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Dedicated to Mieczysław Borowiecki on the occasion of his 70th birthday

ACYCLIC 6-COLOURING OF GRAPHS WITH MAXIMUM DEGREE 5 AND SMALL MAXIMUM AVERAGE DEGREE

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Abstract

A *k-colouring* of a graph G is a mapping c from the set of vertices of G to the set $\{1, \ldots, k\}$ of colours such that adjacent vertices receive distinct colours. Such a k-colouring is called *acyclic*, if for every two distinct colours i and j, the subgraph induced by all the edges linking a vertex coloured with i and a vertex coloured with j is acyclic. In other words, every cycle in G has at least three distinct colours.

Acyclic colourings were introduced by Grünbaum in 1973, and since then have been widely studied. In particular, the problem of acyclic colourings of graphs with bounded maximum degree has been investigated. In 2011, Kostochka and Stocker showed that any graph with maximum degree 5 can be acyclically coloured with at most 7 colours. The question, whether this bound is achieved, remains open. In this note we prove that any graph with maximum degree 5 and maximum average degree at most 4 admits an acyclic 6-colouring. We also provide examples of graphs with these properties.

Keywords: acyclic colouring, bounded degree graph, maximum average degree.

2010 Mathematics Subject Classification: 05C15.

1. Introduction and Notation

In this note we address the problem of acyclic colourings of graphs with maximum degree at most 5 and small maximum average degree. All considered graphs are finite and simple, i.e., without loops or multiple edges. We use standard notation. For a graph G we denote its vertex and edge set by $V(G)$ and $E(G)$, respectively.

For a vertex $x \in V(G)$, $d(x)$ stands for the degree of x. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum and minimum vertex degree of G, respectively. For undefined concepts we refer the reader to [9].

A *k-colouring* of a graph G is a mapping c from the set of vertices of G to the set $\{1, \ldots, k\}$ of colours such that adjacent vertices receive distinct colours. Such k-colouring is called *acyclic*, if for every two distinct colours i and j, the subgraph induced by all the edges linking a vertex coloured with i and a vertex coloured with j is acyclic. In other words, every cycle in G has at least three distinct colours. The minimum k such that G has an acyclic k-colouring is called the *acyclic chromatic number* of G, denoted by $\chi_a(G)$.

Acyclic colourings were introduced by Grünbaum in 1973 [5], and since then have been widely studied and investigated by many authors. In particular, a lot of attention has been paid to the problem of acyclic colourings of graphs with bounded maximum degree. Grünbaum showed in [5] that the acyclic chromatic number of any cubic graph is at most 4. Later, Skulrattanakulchai presented a linear time algorithm that uses at most 4 colours to acyclically colour the vertices of any graph with maximum degree at most 3 [8]. Burstein [3] proved that the acyclic chromatic number of graphs with maximum degree 4 is at most 5. Acyclic colourings of planar and outerplanar graphs were also considered, see, for instance [1, 2]. The problem of determining the acyclic chromatic number of a graph is very difficult in general, even for graphs with bounded degree. Kostochka proves in [6] that it is an NP-complete problem to decide for a given graph G whether $\chi_a(G) \leq 3$.

The paper concerns acyclic colourings of graphs with maximum degree 5. Fertin and Raspaud considered this problem in [4] and proved that any such graph admits an acyclic 9-colouring. Recently, Kostochka and Stocker improved this bound to 7, see [7]. The question, posed in [7], is whether there are graphs for which this bound is achieved? It is known, that there are graphs with maximum degree 5 and acyclic chromatic number 6 (for example K_6, the complete graph on 6 vertices). In this paper we give a non-trivial family of graphs with maximum degree 5, which are acyclically 6 colourable. We use the notion of the *maximum average degree*, $Mad(G)$, of a graph G defined as follows:

$$Mad(G) = \max\left\{2\frac{|E(H)|}{|V(H)|} :\ H \subseteq G\right\}.$$

Our main result is

Theorem 1. *Let G be a graph such that $\Delta(G) \leq 5$ and $Mad(G) \leq 4$. Then*

$$\chi_a(G) \leq 6.$$

We also present examples of graphs that satisfy the assumptions of Theorem 1 and need at least 6 colours in any acyclic colouring.

2. Proof of Theorem 1

Before we proceed, we first state the lemma presenting some structural properties of graphs with maximum degree 5 and maximum average degree at most 4.

Lemma 2. *Let G be a graph such that $\Delta(G) = 5$, $\delta(G) \geq 3$ and $Mad(G) \leq 4$. Then G contains at least one of the following configurations:*

($A1$) a vertex of degree at most 4 adjacent to a vertex of degree 3, or

($A2$) a vertex of degree 5, adjacent to at least two vertices of degree 3.

Proof. We use the discharging method to prove the lemma. Let $G = (V, E)$ be a graph such that $\Delta(G) = 5$, $\delta(G) \geq 3$ and $Mad(G) \leq 4$. Initially, we define a mapping w on V as follows: for each $x \in V$ let $w(x) = d(x)$. Clearly, the fact that $Mad(G) \leq 4$ yields

$$\sum_{x\in V} w(x) \leq 4|V|. \tag{1}$$

In the discharging step, the values of w are distributed between adjacent vertices, according to the rule described below. In this way we obtain a new mapping w'. After this procedure, each $x \in V$ has a new value $w'(x)$, but the sums of values of w' and w, counted over all the vertices, remain the same. We show that if G contains neither (A1) nor (A2), then for each vertex x, we have $w'(x) \geq 4$ and there exists at least one vertex for which this value is strictly greater than 4, obtaining a contradiction with inequality (1). We have only one rule for distributing values between adjacent vertices:

(R1) If x is a vertex of degree 5, then x gives $\frac{1}{2}$ to each its neighbour of degree 3.

Now we compute the values of vertices of G, considering several cases, depending on the degree of x.

If $d(x) = 3$, then $w'(x) = 3 + 3 \cdot \frac{1}{2} = 4\frac{1}{2} > 4$, because G does not contain the configuration (A1).

If $d(x) = 4$, then $w'(x) = w(x) = 4$.

If $d(x) = 5$, then observe that x is adjacent to at most one vertex of degree 3, since otherwise configuration (A2) occurs. Hence, $w'(x) \geq 5 - \frac{1}{2} = 4\frac{1}{2} > 4$.

As we have shown, for each vertex x of G, if $d(x) = 4$, then $w'(x)$ equals 4, and if x is of degree 3 or 5, then $w'(x)$ is greater than 4. This and the fact $\Delta(G) = 5$ implies $\sum_{x\in V} w(x) = \sum_{x\in V} w'(x) > 4|V|$, contrary to inequality (1). ∎

Now we are ready to prove Theorem 1. To this end, we introduce two useful notions. Let G be a graph and assume c is its acyclic k-colouring. For a vertex $x \in$

$V(G)$, we denote by $C(x)$ the multiset of colours assigned by c to the neighbours of x. Let i, j be distinct colours, $x, y \in V(G)$, and P be a path from x to y in G. We call a path P an *alternating* (i, j)*-path* (*from* x *to* y), if the vertices of P are alternately coloured with i and j.

***Proof of Theorem* 1.** Let $G = (V, E)$ be a minimal, with respect to the number of edges, counterexample to the theorem. There is no loss of generality in assuming G is connected. It is easy to see $\delta(G) \geq 2$. If $\Delta(G) \leq 4$, then from the theorem of Burnstein [3] it follows that G has an acyclic 5-colouring. Hence we may assume $\Delta(G) = 5$. Now we provide some additional properties of G.

Claim 1. *G does not contain vertices of degree* 2.

Proof. Assume, contrary to our claim, that there is a vertex x of degree 2. Let y and z be its neighbours. Consider $G' = G - xy$. G being a minimal counterexample implies G' has an acyclic 6-colouring c. We consider two cases.

Case 1. Let $c(x) \neq c(y)$, w.l.o.g., $c(x) = 1$, $c(y) = 2$. We cannot extend this colouring to an acyclic 6-colouring of G only if we have an alternating $(1, 2)$-path from x to y, passing through z (if this is not the case, then adding the edge xy does not create any bichromatic cycle). It follows $c(z) = 2$ and $1 \in C(z)$. Since $d(z) \leq 5$, there is a colour $\alpha \in \{3, 4, 5, 6\}$, $\alpha \notin C(z)$ such that we can recolour x with α, obtaining an acyclic 6-colouring that can be extended.

Case 2. Assume $c(x) = c(y)$, w.l.o.g., $c(x) = 1 = c(y)$. Observe $c(z) \neq 1$. Hence we can recolour x with any colour $\alpha \in \{2, 3, 4, 5, 6\}$, $\alpha \neq c(z)$, obtaining an acyclic 6-colouring that clearly can be extended. □

Claim 2. *G contains no vertex of degree at most* 4 *adjacent to a vertex of degree* 3.

Proof. Assume to the contrary that there is a vertex x of degree at most 4 adjacent to a vertex y of degree 3. We may assume $d(x) = 4$, since similar arguments hold if $d(x) = 3$. Let $G' = G - xy$. From the fact that G is a minimal counterexample it follows G' has an acyclic 6-colouring c. We claim that we can extend this colouring. To see this, we consider two cases. Let y_1, y_2 be the neighbours of y in G'.

Case 1. Assume $c(x) = c(y)$, w.l.o.g., $c(x) = 1 = c(y)$. Observe that if $c(y_1) \neq c(y_2)$, then we can recolour y with any colour $\alpha \in \{2, \ldots, 6\} \setminus \{1, c(y_1), c(y_2)\}$. It is easy to see that the obtained colouring can be extended. Therefore we may assume $c(y_1) = c(y_2) = 2$. If we can recolour y with any colour $\alpha \in \{3, \ldots, 6\}$, then we are done, because again we obtain an acyclic 6-colouring of G' that can be extended. On the other hand, y cannot be recoloured only if for each colour $\alpha \in \{3, \ldots, 6\}$ there is an alternating $(2, \alpha)$-path from y_1 to y_2. It follows $C(y_1) = C(y_2) = \{1, 3, 4, 5, 6\}$. We recolour y_1 with 1 and y with a colour $\beta \in \{3, \ldots, 6\}$, $\beta \notin C(x)$. Such a colour exists, since in $C(x)$ there are at most three distinct colours. Clearly, the obtained colouring can be extended.

Case 2. Suppose $c(x) \neq c(y)$, w.l.o.g., $c(x) = 1$, $c(y) = 2$. Observe that if $1 \notin C(y)$, then the colouring c can be extended, since we do not create any bichromatic cycle. Hence, $1 \in C(y)$ and there are two subcases to consider.

Subcase 2.1. Assume $C(y)$ contains two distinct colours, w.l.o.g., $C(y) = \{1, 3\}$. If we cannot extend the colouring c, then there is an alternating $(1, 2)$-path from x to y. Hence $2 \in C(x)$. It follows $C(x)$ contains at most two colours among $4, 5, 6$. Thus there is a colour $\alpha \in \{4, 5, 6\}$, $\alpha \notin C(x)$. We recolour y with α and obtain an acyclic 6-colouring that can be extended.

Subcase 2.2. Let $C(y) = \{1, 1\}$. Again, there must be an alternating $(1, 2)$-path from x to y and hence $2 \in C(x)$. We may assume that this alternating path is passing through y_1. Observe that if we can recolour x with any colour $\alpha \in \{3, 4, 5, 6\}$, then we are done, because the colouring obtained in this way can be extended. Hence such a recolouring is impossible. Recall that $d_{G'}(x) = 3$ and $2 \in C(x)$. If $C(x)$ contains three distinct colours, then clearly we can recolour x. Thus in $C(x)$ at least one colour occurs more than once. We may assume that one of the following situations holds: $C(x) = \{2, 3, 3\}$ or $C(x) = \{2, 2, 3\}$ or $C(x) = \{2, 2, 2\}$. In all cases, if we can recolour y with a colour $\beta \in \{4, 5, 6\}$, then we are done, because the obtained colouring can be extended. It follows there is an alternating $(1, \beta)$-path from y_1 to y_2, for each $\beta \in \{4, 5, 6\}$. Thus $C(y_1) = \{2, 2, 4, 5, 6\}$. We recolour y_1 with 3 and y with 4. We obtain an acyclic 6-colouring that can be extended. □

Claim 3. *G contains no vertex of degree* 5 *adjacent to at least two vertices of degree* 3*.*

Proof. Assume that there is a vertex x of degree 5 adjacent to vertices y and z, both of degree 3. Let $G' = G - xy$. From the fact that G is a minimal counterexample it follows G' has an acyclic 6-colouring c. We claim that we can extend this colouring. To this end, we consider two cases. Let y_1, y_2 be the neighbours of y in G'.

Case 1. Let $c(x) \neq c(y)$, w.l.o.g., $c(x) = 1, c(y) = 2$. We cannot extend the colouring c only if there is an alternating $(1, 2)$-path from x to y. Hence, $1 \in C(y), 2 \in C(x)$. We may assume that this path passes through y_1. We need to consider two subcases.

Subcase 1.1. Assume $C(y) = \{1, 3\}$. Observe that we can recolour y with any colour $\alpha \in \{4, 5, 6\}$ and what we obtain is again an acyclic 6-colouring of

G'. If any of these colourings can be extended, then we are done. Otherwise, for each $\alpha \in \{4,5,6\}$ there is an alternating $(1,\alpha)$-path from x to y. Thus $C(x)=\{2,4,5,6\}$. Now we focus on the vertex z. If $c(z)\neq c(y)$, then we first recolour y with $c(z)$. (It can be always done, since $c(z)\in\{2,4,5,6\}$.) Now $c(z)=c(y)$. For each $\alpha\in\{2,4,5,6\}$ there is an alternating $(1,\alpha)$-path from x to y, thus $C(z)$ contains 1 at least twice. We try to recolour x with 3. It is easy to see that we obtain an acyclic 6-colouring of G'. If we can extend this colouring, then we are done. Otherwise, there is an alternating $(3,c(y))$-path from z to y. Thus, $C(z)=\{1,1,3\}$. We choose any colour $\beta\in\{2,4,5,6\}$, $\beta\neq c(y)$, and we recolour y with β, x with $c(y)$ and z with β, obtaining an acyclic 6-colouring of G' that can be extended.

Subcase 1.2. Let $C(y)=\{1,1\}$. We start with an easy observation that plays an important role in the rest of the proof. Recall that there is an alternating $(1,2)$-path from x to y. If there were two (or more) such paths, we would have a bichromatic $(1,2)$-cycle in c, what is impossible, since the colouring c is acyclic. Thus we have the following

Observation 3. *There is exactly one alternating* $(1,2)$*-path from* x *to* y*.*

We try to recolour y with a colour $\alpha\in\{3,4,5,6\}$. If any of the obtained colourings is acyclic and can be extended, then we are done. Hence we may assume that either we cannot recolour y with any such α (because the obtained colouring is not acyclic) or we cannot extend this colouring. It follows that for any colour $\alpha\in\{3,4,5,6\}$ there must be an alternating $(1,\alpha)$-path from x to y or an alternating $(1,\alpha)$-path from y_1 to y_2 (or both). According to this, there are five situations which may occur.

Subcase 1.2.1. Assume that for each $\alpha\in\{3,4,5,6\}$ there is an alternating $(1,\alpha)$-path from x to y. Thus $C(x)$ must contain colours $2,3,4,5$ and 6. This is impossible, since $d_{G'}(x)=4$.

Subcase 1.2.2. For exactly three colours α from the set $\{3,4,5,6\}$ we have an alternating $(1,\alpha)$-path from x to y. It follows $C(x)$ contains four distinct colours. We may recolour x with $\beta\in\{3,4,5,6\}$, $\beta\notin C(x)$ and obtain an acyclic 6-colouring that can be extended.

Subcase 1.2.3. We have an alternating $(1,\alpha)$-path from x to y only for two colours from $\{3,4,5,6\}$, say for 3 and 4. Thus there are alternating $(1,5)$- and $(1,6)$-paths from y_1 to y_2. Hence $C(y_1)=\{2,2,5,6,\beta\}$. If $\beta=3$ (or $\beta=4$), then we recolour y_1 with 4 (or with 3). Observation 3 yields there is no alternating $(1,2)$-path from x to y, hence we can extend the obtained colouring. Assume $\beta\in\{2,5,6\}$. It follows that both an alternating $(1,3)$-path and an alternating $(1,4)$-path from x to y must pass through y_2. Hence, $C(y_2)=\{2,3,4,5,6\}$. We recolour y_2 with 2 and y with 3. Clearly, we obtain an acyclic 6-colouring that can be extended.

Subcase 1.2.4. There is only one alternating $(1, \alpha)$-path from x to y, where $\alpha \in \{3, 4, 5, 6\}$. W.l.o.g., we may assume $\alpha = 3$. Hence, $C(y_1) = \{2, 2, 4, 5, 6\}$. We recolour y_1 with 3. Observation 3 yields the obtained acyclic 6-colouring can be extended.

Subcase 1.2.5. Finally, assume that we have all alternating $(1, \alpha)$-paths from y_1 to y_2, for each $\alpha \in \{3, 4, 5, 6\}$, but this is clearly impossible, because $d(y_1) \leq 5$.

Case 2. Assume $c(x) = c(y)$, w.l.o.g., $c(x) = 1 = c(y)$. If we can recolour y with any colour from the set $\{2, \ldots, 6\}$ and obtain an acyclic 6-colouring of G', then we are done, since we get the situation considered in Case 1. Such a recolouring is impossible only if $c(y_1) = c(y_2) = \alpha$, where $\alpha \in \{2, \ldots, 6\}$, and for each colour $\beta \in \{2, \ldots, 6\}$, $\beta \neq \alpha$, there is an alternating (α, β)-path from y_1 to y_2. For simplicity, we assume $\alpha = 2$. Thus, $C(y_1) = C(y_2) = \{1, 3, 4, 5, 6\}$. Assume for a moment that there exists a colour $\beta' \in \{3, 4, 5, 6\}$, $\beta' \notin C(x)$. Then we can recolour y_1 with 1, y with β' and we are done, because the obtained colouring can be extended. Hence $C(x) = \{3, 4, 5, 6\}$. We focus on z. W.l.o.g., $c(z) = 3$. We recolour y_1 with 1, y with 3. It is obvious that we obtain an acyclic 6-colouring of G'. We cannot extend this colouring only if there is an alternating $(1, 3)$-path. Hence $1 \in C(z)$. In this case we recolour x with 2. If we still cannot extend the colouring, then there is an alternating $(2, 3)$-path. It follows $C(z) = \{1, 1, 2\}$. We recolour z with 4, x with 3, y with 4 and y_1 with 1. We obtain an acyclic 6-colouring of G' that can be extended. □

To finish the proof it is enough to observe that Claim 1 implies G satisfies the assumptions of Lemma 2. Hence G contains (A1) or (A2), but by Claims 2 and 3 this is impossible. ■

3. Concluding Remarks

We have proved that any graph with maximum degree at most 5 and with maximum average degree at most 4 has an acyclic 6-colouring. Now we indicate that there are graphs that satisfy these three conditions. For example, a graph G_1 presented in Figure 2. It is easy to check that $\Delta(G_1) = 5$ and $Mad(G_1) = 4$. Hence $\chi_a(G_1) \leq 6$, by Theorem 1. We prove that G_1 does not have any acyclic 5-colouring. To this end, we first consider a graph G_0 (see Figure 1). Let the vertices of G_0 be denoted as in Figure 1. It is easy to see that $\chi_a(G_0) = 5$. Moreover, in any acyclic 5-colouring of G_0, there is exactly one index $i \in \{1, 2, 3\}$, such that both x_i and y_i have the same colour (w.l.o.g., $i = 1$), and all other vertices have distinct colours (see Figure 1). Now let us try to extend this colouring to

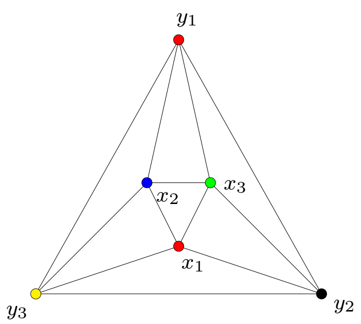

Figure 1. Graph G_0.

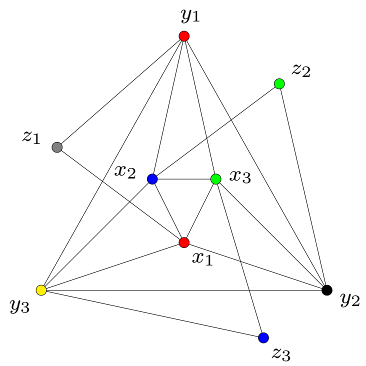

Figure 2. Graph G_1.

an acyclic 5-colouring of G_1. We focus on z_1. Clearly, z_1 cannot be coloured with the same colour as y_1. Furthermore, we cannot use any of the other four colours, because we create a bichromatic cycle of length 4. The fact that c was an arbitrary acyclic 5-colouring of G_0 implies $\chi_a(G_1) \geq 6$. Hence, $\chi_a(G_1) = 6$ (its acyclic 6-colouring is presented in Figure 2). There are also other graphs satisfying the above mentioned conditions, for instance, the graph presented in Figure 3. Nevertheless, for all such graphs, which we know, their maximum average degree equals 4. We conclude with the following problem: Are there graphs with maximum degree 5, acyclic chromatic number 6 and maximum average degree less than 4?

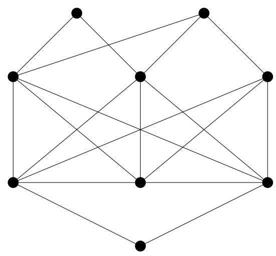

Figure 3. Graph G_2 satisfying $\Delta(G_2) = 5, Mad(G_2) = 4$ and $\chi_a(G_2) = 6$.

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