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EXACT EXPECTATION AND VARIANCE OF MINIMAL BASIS OF RANDOM MATROIDS

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Abstract

We formulate and prove a formula to compute the expected value of the minimal random basis of an arbitrary finite matroid whose elements are assigned weights which are independent and uniformly distributed on the interval $[0, 1]$. This method yields an exact formula in terms of the Tutte polynomial. We give a simple formula to find the minimal random basis of the projective geometry $PG(r-1, q)$.

Keywords: minimal basis, q-analog, finite projective geometry, Tutte polynomial.

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1. Introduction

This paper is devoted to compute the expected value of the minimal random basis of an arbitrary finite matroid whose elements are assigned weights which are independent and uniformly distributed on the interval $[0, 1]$. Contents of

this paper were originated in [10]. Our main aim is to give a counterpart of the results of Steele and Fill given in [15] (see also [3]). We reformulate their method to avoid such terms as vertex, component and similar ones. This enabled us to transfer results from graph to matroids (W. Tutte: "If a theorem about graphs can be expressed in terms of edges and circuits only it probably exemplifies a more general theorem about matroids" – see Oxley [14]).

Note however that investigations of minimal basis of random graphs were originated by Frieze in [4] (see also [1], [5] and [6]). In those papers authors considered r-regular n-vertex graph G with random independent edge lengths, each uniformly distributed on $[0,1]$. They gave a formula of expected length of a minimum spanning tree $mst\,(G)$.

In Section 2 we give some basic definition and notation needed in the next section of the paper. In Section 3 we find the expected value of the minimal basis of matroid in general and estimation of its variance. In Section 4 we consider more detailed the case of projective geometry $PG\,(r-1,q)$. Using the Tutte polynomial we are interested in finding the expected value of the minimal basis and moreover its variance in a more effective way. We investigate the weight of the first k elements of minimal basis.

2. Definitions and Notations

Let $M=(E,\mathcal{B})$ be a matroid on the finite ground set E (where $|E|=m$) with collection of basis $\mathcal{B}$. The rank of a set $A\subseteq E$ is denoted $\rho\,(A)$.

The Tutte polynomial is expressed by the formula

$$T\,(M;x,y)=\sum_{A\subseteq E}(x-1)^{\rho(E)-\rho(A)}\,(y-1)^{|A|-\rho(A)}\,. \tag{1}$$

For any matroid M, if (x,y) belongs to the hyperbola $(x-1)\,(y-1)=1$ then

$$T\,(M;x,y)=x^{|E|}\,(x-1)^{\rho(E)-|E|}\,. \tag{2}$$

We refer the reader the chapter [2] written by Brylawski and Oxley as a source of very important properties of Tutte polynomial for matroids.

Finite projective geometries in matroid theory are analogous to complete graphs in graph theory. Let $GF\,(q)$ be a Galois field, where q is the power of prime. Let $V\,(r,q)$ be an r-dimensional vector space over $GF\,(q)$. There exists a one-to-one correspondence between subspaces of projective geometry $PG\,(r-1,q)$ and subspaces of space $V\,(r,q)$. "Directions" in $V\,(r,q)$ are points of the projective geometry $PG\,(r-1,q)$ of dimension $r-1$. The monograph [7] gives a detailed exposition of this subject, see also [13] or [16]. Let q be fixed

and n a nonnegative integer. We use the standard notation $[n] = \dfrac{q^n - 1}{q - 1}$ (see for example [8] or [9]). The Gaussian coefficients are defined as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q^n - 1)(q^n - q)\dots(q^n - q^{k-1})}{(q^k - 1)(q^k - q)\dots(q^k - q^{k-1})}$$

for $1 \leq k \leq n$ and $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$. It is well known (see [7]) that the projective geometry $PG(r-1, q)$ has $[r]$ elements and $\begin{bmatrix} r \\ k \end{bmatrix}$ subspaces of rank-k.

Let $\{X_e, e \in E\}$ be a family of independent random variables uniformly distributed on the interval $[0, 1]$. The variable X_e is the weight associated with the element $e \in E$. Let $\mathbb{I}(A)$ denote the indicator of a set A. The weight $W_{\mathrm{MB}}(M)$ of the minimal basis $\mathrm{MB}(M)$ of the matroid M is given by

$$W_{\mathrm{MB}}(M) = \min_{B \in \mathcal{B}} \sum_{e \in E} X_e \mathbb{I}(e \in B).$$

We define

$$e_t(M) = \{e \in E : X_e \leq t\}$$

and

$$mb(M) = \mathbb{E}(W_{\mathrm{MB}}(M)).$$

3. Minimal Basis

The proof of Theorem 1 demonstrates that the method used by Steele and Fill takes advantage only of matroidal properties of graphs such as rank and independence.

Theorem 1. *For an arbitrary finite matroid M*

$$W_{\mathrm{MB}}(M) = r - \int_0^1 \rho(e_t(M))\, dt. \tag{3}$$

Proof. For an arbitrary set A and any $x \in [0, 1]$ the following equality holds

$$x\mathbb{I}(a \in A) = \int_0^1 \mathbb{I}(t < x, a \in A)\, dt,$$

then we have

$$\begin{aligned} W_{\mathrm{MB}}(M) &= \sum_{e\in M} X_e \mathbb{I}(e \in \mathrm{MB}(M)) \\ &= \sum_{e\in M} \int_0^1 \mathbb{I}(t < X_e, e \in \mathrm{MB}(M))\, dt = r - \int_0^1 \rho(e_t(M))\, dt. \end{aligned}$$

■

Now we rewrite the formula (1) to the form more usable in the next theorem to express $W_{\mathrm{MB}}(M)$ by Tutte polynomial. Let us denote $\eta(A) = \rho(E) - \rho(A)$. Then from the formula (1) we get

$$\begin{aligned} T(M;x,y) &= \frac{1}{(y-1)^r} \sum_{A\subseteq E} (y-1)^{|A|} ((x-1)(y-1))^{\eta(A)} \\ &= \frac{y^m}{(y-1)^r} \sum_{A\subseteq E} \left(\frac{y-1}{y}\right)^{|A|} \left(\frac{1}{y}\right)^{m-|A|} ((x-1)(y-1))^{\eta(A)}. \end{aligned} \tag{4}$$

Substituting $y = \frac{1}{1-p}$ and $x = 1 + \frac{1-p}{p}e^t$, and $p = \frac{y-1}{y}$ and $1-p = \frac{1}{y}$ into (4) we get

$$T(M;x,y) = \left(\frac{1}{p}\right)^r \left(\frac{1}{1-p}\right)^{m-r} \sum_{A\subseteq E} p^{|A|} (1-p)^{m-|A|} e^{t\eta(A)}. \tag{5}$$

Theorem 2. *For an arbitrary finite matroid M*

$$mb(M) = \mathbb{E}(W_{\mathrm{MB}}(M)) = \int_0^1 \frac{1-p}{p} \frac{T_x\left(M; \frac{1}{p}, \frac{1}{1-p}\right)}{T\left(M; \frac{1}{p}, \frac{1}{1-p}\right)}\, dp \tag{6}$$

and

(7)

$$\mathrm{Var}(W_{\mathrm{MB}}(M)) \le \int_0^1 \left(\frac{1-p}{p}\right)^2 \frac{T_{xx}\left(M; \frac{1}{p}, \frac{1}{1-p}\right)}{T\left(M; \frac{1}{p}, \frac{1}{1-p}\right)}\, dp + 2mb(M) - (mb(M))^2,$$

where $T_x(x,y)$ denotes the partial derivative of $T(x,y)$ with respect to x.

Proof. The moment generating function for the random variable $\eta(e_p(M))$ is determined by the formula

$$\varphi(t) = \mathbb{E}\left(e^{t\eta(e_p(M))}\right).$$

Hence

$$\varphi(t) = \sum_{A\subseteq E} p^{|A|}(1-p)^{m-|A|} e^{t\eta(A)}. \tag{8}$$

Comparing (8) with (5) we obtain

$$\varphi(t) = p^r(1-p)^{m-r} T\left(M; 1+\frac{1-p}{p}e^t, \frac{1}{1-p}\right).$$

To evaluate $\mathbb{E}(\eta(e_p(M)))$ we calculate $\varphi'(t)|_{t=0}$.

$$\varphi'(t) = \varphi(t)\, e^t \frac{1-p}{p} \frac{T_x\left(M; 1+\frac{1-p}{p}e^t, \frac{1}{1-p}\right)}{T\left(M; 1+\frac{1-p}{p}e^t, \frac{1}{1-p}\right)}.$$

Substituting $t=0$ we get

$$\mathbb{E}(\eta(e_t(M))) = \frac{1-p}{p} \frac{T_x\left(M; \frac{1}{p}, \frac{1}{1-p}\right)}{T\left(M; \frac{1}{p}, \frac{1}{1-p}\right)}.$$

We now turn to the formula (3) and we get

$$\mathbb{E}(W_{MB}(M)) = \int_0^1 \frac{1-p}{p} \frac{T_x\left(M; \frac{1}{p}, \frac{1}{1-p}\right)}{T\left(M; \frac{1}{p}, \frac{1}{1-p}\right)}\, dp,$$

which gives (6).

To estimate $\mathrm{Var}(W_{MB}(M))$ we shall first calculate $\mathbb{E}\left(\eta^2(e_p(M))\right)$.

$$\varphi''(t) = \varphi(t)\, e^{2t}\left(\frac{1-p}{p}\right)^2 \frac{T_{xx}\left(M; 1+\frac{1-p}{p}e^t, \frac{1}{1-p}\right)}{T\left(M; 1+\frac{1-p}{p}e^t, \frac{1}{1-p}\right)}$$
$$+ \varphi(t)\, e^t \frac{1-p}{p} \frac{T_x\left(M; 1+\frac{1-p}{p}e^t, \frac{1}{1-p}\right)}{T\left(M; 1+\frac{1-p}{p}e^t, \frac{1}{1-p}\right)}.$$

Hence

$$\begin{aligned}\mathbb{E}\left(\eta^2\left(e_p\left(M\right)\right)\right) &= \varphi''\left(0\right)+\varphi'\left(0\right)\\ &= \left(\frac{1-p}{p}\right)^2\frac{T_{xx}\left(M;\frac{1}{p},\frac{1}{1-p}\right)}{T\left(M;\frac{1}{p},\frac{1}{1-p}\right)}+2\frac{1-p}{p}\frac{T_x\left(M;\frac{1}{p},\frac{1}{1-p}\right)}{T\left(M;\frac{1}{p},\frac{1}{1-p}\right)}.\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{E}\left(W_{\mathrm{MB}}^2\left(M\right)\right) &= \mathbb{E}\left(\int_0^1\eta\left(e_t\left(M\right)\right)dt\right)^2\\ &\le \mathbb{E}\int_0^1\eta^2\left(e_t\left(M\right)\right)dt=\int_0^1\mathbb{E}\left(\eta^2\left(e_t\left(M\right)\right)\right)dt\,.\end{aligned}\tag{9}$$

The formula (9) gives (7), then we obtain the assertion. ■

The point $\left(\frac{1}{p},\frac{1}{1-p}\right)$ belongs to the hyperbola $(x-1)(y-1)=1$. Then applying equality (2) to (6) and (7), we obtain a following version of Theorem 2.

Theorem 3. *For an arbitrary matroid M*

$$mb\left(M\right)=\rho\left(E\right)-\int_0^1(1-p)^{|E|}\sum_{A\subseteq E}\rho\left(A\right)\left(\frac{1-p}{p}\right)^{-|A|}dp\tag{10}$$

and

$$\begin{aligned}\mathrm{Var}\left(W_{\mathrm{MB}}\left(M\right)\right) \le{}& \left(\rho\left(E\right)\right)^2-\rho\left(E\right)\int_0^1(1-p)^{|E|}\left(\frac{1-p}{p}\right)^{-\rho(E)}dp\\ &-\left(2\rho\left(E\right)-1\right)\int_0^1(1-p)^{|E|}\sum_{A\subseteq E}\rho\left(A\right)\left(\frac{1-p}{p}\right)^{-|A|}dp\\ &+\int_0^1(1-p)^{|E|}\sum_{A\subseteq E}\left(\rho\left(A\right)\right)^2\left(\frac{1-p}{p}\right)^{-|A|}dp\\ &+2mb\left(M\right)-\left(mb\left(M\right)\right)^2\,.\end{aligned}\tag{11}$$

Proof. Substituting $x = \dfrac{1}{p}$ and $y = \dfrac{1}{1-p}$ to (2) we get

$$T\left(M; \frac{1}{p}, \frac{1}{1-p}\right) = \left(\frac{1}{p}\right)^{|E|} \left(\frac{1-p}{p}\right)^{\rho(E)-|E|} = p^{-\rho(E)} (1-p)^{\rho(E)-|E|} .$$

We calculate the derivatives from (1):

$$T_x(M; x, y) = \frac{\rho(E)}{x-1} T(M; x, y) - \frac{1}{x-1} \sum_{A \subseteq E} \rho(A) (x-1)^{\rho(E)-\rho(A)} (y-1)^{|A|-\rho(A)}$$

and

$$\begin{aligned} T_{xx}(M; x, y) = & -\rho(E)(x-1)^{-2} T(M; x, y) + \rho(E)(x-1)^{-1} T_x(M; x, y) \\ & + (x-1)^{-2} \sum_{A \subseteq E} \rho(A)(x-1)^{\rho(E)-\rho(A)} (y-1)^{|A|-\rho(A)} \\ & - (x-1)^{-1} \sum_{A \subseteq E} \rho(A)(\rho(E) - \rho(A))(x-1)^{\rho(E)-\rho(A)-1} (y-1)^{|A|-\rho(A)} . \end{aligned}$$

Therefore

$$\begin{aligned} T_{xx}(M; x, y) = & \frac{\rho(E)}{(x-1)^2} (\rho(E) T(M; x, y) - 1) \\ & - \frac{1}{(x-1)^2} (\rho(E) \sum_{A \subseteq E} \rho(A)(x-1)^{\rho(E)-\rho(A)} (y-1)^{|A|-\rho(A)} \\ & - \sum_{A \subseteq E} \rho(A)(x-1)^{\rho(E)-\rho(A)} (y-1)^{|A|-\rho(A)} \\ & + \sum_{A \subseteq E} \rho(A)(\rho(E) - \rho(A))(x-1)^{\rho(E)-\rho(A)} (y-1)^{|A|-\rho(A)}). \end{aligned}$$

Since (x, y) belongs to the hyperbola $(x-1)(y-1) = 1$, then for $y = x/(x-1)$ and after substitution $x = \dfrac{1}{p}$ and easy computations we obtain

$$\begin{aligned} & T_x\left(M; \frac{1}{p}, \frac{1}{1-p}\right) = \rho(E) p^{-|E|} \left(\frac{1-p}{p}\right)^{-|E|} \left(\frac{1-p}{p}\right)^{\rho(E)-1} \\ & - \left(\frac{1-p}{p}\right)^{\rho(E)-1} \sum_{A \subseteq E} \rho(A) \left(\frac{1-p}{p}\right)^{-|A|} \\ & = \left(\frac{1-p}{p}\right)^{\rho(E)-1} \left(\rho(E)(1-p)^{-|E|} - \sum_{A \subseteq E} \rho(A) \left(\frac{1-p}{p}\right)^{-|A|}\right) \end{aligned} \tag{12}$$

and

(13)
$$T_{xx} = \left(\frac{1-p}{p}\right)^{-2} \left(\frac{1-p}{p}\right)^{\rho(E)} \left((\rho(E))^2 (1-p)^{-|E|} - \rho(E) \left(\frac{1-p}{p}\right)^{-\rho(E)} \right.$$
$$\left. - \sum_{A \subseteq E} \rho(A) \left(\frac{1-p}{p}\right)^{-|A|} \left(2\rho(E) - 1\right) + \sum_{A \subseteq E} \rho(A)^2 \left(\frac{1-p}{p}\right)^{-|A|} \right).$$

Substituting (12) to (6) we get

$$mb(M) = \int_0^1 \left(\rho(E) - (1-p)^{|E|} \sum_{A \subseteq E} \rho(A) \left(\frac{1-p}{p}\right)^{-|A|} \right) dp \tag{14}$$

which gives (10). Substituting (13) to (7) we get

$$V(W_{\mathrm{MB}}(M)) \leq \int_0^1 (1-p)^{|E|} \left((\rho(E))^2 (1-p)^{-|E|} - \rho(E) \left(\frac{1-p}{p}\right)^{-\rho(E)} \right.$$
$$\left. - \sum_{A \subseteq E} \rho(A) \left(\frac{1-p}{p}\right)^{-|A|} \left(2\rho(E) - 1\right) + \sum_{A \subseteq E} \rho(A)^2 \left(\frac{1-p}{p}\right)^{-|A|} \right) dp$$
$$+ 2mb(M) - (mb(M))^2,$$

which gives (11). ■

The Tutte polynomial $T(M; x, y)$ on the hyperbola $(x-1)(y-1) = 1$ is a function of x depending only on $\rho(E)$ and $|E|$. This property is not shared by the first derivative $T_x(M; x, y)$ as is shown by the following example.

4. The Case of Projective Geometries

Now, we consider more carefully the case where the matroids are projective geometries.

From Theorem 1 we obtain the exact value of expectation of minimal basis of $PG(r-1, q)$.

Corollary 4. *For $M = PG(r-1, q)$*

$$mb(M) = r - \sum_{k=1}^{r} k \begin{bmatrix} r \\ k \end{bmatrix} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j q^{\binom{j}{2}} \frac{1}{[r] - [k-j] + 1}. \tag{15}$$

Proof. The expected value of the rank of a random subset of the projective geometry is given by

$$\begin{aligned}\mathbb{E}\left(\rho\left(M\right)\right) &= rP^{(r)} + \left(r-1\right)P^{(r-1)}\begin{bmatrix} r \\ r-1 \end{bmatrix}\left(1-p\right)^{[r]-[r-1]} + \cdots \\ &= \sum_{k=1}^{r} k\begin{bmatrix} r \\ k \end{bmatrix}\left(1-p\right)^{[r]-[k]}P^{(k)},\end{aligned} \tag{16}$$

where $P^{(k)}$ is the probability that M has the rank k. Since

$$P^{(k)} = \sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}\left(-1\right)^{j} q^{\binom{j}{2}}\left(1-p\right)^{[k]-[k-j]}$$

(see [9], formula (5.1.6)) then using (16) we obtain after simple computations

$$\int_0^1 \mathbb{E}\left(r\left(M\right)\right)\,dp = \sum_{k=1}^{r} k\begin{bmatrix} r \\ k \end{bmatrix}\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}\left(-1\right)^{j} q^{\binom{j}{2}}\frac{1}{[r]-[k-j]+1}\,.$$

■

Lemma 5. *Let $M = PG(r-1,q)$. Then Tutte polynomials and its derivatives have the form:*

$$T\left(M;x,y\right) = \frac{1}{\left(y-1\right)^{r}}\left(y^{[r]} + \sum_{j=0}^{r-1} y^{[j]}\begin{bmatrix} r \\ j \end{bmatrix}\prod_{i=0}^{r-j-1}\left(\left(x-1\right)\left(y-1\right)-q^{i}\right)\right), \tag{17}$$

(18)

$$T_x\left(M;x,y\right) = \frac{1}{\left(y-1\right)^{r}}\sum_{j=0}^{r-1} y^{[j]}\begin{bmatrix} r \\ j \end{bmatrix}\sum_{k=0}^{r-j-1}\left(y-1\right)\prod_{\substack{i=0\\ i\neq k}}^{r-j-1}\left(\left(x-1\right)\left(y-1\right)-q^{i}\right),$$

$$\begin{aligned}T_{xx}\left(M;x,y\right) &= \frac{1}{\left(y-1\right)^{r-1}}\sum_{j=0}^{r-1} y^{[j]}\begin{bmatrix} r \\ j \end{bmatrix}\sum_{(k,l)\in A_j}\left(y-1\right) \\ &\quad\times\prod_{\substack{i=0\\ i\neq k, i\neq l}}^{r-j-1}\left(\left(x-1\right)\left(y-1\right)-q^{i}\right),\end{aligned} \tag{19}$$

where

$$A_j = \{(k,l) : k\in\{0,\ldots,r-j-1\}, l\in\{0,\ldots,r-j-1\}, k\neq l\}.$$

Proof. The formula (17) was proved in [12]. By simple differentiation we have formulas (18) and (19). ■

If $(x-1)(y-1)=1$, then letting $y'=x/(x-1)$ and $x=\frac{1}{p}$ we have

$$T\left(M;\frac{1}{p},\frac{1}{(1-p)}\right)=p^{-r}(1-p)^{r-[r]},$$

$$T_x\left(M;\frac{1}{p},\frac{1}{1-p}\right)=\left(\frac{1-p}{p}\right)^{r-1}\sum_{j=0}^{r-1}(1-p)^{-[j]}\begin{bmatrix}r\\j\end{bmatrix}\prod_{i=1}^{r-j-1}\left(1-q^i\right),$$

$$T_{xx}\left(M;\frac{1}{p},\frac{1}{1-p}\right)=2\left(\frac{1-p}{p}\right)^{r-2}\sum_{j=0}^{r-1}(1-p)^{-[j]}\begin{bmatrix}r\\j\end{bmatrix}\sum_{l=1}^{r-j-1}\prod_{i=1 i\neq l}^{r-j-1}\left(1-q^i\right).$$

Proposition 6. *For $M=PG(r-1,q)$,*

$$mb(M)=\sum_{j=0}^{r-1}\frac{\begin{bmatrix}r\\j\end{bmatrix}\prod_{i=1}^{r-j-1}\left(1-q^i\right)}{[r]-[j]+1}$$

and

$$\operatorname{Var}\left(W_{\mathrm{MB}}(M)\right)\leq 2\sum_{j=0}^{r-1}\frac{\begin{bmatrix}r\\j\end{bmatrix}\sum_{l=1}^{r-j-1}\prod_{\substack{i=1\\i\neq l}}^{r-j-1}\left(1-q^i\right)}{[r]-[j]+1}+2mb(M)-(mb(M))^2.$$

Proof.

$$\begin{aligned}mb(M)&=\int_0^1\frac{1-p}{p}\frac{\left(\frac{1-p}{p}\right)^{r-1}\sum_{j=0}^{r-1}(1-p)^{-[j]}\begin{bmatrix}r\\j\end{bmatrix}\prod_{i=1}^{r-j-1}\left(1-q^i\right)}{p^{-r}(1-p)^{r-[r]}}\,dp\\&=\int_0^1(1-p)^{[r]}\sum_{j=0}^{r-1}(1-p)^{-[j]}\begin{bmatrix}r\\j\end{bmatrix}\prod_{i=1}^{r-j-1}\left(1-q^i\right)\,dp\\&=\sum_{j=0}^{r-1}\frac{\begin{bmatrix}r\\j\end{bmatrix}\prod_{i=1}^{r-j-1}\left(1-q^i\right)}{[r]-[j]+1}.\end{aligned}$$

$$\begin{aligned}
&\operatorname{Var}\left(W_{\mathrm{MB}}(M)\right)\\
&\leq \int_0^1 \left(\frac{1-p}{p}\right)^2 \frac{2\left(\frac{1-p}{p}\right)^{r-2} \sum_{j=0}^{r-1} (1-p)^{-[j]} \left[{r \atop j}\right] \sum_{l=1}^{r-j-1} \prod_{\left[{i=1 \atop i\neq l}\right]}^{r-j-1} \left(1-q^i\right)}{p^{-r}(1-p)^{r-[r]}}\, dp\\
&+ 2mb(M) - (mb(M))^2\\
&= \sum_{j=0}^{r-1} \frac{\left[{r \atop j}\right] \sum_{l=1}^{r-j-1} \prod_{\left[{i=1 \atop i\neq l}\right]} \left(1-q^i\right)}{[r]-[j]+1} + 2mb(M) - (mb(M))^2 .
\end{aligned}$$

■

Order the elements $e_1, e_2, \ldots, e_{[r]}$ of $PG(r-1,q)$ so that e_i has weight Z_i. Let $(Y_1, Y_2, \ldots, Y_r)$ be the subsequence of the sequence $(Z_1, Z_2, \ldots, Z_n)$ such that $Y_i^{(n)} = Z_{k_i}^{(n)}$ and k_i is the least index such that $e_{k_i} \notin \sigma\{e_{k_1}, e_{k_2}, \ldots, e_{k_{i-1}}\}$, where $\sigma(A)$ denotes the subspace spanned by A. Note that $k_1 = 1, k_2 = 2$ and $k_i \geq i$ for $i \geq 3$. We call the random variables $Y_1, Y_2, \ldots, Y_r$ *the q-analogs of the order statistics.* Problem of limiting distribution of $Y_{k(r)}$ for $r \to \infty$ was considered in [11]. In this paper we restrict ourselves to a finite n.

Let

$$P_{i_{m-1}}^{(m)} = \left(\prod_{n=i_{m-2}+2}^{i_{m-1}} \frac{[m]-n}{[r]-n}\right) \frac{[r]-[m]}{[r]-(i_{m-1}+1)}$$

be a probability that to find the minimal weight of the first $k+1$ increaseable ordered elements of basis in $PG(r-1,q)$ was chosen $i_{m-1}+2$ points. The first factor determines a probability that successively chosen points belong to a space which was generated by previous chosen points.

Laborious but routine calculations give the following theorem.

Theorem 7. *Let W_k denote a minimal weight of the first $k+1$ elements of a basis in $PG(r-1,q)$. Then for $k \geq 2$*

$$\mathbb{E}W_k = \mathbb{E}W_{k-1} + \sum_{\left[{1\leq i_1<i_2<\cdots<i_{k-1}, \atop i_{l-1}\leq [l]-1,\ l=2,\ldots,k}\right]} \prod_{m=2}^{k} P_{i_{m-1}}^{(m)} \mathbb{E}Z_{i_{m-1}+2},$$

besides $\mathbb{E}W_1 = \mathbb{E}Z_1 + \mathbb{E}Z_2$.

Note finally that $\mathbb{E}W_{r-1} = mb(PG(r-1,q))$, where $mb(PG(r-1,q))$ is calculated from (15) or (6).

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