(Q_1, \ldots, Q_t)$ be a generalised Θ-graph of order n, where $t \geq 2$, with source x and sink y. Then $\lceil \frac{n-t+1}{2} \rceil \leq f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$. Hence, if t is fixed, then $f(G) = \frac{n}{2} + O(1)$.*

Furthermore, there exists a balanced decomposition $\mathcal{P}$ for G of size at most $\lfloor \frac{n}{2} \rfloor + 1$ with one of the following forms.

(i) $\mathcal{P} = \{V_1, V_2, V_3\}$, *where $x \in V_1$, $y \in V_2$, and $V_3 \subset V(\operatorname{int} Q_i)$ (possibly empty, whence $\mathcal{P} = \{V_1, V_2\}$) for some i.*

(ii) $\mathcal{P} = \{V_1, V_2\}$, *where $x, y \in V_1$; $V_2 \subset V(\operatorname{int} Q_i)$ for some i with $|V(Q_i)| \geq \lfloor \frac{n}{2} \rfloor + 2$; and $|V_2| = \lfloor \frac{n}{2} \rfloor$ or $|V_2| = \lfloor \frac{n}{2} \rfloor + 1$.*

(iii) $\mathcal{P} = \{V(Q_1), V(\operatorname{int} Q_2), \ldots, V(\operatorname{int} Q_t)\}$.

Finally, we have partial results when the number of coloured vertices of G is small [5, 6].

Theorem 4 [5, 6]**.** *If G is a 2-connected graph of order $n \geq \max(2k, 3)$, then $f(k, G) \leq \lfloor \frac{n}{2} \rfloor + 1$ for $k = 1, 2, 3$.*

Conjecture 1 remains open. In Section 3, we shall prove the partial result in the case when G is a subdivided K_4, which we denote by TK_4.

Theorem 5. *If G is a TK_4 of order n, then $f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

A graph is a *series-parallel* (*SP*) *graph* if it can be obtained as follows. Start with a path of length at least 1. Perform a sequence of operations of the following type successively.

($*$) Replace an edge with a generalised Θ-graph, by identifying the vertices of the edge with the source and the sink of the generalised Θ-graph.

The end-vertices of the initial path are the *source* and the *sink* of the SP graph.

There are many other formulations of SP graphs (see, for example [2, 3]), and they are easily seen to be equivalent to the above. By a result of Duffin [2], an SP graph is 2-connected if and only if it can be obtained as described above, with at least one operation, and the initial path has length 1. In Section 4, we shall prove the following case of Conjecture 1.

Theorem 6. *If G is a 2-connected series-parallel graph of order n, then $f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

Note that Theorems 5 and 6 are exclusive from each other. Theorem 6 has an instant corollary. A result of Elmallah and Colbourn [3] says that if G is a 3-connected planar graph, then G has a spanning 2-connected SP graph.

Corollary 7. *If G is a 3-connected planar graph of order n, then $f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

We remark that if the "2-connected" assumption on G is neglected in Theorem 6, then $f(G)$ can vary greatly. For example, if G is a path, then $f(G) = n$. On the other hand, if G is a generalised Θ-graph with paths of length 2, and with a "pendant" edge attached to the source, then it can be shown that $f(G) \leq \lfloor \frac{n}{2} \rfloor + 2$.

2. TOOLS

In this section, we develop some tools which we will need in the proofs of Theorems 5 and 6. Firstly, Lemma 8 below will be needed for both proofs.

Lemma 8. *Let G be a connected graph of order n. Suppose that there is a numbering of $V(G)$ with $1, \ldots, n$ such that the subgraph of G induced by any set of at least $\lceil \frac{n}{2} \rceil - 1$ consecutive vertices (modulo n) is connected. Then $f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

Proof. Let (R, B) be a balanced colouring of G. For $1 \leq i \leq n$, let A_i be the vertices numbered $i, i+1, \ldots, i + \lfloor \frac{n}{2} \rfloor - 1$ (modulo n), and $g(i) = |A_i \cap R| - |A_i \cap B|$. We have $|g(i) - g(i+1)| \leq 2$ (where $g(n+1) = g(1)$ by convention) for every i, and $\sum_{i=1}^{n} g(i) = 0$. Hence for some i, either $g(i) = 0$, or without loss of generality, $g(i) = -1$ and $g(i+1) = 1$. If the former, then $\{A_i, A_i^c\}$ is a suitable balanced decomposition, since $|A_i|, |A_i^c| \geq \lceil \frac{n}{2} \rceil - 1$. If the latter, let w be the vertex numbered $i + \lfloor \frac{n}{2} \rfloor$ (modulo n). Then, $w \in R$, and $|(A_i \cup \{w\}) \cap R| = |(A_i \cup \{w\}) \cap B|$. Hence, $\{A_i \cup \{w\}, (A_i \cup \{w\})^c\}$ is a suitable balanced decomposition, since $|A_i \cup \{w\}|, |(A_i \cup \{w\})^c| \geq \lceil \frac{n}{2} \rceil - 1$. ■

Next, we shall develop some ideas about SP graphs. This part of Section 2 can be interesting in its own right.

We first recall the well-known series and parallel compositions of SP graphs. Let G_1 and G_2 be two SP graphs, with sources a_1, a_2 and sinks b_1, b_2. Then, their *series composition* is the graph $G_1 +_s G_2$, formed by identifying b_1 and a_2. Their *parallel composition* is the graph $G_1 +_p G_2$, formed by identifying a_1, a_2, and b_1, b_2. Both of these compositions can be extended to three or more SP graphs in the obvious way. Observe that $G_1 +_s G_2$ is connected, but not 2-connected, while $G_1 +_p G_2$ is 2-connected.

For the rest of the paper, we assume that all SP graphs are obtained as follows. Start with a path G_0 with end-vertices x_0 and y_0, and replace edges successively with generalised Θ-graphs by the operation $(*)$ m times, for some $m \geq 1$. Let $T_1, \dots, T_m$ be the generalised Θ-graphs. For each i, let x_i and y_i be the source and sink of T_i. We make the following assumptions.

(†) No T_i replaces an edge e of some T_j $(j < i)$ which joins x_j and y_j.

Otherwise, the same final SP graph can be obtained by appending T'_j instead of T_j when T_j was appended, where T'_j is the graph obtained from T_j by replacing e with T_i (by the operation $(*)$).

(‡) For any i, T_i is appended as follows. T_i replaces the edge ab which appeared in some first T_j $(j < i)$, or in G_0. If the former, assume that $d_{T_j}(a, x_j) < d_{T_j}(b, x_j)$. If the latter, assume that $d_{G_0}(a, x_0) < d_{G_0}(b, x_0)$. In both cases, identify x_i with a, and y_i with b.

Now, for an SP graph G, we shall define a linear ordering $\prec$ on $V(G)$. First, for a generalised Θ-graph $T = \Theta(Q_1, \dots, Q_t)$ (for some $t \geq 2$) with source a and sink b, define a linear ordering $\prec_T$ on $V(T)$ as follows. We have $u \prec_T v$ if either $u = a$, or $v = b$, or $u \in V(\operatorname{int} Q_i)$ and $v \in V(\operatorname{int} Q_j)$ for some $i < j$, or $u, v \in V(\operatorname{int} Q_i)$ with $d_{Q_i}(u, a) < d_{Q_i}(v, a)$ for some i. Next, for $1 \leq i \leq m$, let G_i be the SP graph obtained after $T_1, \dots, T_i$ have been appended (so that $G = G_m$). We define a linear ordering $\prec_i$ on $V(G_i)$ for each i. Proceed inductively. Initially, define the linear ordering $\prec_0$ on $V(G_0)$ by $u \prec_0 v$ if $d_{G_0}(u, x_0) < d_{G_0}(v, x_0)$. Now for $i \geq 1$, suppose that we have defined the linear ordering $\prec_{i-1}$ on $V(G_{i-1})$. The graph T_i has a linear ordering $\prec_{T_i}$. The vertices x_i, y_i are identified with an edge $ab \in E(G_{i-1})$, and ab first appeared either as an edge of G_0, or when some T_j $(j < i)$ was appended. Define the linear ordering $\prec_i$ on $V(G_i)$ as follows.

- If $u, v \notin V(T_i - \{a, b\})$ and $u \prec_{i-1} v$ in G_{i-1}, then $u \prec_i v$.
- If $u, v \in V(T_i - \{a, b\})$ and $u \prec_{T_i} v$ in T_i, then $u \prec_i v$.
- Suppose that $u \notin V(T_i - \{a, b\})$ and $v \in V(T_i - \{a, b\})$.
  - If $ab \in E(G_0)$, or $ab \in E(T_j)$ with $a \neq x_j$, $b \neq y_j$, then $u \prec_i v$ if $u \prec_{i-1} a$ or $u = a$ in G_{i-1}, and $v \prec_i u$ otherwise.
  - If $ab \in E(T_j)$ with $a = x_j$, $b \neq y_j$, then $u \prec_i v$ if $u \prec_{i-1} b$ in G_{i-1}, and $v \prec_i u$ otherwise.
  - If $ab \in E(T_j)$ with $a \neq x_j$, $b = y_j$, then $u \prec_i v$ if $u \prec_{i-1} a$ or $u = a$ in G_{i-1}, and $v \prec_i u$ otherwise.

Finally, set $\prec = \prec_m$. Note that $\prec$ is well-defined, in view of (†) and (‡). In practice, the linear ordering $\prec$ is quite simple. Figure 1 shows an example.

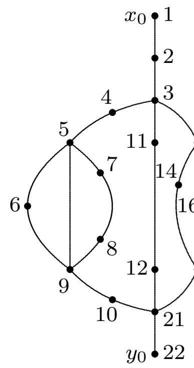

Figure 1. The linear ordering $\prec$.

With the linear ordering $\prec$ now defined, we have the following lemma.

Lemma 9. *Let G be an SP graph, with the linear ordering $\prec$ on $V(G)$ as defined. Then, every subgraph of G induced by an initial segment or a final segment of $\prec$ is connected.*

Proof. We use all the terms that we have already defined. We show inductively that the lemma holds for each $\prec_i$ on G_i. The lemma clearly holds for G_0. Now for $1 \leq i \leq m$, suppose that it holds for G_{i-1}. G_i is obtained from G_{i-1} by replacing an edge $ab \in E(G_{i-1})$ with the graph T_i, by identifying x_i, y_i with a, b in such a way that (†) and (‡) are satisfied. Note that $a \prec_{i-1} b$ and $a \prec_i b$. We also have the linear ordering $\prec_{T_i}$ on $V(T_i)$.

Observe that any initial segment and final segment of $\prec_{T_i}$ induces a connected subgraph of T_i. Also, by the definition of $\prec_i$, $V(T_i - \{x_i, y_i\})$ is a single segment in $V(G_i)$. Let $I \subset V(G_i)$ be an initial segment of $\prec_i$ and $I^c = V(G_i) \setminus I$ be the corresponding final segment. It suffices to show that both $G_i[I]$ and $G_i[I^c]$ are connected.

- If $a, b \notin I$, then I is also an initial segment of $\prec_{i-1}$ in $V(G_{i-1})$. Let $I' = V(G_{i-1}) \setminus I$ be the corresponding final segment. We have $G_i[I] = G_{i-1}[I]$, and $G_i[I^c]$ is obtained from $G_{i-1}[I']$ by replacing ab with T_i, so is connected. A similar argument applies if $a, b \in I$.

- If $a \in I$ and $b \notin I$, then $G_i[I]$ is formed by attaching $T_i[J']$ to $G_{i-1}[J]$ at a, where J is some initial segment of $\prec_{i-1}$ in $V(G_{i-1})$ containing a but not b, and J' is some initial segment of $\prec_{T_i}$ in $V(T_i)$. Clearly, $G_i[I]$ is connected. $G_i[I^c]$ has a similar structure, so it is also connected.

■

3. SUBDIVISION OF K_4

***Proof of Theorem* 5.** Let G be a TK_4 of order n. Let $x_1, \ldots, x_4$ be the branch vertices of G, and $Q_1, \ldots, Q_6$ be the sub-paths joining them, with $V(Q_1) \cap V(Q_5) \cap V(Q_6) = \{x_1\}$, $V(Q_1) \cap V(Q_2) \cap V(Q_3) = \{x_2\}$, $V(Q_2) \cap V(Q_4) \cap V(Q_6) = \{x_3\}$, $V(Q_3) \cap V(Q_4) \cap V(Q_5) = \{x_4\}$. Assume that $|V(Q_i)| \geq 3$ for each i (so that $n \geq 10$), otherwise the result follows from Theorem 3. Let (R, B) be a balanced colouring of G.

Case 1. $|V(Q_i)| \leq \lfloor \frac{n}{2} \rfloor$ for all $1 \leq i \leq 6$. We proceed by proving several claims.

Claim 10. *There exist partitions* $V(\operatorname{int} Q_4) = S_1 \dot\cup S_2$, $V(\operatorname{int} Q_5) = T_1 \dot\cup T_2$ *and* $V(\operatorname{int} Q_6) = U_1 \dot\cup U_2$ *such that the graphs* $H_1 = G[V(Q_1) \cup T_1 \cup U_2]$, $H_2 = G[V(Q_2) \cup U_1 \cup S_2]$ *and* $H_3 = G[V(Q_3) \cup S_1 \cup T_2]$ *are connected, with* $|V(H_i)| \leq \lfloor \frac{n}{2} \rfloor$ *for each* i.

Proof. Recall that $3 \leq |V(Q_i)| \leq \lfloor \frac{n}{2} \rfloor$ for each i. If there is no partition $V(\operatorname{int} Q_4) = S_1 \dot\cup S_2$ such that $G[V(Q_2) \cup S_2]$ and $G[V(Q_3) \cup S_1]$ are connected and have order at most $\lfloor \frac{n}{2} \rfloor$, then we would have $|V(G)| \geq 2\lfloor \frac{n}{2} \rfloor + 4 > n$, a contradiction. Hence, take a suitable partition $V(\operatorname{int} Q_4) = S_1 \dot\cup S_2$. If partitions $V(\operatorname{int} Q_5) = T_1 \dot\cup T_2$ and $V(\operatorname{int} Q_6) = U_1 \dot\cup U_2$ do not simultaneously exist such that $G[V(Q_1) \cup T_1 \cup U_2]$, $G[V(Q_2) \cup U_1 \cup S_2]$ and $G[V(Q_3) \cup S_1 \cup T_2]$ are connected and have order at most $\lfloor \frac{n}{2} \rfloor$, then a similar counting argument gives $|V(G)| \geq 2\lfloor \frac{n}{2} \rfloor + 2 > n$, another contradiction. □

From here, H_1, H_2 and H_3 are defined as in Claim 10.

Claim 11. *For some* i*, there exists a balanced set* $A \subset V(H_i)$*, with* $x_2 \in A$.

Proof. The claim holds if $x_2 \in (R \cup B)^c$. Without loss of generality, let $x_2 \in R$. If such a set A does not exist, then $|V(H_i) \cap R| > |V(H_i) \cap B|$ for every i, so $|R| = 1 + \sum_{i=1}^{3} |V(H_i - x_2) \cap R| \geq 1 + \sum_{i=1}^{3} |V(H_i) \cap B| > |B|$, a contradiction. □

Claim 12. *Let* $A \subsetneq V(Q_4 \cup Q_5 \cup Q_6)^c$ *be a balanced set with* $x_2 \in A$, $|A| \leq \lfloor \frac{n}{2} \rfloor - 1$, *and* $N(A) \subset R$ *or* $N(A) \subset B$. *Then for some* i*, there exists a balanced set* $C \subset V(H_i - A)$ *with* $N(C) \cap V(H_i) \cap A \neq \emptyset$.

Proof. It suffices to prove the claim with $N(A) \subset R$. If such a set C does not exist, then $|V(H_i - A) \cap R| > |V(H_i - A) \cap B|$ for every i. But, $|A^c \cap R| = \sum_{i=1}^{3} |V(H_i - A) \cap R| > \sum_{i=1}^{3} |V(H_i - A) \cap B| = |A^c \cap B|$, a contradiction, since A^c is a balanced set. □

Claim 13. *Let A be a balanced set with $V(Q_1) \subset A \subset V(Q_4)^c$, $|A| \le \lfloor \frac{n}{2} \rfloor - 1$, $N(A) \setminus \{x_3, x_4\} \subset R$ or $N(A) \setminus \{x_3, x_4\} \subset B$, and $A \not\supset V(\mathrm{int}\, Q_2 \cup \mathrm{int}\, Q_6)$, $A \not\supset V(\mathrm{int}\, Q_3 \cup \mathrm{int}\, Q_5)$. Then, there exists a balanced set $C \subset X$ with $N(C) \cap A \neq \emptyset$, $|C| \le \lfloor \frac{n}{2} \rfloor$ and $(A \cup C)^c$ is connected, where X is either $V((Q_3 \cup Q_5 \cup \mathrm{int}\, Q_4) - A)$ or $V((Q_2 \cup Q_6 \cup \mathrm{int}\, Q_4) - A)$.*

Proof. It suffices to prove the claim with $N(A) \setminus \{x_3, x_4\} \subset R$. Let $V_i = V(\mathrm{int}\, Q_i - A)$ for $i \in \{2, 3, 5, 6\}$, and $I = \{i \in \{2, 3, 5, 6\} : V_i \neq \emptyset\}$.

If we cannot find a suitable set $C \subset V_i \cup \{x_4\}$ for some $i \in \{3, 5\}$, then $|(V_3 \cup V_5 \cup \{x_4\}) \cap R| > |(V_3 \cup V_5 \cup \{x_4\}) \cap B|$ (whether $3, 5 \in I$, or $3 \in I$, $5 \notin I$, or $3 \notin I$, $5 \in I$). Similarly, $|(V_2 \cup V_6 \cup \{x_3\}) \cap R| > |(V_2 \cup V_6 \cup \{x_3\}) \cap B|$. Since A^c is a balanced set, it is clear that there exists a partition $V(Q_4) = W_1 \dot\cup W_2$ such that $V_3 \cup V_5 \cup W_1$ and $V_2 \cup V_6 \cup W_2$ are balanced sets. One of these has at most $\lfloor \frac{n}{2} \rfloor$ vertices and hence is a suitable set for C. □

Claim 14. *Let A be a balanced set with $V(Q_1 \cup Q_3 \cup Q_5) \setminus \{x_4\} \subset A \subset V(G - x_3)$, $|A| \le \lfloor \frac{n}{2} \rfloor - 1$, and $N(A) \setminus \{x_3\} \subset R$ or $N(A) \setminus \{x_3\} \subset B$. Then, there exists a balanced set $C \subset X$ with $N(C) \cap X \cap A \neq \emptyset$ and $|C| \le \lfloor \frac{n}{2} \rfloor$, where X is either $V(\mathrm{int}\, Q_2 - A)$, $V(Q_4 - (A \cup \{x_3\}))$ or $V(\mathrm{int}\, Q_6 - A)$.*

Proof. It suffices to prove the claim with $N(A) \setminus \{x_3\} \subset R$. For $i \in \{2, 4, 6\}$, let $X_i = V(Q_i - (A \cup \{x_3\}))$. Let $J = \{i \in \{2, 4, 6\} : X_i \neq \emptyset\}$. Note that $|J| \ge 2$, otherwise we have $|A| \ge n - \lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil$, a contradiction. If we cannot find a suitable $C \subset X_i$ for some $i \in J$, then $|X_i \cap R| > |X_i \cap B|$ for each $i \in J$. But, $|A^c \cap R| \ge \sum_{i \in J} |X_i \cap R| \ge \sum_{i \in J} |X_i \cap B| + 2 > |A^c \cap B|$, a contradiction, since A^c is a balanced set. □

We now describe an algorithm. Take a balanced set A_1 as given by Claim 11. Without loss of generality, $A_1 \subset V(H_1)$. We have $|A_1| \le \lfloor \frac{n}{2} \rfloor$.

Step 1. If $|A_1| = \lfloor \frac{n}{2} \rfloor$, stop. $\{A_1, A_1^c\}$ is a suitable balanced decomposition for G. Otherwise, $|A_1| \le \lfloor \frac{n}{2} \rfloor - 1$. If $V(Q_1) \subset A_1$, go to Step 3. If $A_1 = V(Q_4 \cup Q_5 \cup Q_6)^c$, stop. $V(Q_4 \cup Q_5 \cup Q_6)$ is a balanced set, so by Theorem 2, $Q_4 \cup Q_5 \cup Q_6$ has a balanced decomposition $\mathcal{P}$ with size at most $\lfloor \frac{n}{2} \rfloor + 1$. Hence, $\{A_1\} \cup \mathcal{P}$ is a suitable balanced decomposition for G. Otherwise, $A_1 \subsetneq V(Q_4 \cup Q_5 \cup Q_6)^c$. If there is an uncoloured vertex $u \in N(A_1)$, or red and blue vertices $v, w \in N(A_1)$, let $A_2 = A_1 \cup \{u\}$ or $A_2 = A_1 \cup \{v, w\}$ accordingly; if not, go to Step 2. We can choose u, or v and w, such that at most one of x_1, x_3, x_4 is appended to A_1. A_2 is another balanced set. If $|A_2| = \lfloor \frac{n}{2} \rfloor$ or $|A_2| = \lfloor \frac{n}{2} \rfloor + 1$, stop; $\{A_2, A_2^c\}$ is a suitable balanced decomposition for G. Otherwise, $|A_2| \le \lfloor \frac{n}{2} \rfloor - 1$. If we have appended exactly one of x_1, x_3, x_4, go to Step 3, using A_2 for A_1. Otherwise, repeat Step 1, using A_2 for A_1.

Step 2. We have either $N(A_1) \subset R$ or $N(A_1) \subset B$. The set A_1 satisfies the conditions of Claim 12, and we can find a balanced set C as described. If $|A_1 \cup C| \geq \lfloor \frac{n}{2} \rfloor$, stop. We have a suitable balanced decomposition $\{A_1, C, (A_1 \cup C)^c\}$ for G. Otherwise, $|A_1 \cup C| \leq \lfloor \frac{n}{2} \rfloor - 1$. If we have exactly one of x_1, x_3, x_4 in $A_1 \cup C$, go to Step 3, using $A_1 \cup C$ for A_1. Otherwise, go back to Step 1, using $A_1 \cup C$ for A_1.

Step 3. Re-label the Q_i, x_j and H_k by cycling $Q_1 \to Q_2 \to Q_3 \to Q_1$, $Q_4 \to Q_5 \to Q_6 \to Q_4$, $x_1 \to x_3 \to x_4 \to x_1$, and $H_1 \to H_2 \to H_3 \to H_1$, so that $V(Q_1) \subset A_1 \subset V(Q_4)^c$ in the re-labelling. If $A_1 \supset V(\operatorname{int} Q_3 \cup \operatorname{int} Q_5)$ or $A_1 \supset V(\operatorname{int} Q_2 \cup \operatorname{int} Q_6)$, go to Step 5. Otherwise, if there is an uncoloured vertex $u \in N(A_1) \setminus \{x_3, x_4\}$, or red and blue vertices $v, w \in N(A_1) \setminus \{x_3, x_4\}$, let $A_3 = A_1 \cup \{u\}$ or $A_3 = A_1 \cup \{v, w\}$ accordingly; if not, go to Step 4. If $|A_3| = \lfloor \frac{n}{2} \rfloor$ or $|A_3| = \lfloor \frac{n}{2} \rfloor + 1$, stop. We have a suitable balanced decomposition $\{A_3, A_3^c\}$ for G. Otherwise, $|A_3| \leq \lfloor \frac{n}{2} \rfloor - 1$. If $A_3 \supset V(\operatorname{int} Q_3 \cup \operatorname{int} Q_5)$ or $A_3 \supset V(\operatorname{int} Q_2 \cup \operatorname{int} Q_6)$, go to Step 5, using A_3 for A_1. Otherwise, repeat Step 3, using A_3 for A_1.

Step 4. We have either $N(A_1) \setminus \{x_3, x_4\} \subset R$ or $N(A_1) \setminus \{x_3, x_4\} \subset B$. The set A_1 satisfies the conditions of Claim 13, and we can find a balanced set C as described. If $|A_1 \cup C| \geq \lfloor \frac{n}{2} \rfloor$, stop. We have a suitable balanced decomposition $\{A_1, C, (A_1 \cup C)^c\}$ for G. Otherwise, $|A_1 \cup C| \leq \lfloor \frac{n}{2} \rfloor - 1$. If we have exactly one of x_3, x_4 in $A_1 \cup C$, go to Step 5, using $A_1 \cup C$ for A_1. Otherwise, go back to Step 3, using $A_1 \cup C$ for A_1.

Step 5. Re-label the Q_i, x_j and H_k by $Q_2 \leftrightarrow Q_3$, $Q_5 \leftrightarrow Q_6$, $x_3 \leftrightarrow x_4$ and $H_2 \leftrightarrow H_3$, so that $V(Q_1 \cup Q_3 \cup Q_5) \setminus \{x_4\} \subset A_1 \subset V(G - x_3)$ in the re-labelling. If there is an uncoloured vertex $u \in N(A_1) \setminus \{x_3\}$, or red and blue vertices $v, w \in N(A_1) \setminus \{x_3\}$, let $A_4 = A_1 \cup \{u\}$ or $A_4 = A_1 \cup \{v, w\}$ accordingly. If $|A_4| = \lfloor \frac{n}{2} \rfloor$ or $|A_4| = \lfloor \frac{n}{2} \rfloor + 1$, stop. We have a suitable balanced decomposition $\{A_4, A_4^c\}$ for G. Otherwise, $|A_4| \leq \lfloor \frac{n}{2} \rfloor - 1$. Repeat Step 5, using A_4 for A_1. If $N(A_1) \setminus \{x_3\} \subset R$ or $N(A_1) \setminus \{x_3\} \subset B$, then A_1 satisfies the conditions of Claim 14, and we can find a balanced set C as described. If $|A_1 \cup C| \geq \lfloor \frac{n}{2} \rfloor$, stop. We have a suitable balanced decomposition $\{A_1, C, (A_1 \cup C)^c\}$ for G. Otherwise, $|A_1 \cup C| \leq \lfloor \frac{n}{2} \rfloor - 1$. Repeat Step 5, using $A_1 \cup C$ for A_1.

The algorithm must terminate, since whenever we append new vertices, we are increasing the number of vertices in A_1. When the algorithm terminates, we will obtain a suitable balanced decomposition for G.

Case 2. Without loss of generality, $|V(Q_1)| \geq \lfloor \frac{n}{2} \rfloor + 1$. Number the vertices of G with $1, \ldots, n$ as follows. Start at x_1 and move along Q_1 to x_2. Then, move along Q_2 to the vertex adjacent to x_3. Then, move along Q_3 from the vertex adjacent to x_2 to the vertex adjacent to x_4. Then, move along Q_4 from x_3 to x_4. Then, move along Q_5 from the vertex adjacent to x_4 to the vertex adjacent to x_1.

Finally, move along Q_6, from the vertex adjacent to x_3 to the vertex adjacent to x_1.

This numbering satisfies the condition of Lemma 8. Indeed, let $A \subset V(G)$ be a set of consecutive vertices (modulo n), with first vertex v, and $|A| \geq \lceil \frac{n}{2} \rceil - 1$. Every initial segment induces a connected subgraph of G. If $v \in V(Q_1)$, then $G[A]$ is connected. If $v \in V(Q_1)^c$, then note that $|V(Q_1)^c| \leq n - (\lfloor \frac{n}{2} \rfloor + 1) \leq |A|$. Hence, either $A = V(Q_1)^c$, or A is the union of an initial segment and a final segment. In either case, $G[A]$ is connected. Hence by Lemma 8, we have $f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$.

The proof of Theorem 5 is now complete. ∎

4. Series-parallel Graphs

***Proof of Theorem* 6.** Let G be a 2-connected SP graph of order n. G can be obtained as described in Section 2, where (†) and (‡) are satisfied. Let $T_1, \ldots, T_m$ be the generalised Θ-graphs, for some $m \geq 1$. Let $T_1 = \Theta(Q_1, \ldots, Q_t)$, for some $t \geq 2$, with source x and sink y. For a subgraph $F \subset T_1$, let $\langle F \rangle \subset G$ be the subgraph of G that F has been transformed to.

Let (R, B) be a balanced colouring of G. We shall prove a stronger assertion. There exists a balanced decomposition $\mathcal{P}$ for G of size at most $\lfloor \frac{n}{2} \rfloor + 1$, with one of the following forms.

(i) $\mathcal{P} = \{V_1, V_2, V_3\}$, where $x \in V_1$, $y \in V_2$, and $V_3 \subset V(\langle Q_i \rangle - \{x, y\})$ (possibly empty, whence $\mathcal{P} = \{V_1, V_2\}$) for some i.

(ii) $\mathcal{P} = \{V_1, V_2\}$, where $x, y \in V_1$, and $V_2 \subset V(\langle Q_i \rangle - \{x, y\})$ for some i with $|V(\langle Q_i \rangle)| \geq \lfloor \frac{n}{2} \rfloor + 2$, and $|V_2| = \lfloor \frac{n}{2} \rfloor$ or $|V_2| = \lfloor \frac{n}{2} \rfloor + 1$.

(iii) $\mathcal{P} = \{V(\langle Q_1 \rangle), V(\langle Q_2 \rangle - \{x, y\}), \ldots, V(\langle Q_t \rangle - \{x, y\})\}$.

Case 1. $|V(\langle Q_i \rangle)| \leq \lfloor \frac{n}{2} \rfloor + 1$ for all $1 \leq i \leq t$. We use induction on m. By Theorem 3, the stronger assertion holds for $m = 1$. Now let $m \geq 2$ and suppose that the result holds for any 2-connected SP graph that can be obtained from $m - 1$ applications of the operation $(*)$.

Let $T_m = \Theta(R_1, \ldots, R_s)$ for some $s \geq 2$, with source a and sink b. T_m has a linear ordering $\prec_{T_m}$ as described in Section 2. Obtain the graph H from G as follows. Replace T_m with a path P of order $|V(T_m)|$ by identifying the end-vertices of P with a and b, and with the vertex $u \in V(T_m)$ corresponding to the vertex $u' \in V(P)$ by $d_P(u', a) + 1$ being the position of u in $\prec_{T_m}$. Also, let u' inherit the colour of u, and let $\prec_P$ be the corresponding linear ordering on $V(P)$.

The graph H can be obtained by $m - 1$ applications of the operation $(*)$, so by induction, H has a balanced decomposition $\mathcal{P}'$ of size at most $\lfloor \frac{n}{2} \rfloor + 1$, with one of the forms (i) to (iii) as described above. If $\mathcal{P}'$ is of form (iii), then

$\mathcal{P}'$ is a suitable balanced decomposition for G of form (iii), in view of (†) (since $\{a,b\} \neq \{x,y\}$). If $\mathcal{P}'$ is of form (i) or (ii), then the path P is partitioned into at most three sub-paths. If P is divided into one or two sub-paths, then by Lemma 9, $\mathcal{P}'$ is still a balanced decomposition in G and is of form (i) or (ii). If P is divided into three sub-paths and $\mathcal{P}'$ is of form (ii), then $\mathcal{P}' = \{V_1, V_2\}$ as described. The end-vertices of P must be in V_1. This means that $|V(P)| \geq |V_2| + 2 \geq \lfloor \frac{n}{2} \rfloor + 2$, a contradiction. Now, assume that $\mathcal{P}'$ is of form (i). Then, $\mathcal{P}' = \{V_1, V_2, V_3\}$ as described. $G[V_1]$ has the following structure: Take $H[V_1]$, remove an initial segment of P (w.r.t. $\prec_P$), and replace with the corresponding initial segment of $V(T_m)$ (w.r.t. $\prec_{T_m}$). By Lemma 9, $G[V_1]$ is connected. Similarly, $G[V_2]$ is also connected. Now, V_3 is a middle segment of $V(T_m)$ (w.r.t. $\prec_{T_m}$), so $G[V_3]$ consists of possibly several disjoint paths, each one being a sub-path of R_j for some j.

We now describe an algorithm.

Step 1. If $|V_1| = \lfloor \frac{n}{2} \rfloor$ or $|V_1| = \lfloor \frac{n}{2} \rfloor + 1$, stop. We have a suitable balanced decomposition $\{V_1, V_1^c\}$ for G. Otherwise, $|V_1| \leq \lfloor \frac{n}{2} \rfloor - 1$. If $N(V_1) \cap V(T_m - b) \subset R$ or $N(V_1) \cap V(T_m - b) \subset B$, go to Step 2. Otherwise, there is an uncoloured vertex $u \in N(V_1) \cap V(T_m - b)$, or red and blue vertices $v, w \in N(V_1) \cap V(T_m - b)$. Let $V_1' = V_1 \cup \{u\}$ or $V_1' = V_1 \cup \{v, w\}$, and $V_3' = V_3 \setminus \{u\}$ or $V_3' = V_3 \setminus \{v, w\}$ accordingly. Repeat Step 1, using V_1' for V_1, and V_3' for V_3.

Step 2. Note that $G[V_3]$ consists of paths $A_1, \ldots, A_r$, where for each i, $A_i \subset \operatorname{int} R_j$ for some j, and A_i has one end-vertex adjacent to a vertex in V_1, the other adjacent to a vertex in V_2. Let $a_1, \ldots, a_r$ be the end-vertices adjacent to vertices in V_1. Since $N(V_1) \cap V(T_m - b) \subset R$ (resp. $N(V_1) \cap V(T_m - b) \subset B$), we have $a_1, \ldots, a_r \in R$ (resp. $a_1, \ldots, a_r \in B$). Since $|V_3 \cap R| = |V_3 \cap B|$, for some i and $x \in V(A_i)$, the path $Q = a_i \cdots x \subset A_i$ satisfies $|V(Q) \cap R| = |V(Q) \cap B|$. If $|V_1 \cup V(Q)| \geq \lfloor \frac{n}{2} \rfloor + 2$, stop. $\{V_1, V(Q), (V_1 \cup V(Q))^c\}$ is a suitable balanced decomposition for G, since $|V(Q)| < |V(T_m)| \leq \lfloor \frac{n}{2} \rfloor$. Otherwise, return to Step 1, using $V_1 \cup V(Q)$ for V_1 and $V_3 \setminus V(Q)$ for V_3.

This algorithm must terminate, and when it does so, we have a balanced decomposition of size at most $\lfloor \frac{n}{2} \rfloor + 1$ for G, and with a structure of form (i) or (ii). This completes the proof of Case 1.

Case 2. Without loss of generality, $|V(\langle Q_1 \rangle)| \geq \lfloor \frac{n}{2} \rfloor + 2$. For $a, b \in V(Q_1)$, let $a \cdots b \subset Q_1$ be the sub-path with end-vertices a and b. Let $Q_1 = u_1 \cdots u_s$ for some $s \geq 3$, where $u_1 = x$, $u_s = y$, and let $Q' = \langle Q_2 \rangle +_p \cdots +_p \langle Q_t \rangle$. Note that $|V(Q')| \leq \lceil \frac{n}{2} \rceil$. We have

(a) either $|V(\langle u_j u_{j+1} \rangle)| \leq \lceil \frac{n}{2} \rceil$ for every $1 \leq j < s$,

(b) or $|V(\langle u_j u_{j+1} \rangle)| \geq \lceil \frac{n}{2} \rceil + 1$ for some $1 \leq j < s$.

If (b) holds, let $Q'' = \langle u_{j+1} \cdots u_s \rangle +_s Q' +_s \langle u_1 \cdots u_j \rangle$, so that $|V(Q'')| \le \lfloor \frac{n}{2} \rfloor + 1$. Now, $\langle u_j u_{j+1} \rangle$ is a 2-connected SP graph, so we have $\langle u_j u_{j+1} \rangle = H_1 +_p \cdots +_p H_r$ for some $r \ge 2$, and for every k, we have H_k is SP, and not a parallel composition. Consider G as $G = H_1 +_p \cdots +_p H_r +_p Q''$. If $|V(H_k)| \le \lfloor \frac{n}{2} \rfloor + 1$ for every k, then we are done by applying Case 1. Otherwise, if $|V(H_\ell)| \ge \lfloor \frac{n}{2} \rfloor + 2$ for some ℓ, then we can go back to the start of Case 2. This procedure cannot be repeated infinitely often, since each time that we have to restart Case 2, with (b) holding and we obtain the corresponding graph H_ℓ, the graphs H_ℓ have strictly decreasing orders.

Hence after applying the above procedure finitely many times, G will have a structure such that (a) holds. We can describe this structure as follows. There are vertices $v_1, \ldots, v_q \in V(G)$, where $q \ge 3$, and SP graphs $F_1, \ldots, F_q$ such that for every $1 \le i \le q$, F_i has source v_i and sink v_{i+1} (where $v_{q+1} = v_1$ by convention), $|V(F_i)| \le \lceil \frac{n}{2} \rceil$, and G is the union of the F_i in this way. Number $V(G)$ with $1, \ldots, n$ as follows. Each F_i has a linear ordering $\prec_i$ as described in Section 2. We have u precedes v in the numbering if either $u = v_1$, or $u, v \in V(F_i - v_i)$ and $u \prec_i v$ for some i, or $u \in V(F_i - v_i)$ and $v \in V(F_j - v_j)$ for some $i < j$. This numbering satisfies the condition of Lemma 8. Indeed, let $A \subset V(G)$ be a set of consecutive vertices (modulo n) with $|A| \ge \lceil \frac{n}{2} \rceil - 1$. Since $|V(F_i)| \le \lceil \frac{n}{2} \rceil$ for every i, the vertices of $v_1, \ldots, v_q$ in A are consecutive. Without loss of generality, $v_1, \ldots, v_p \in A$, where $p \ge 1$. Then, $A = V(F_1 \cup \cdots \cup F_{p-1}) \cup I \cup J$, where I is an initial segment of $F_p - v_{p+1}$ (w.r.t. $\prec_p$), and J is a final segment of $F_q - v_q$ (w.r.t. $\prec_q$). By Lemma 9, $G[I]$ and $G[J]$ are both connected, so $G[A]$ is also connected. Hence, applying Lemma 8, we have $f(G) \le \lfloor \frac{n}{2} \rfloor + 1$, and we are done for Case 2.

This completes the proof of Theorem 6. ■

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