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FRUCHT'S THEOREM FOR THE DIGRAPH FACTORIAL

Richard H. Hammack

Department of Mathematics and Applied Mathematics
Virginia Commonwealth University
Richmond, VA 23284 USA

e-mail: rhammack@vcu.edu

Abstract

To every graph (or digraph) A, there is an associated automorphism group $\mathrm{Aut}(A)$. Frucht's theorem asserts the converse association; that for any finite group G there is a graph (or digraph) A for which $\mathrm{Aut}(A) \cong G$.

A new operation on digraphs was introduced recently as an aid in solving certain questions regarding cancellation over the direct product of digraphs. Given a digraph A, its *factorial* $A!$ is certain digraph whose vertex set is the permutations of $V(A)$. The arc set $E(A!)$ forms a group, and the loops form a subgroup that is isomorphic to $\mathrm{Aut}(A)$. (So $E(A!)$ can be regarded as an extension of $\mathrm{Aut}(A)$.)

This note proves an analogue of Frucht's theorem in which $\mathrm{Aut}(A)$ is replaced by the group $E(A!)$. Given any finite group G, we show that there is a graph A for which $E(A!) \cong G$.

Keywords: Frucht's theorem, digraphs, graph automorphisms, digraph factorial.

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1. Introduction

We regard a *digraph* A is a binary relation $E(A)$ on a finite vertex set $V(A)$, that is, a subset $E(A) \subseteq V(A) \times V(A)$. An ordered pair in $E(A)$ is called an *arc*; it is denoted as $[x, y]$ and visualized as an arrow pointing from x to y. A reflexive arc $[x, x]$ is called a *loop*. A *graph* is a digraph that is symmetric (as a relation); thus we view any edge xy in a graph as consisting of two arcs $[x, y]$ and $[y, x]$. But in drawing a graph, we sometimes represent these two arcs as a single edge (without an arrow) joining x to y.

The *degree* of a vertex x of a digraph is the ordered pair $(\mathrm{id}(x), \mathrm{od}(x))$ of its in-degree and out-degree. (We also use id to denote identity maps; the distinction should always be clear from context.) For a graph, the in- and out-degrees of a vertex are always equal; we call this common value the *degree* of the vertex.

Frucht's theorem (see Theorem 5.6 of [1]) states that for any finite group G, there is a graph whose automorphism group is isomorphic to G.

Recently a new group invariant of a digraph was introduced. Given a digraph A, its *factorial* $A!$ is a certain digraph whose vertices are the permutations of $V(A)$. (Its formal definition is given in Section 3 below.) The arc set $E(A!)$ forms a group, where the operation is pairwise multiplication of endpoints. In fact (as we shall see) $\mathrm{Aut}(A)$ can be naturally identified with a subgroup of $E(A!)$. Thus $E(A!)$ is an extension of $\mathrm{Aut}(A)$; it carries all the information of the automorphism group, plus more.

Curiosity compels us to ask if an analogue of Frucht's theorem holds for $E(A!)$. Given an arbitrary finite group G, is there a digraph (or graph) A for which $E(A!) \cong G$? This note gives an affirmative answer.

In what follows, we first sketch a proof of Frucht's theorem (Though it is a standard result, we will need to modify the standard construction slightly.) Next we define the notion of the digraph factorial, and note that $E(A!)$ is a group. Finally, our main results are proved.

Although it is not an essential ingredient of this note, the idea of the direct product of graphs may help the reader appreciate the significance of digraph factorial. The *direct product* of two digraphs A and B is the digraph $A \times B$ whose vertex set is the Cartesian product $V(A) \times V(B)$ and whose arcs are the pairs $[(x, y), (x', y')]$ with $[x, x'] \in E(A)$ and $[y, y'] \in E(B)$.

2. Frucht's Theorem

We now briefly recall the proof of Frucht's theorem, that to any finite group G there is a graph A with $\mathrm{Aut}(A) \cong G$. Our approach and notation follows that of [1], though our encoding of the arcs differs slightly, so that no vertex of A has degree 1.

Given G, let $\Delta = \{h_1, h_2, \dots, h_k\}$ be a set of its generators. Form the Cayley color graph $D_\Delta(G)$. This is a digraph whose vertices are the elements of G and whose arcs have form $[g, gh_i]$, where $g \in G$. We view an arc of form $[g, gh_i]$ as having color i.

There is a subgroup of $\mathrm{Aut}(D_\Delta(G))$ consisting of automorphisms that preserve the colors of the arcs. Call this the *group of color-preserving automorphisms*. It is easy to check that this group is precisely $\{\varphi_g : g \in G\}$, where $\varphi_g(x) = gx$, and that it is isomorphic to G.

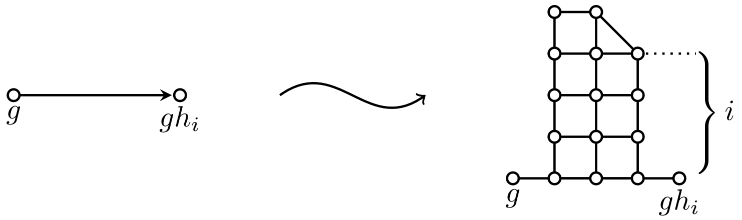

Figure 1. Replace each arc of color i with a length-4 path supporting a tower of height i.

Next, convert digraph $D_\Delta(G)$ to a graph A by encoding its arcs as shown in Figure 1. Replace each arc $[g, gh_i]$ of color i with a path of length four, on which is placed an asymmetric "tower" whose shorter side has height i. It is easy to confirm that $\mathrm{Aut}(A)$ is isomorphic to the group of color-preserving automorphisms of $D_\Delta(G)$, that is, $\mathrm{Aut}(A) \cong G$.

Note that, so long as $|G| \neq 2$, no vertex of A has degree 1.

3. The Digraph Factorial

We now recall the factorial operation on digraphs, a construction that has appeared in [2, 3, 4, 5] and [6]. Given a digraph A, we let $S_{V(A)}$ denote the symmetric group on $V(A)$, that is, the set of permutations of the vertices of A, the set of bijections $V(A) \to V(A)$.

Definition. Given a digraph A, its *factorial* is another digraph, denoted as $A!$, and defined on the vertex set $V(A!) = S_{V(A)}$. For the arcs, $[\alpha, \beta] \in E(A!)$ provided that $[x, y] \in E(A) \iff [\alpha(x), \beta(y)] \in E(A)$ for all pairs $x, y \in V(A)$.

The definition implies that there is a loop at $\alpha \in V(A!)$ if and only if α is an automorphism of A. In particular, $E(A!)$ always contains the loop $[\mathrm{id}, \mathrm{id}]$.

It is immediate that the arc set $E(A!)$ is a group with identity $[\mathrm{id}, \mathrm{id}]$ and multiplication $[\alpha, \beta][\gamma, \delta] = [\alpha\gamma, \beta\delta]$. We also have $[\alpha, \beta]^{-1} = [\alpha^{-1}, \beta^{-1}]$. Observe that $\mathrm{Aut}(A)$ embeds as a subgroup of $E(A!)$, for it is the set of loops $[\alpha, \alpha]$ of $E(A!)$. In this sense, $E(A!)$ can be regarded as an extension of $\mathrm{Aut}(A)$.

Our first example explains the origins of the term "factorial." Let K_n^* be the complete (symmetric) graph with a loop at each vertex. Our definition yields $K_n^*! \cong K_{n!}^*$. Readers familiar with the direct product will recognize that

$$K_n^*! \cong K_{n!}^* \cong K_n^* \times K_{n-1}^* \times K_{n-2}^* \times \cdots \times K_3^* \times K_2^* \times K_1^*.$$

Note that $E(K_n^*!)$ consists of all elements $[\alpha, \beta]$ where $\alpha, \beta \in S_n$, so we see that $E(K_n^*!)$ is the group product $S_n \times S_n$. (Here S_n is the symmetric group on the n vertices of K_n^*.)

The computation of $E(K_n^*!)$ was easy, because every pair of vertices in K_n^* is an arc, so the conditions in Definition 3 for $[\alpha,\beta]$ to be an arc cannot fail to hold. For less obvious computations, it is helpful to keep in mind the following interpretation of $E(A!)$. Any arc $[\alpha,\beta]\in E(A!)$ can be regarded as a permutation of the arcs of A, where $[\alpha,\beta]([x,y])=[\alpha(x),\beta(z)]$. This permutation preserves in-incidences and out-incidences in the following sense: Given two arcs $[x,y],[x,z]$ of A that have a common tail, $[\alpha,\beta]$ carries them to the two arcs $[\alpha(x),\beta(y)]$, $[\alpha(x),\beta(z)]$ of A with a common tail. Given two arcs $[x,y],[z,y]$ with a common tip, $[\alpha,\beta]$ carries them to the two arcs $[\alpha(x),\beta(y)]$, $[\alpha(z),\beta(y)]$ of A with a common tip.

Bear in mind, however, that even if the tip of $[x,y]$ meets the tail of $[y,z]$, then the arcs $[\alpha,\beta]([x,y])$ and $[\alpha,\beta]([y,z])$ need not meet; they can be far apart in A. To illustrate these ideas, Figure 2 shows the effect of a typical $[\alpha,\beta]$ on the arcs incident with a typical vertex z of A. Observe that for any vertex z of A, the out-degrees of z and $\alpha(z)$ are the same, as are the in-degrees of z and $\beta(z)$.

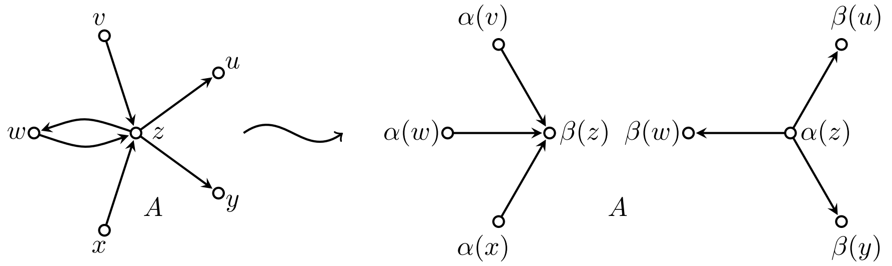

Figure 2. Action of an arc $[\alpha,\beta]\in E(A!)$ on the neighborhood of a vertex $z\in E(A)$.

By an *alternating walk* in A we mean a walk in which any two successive arcs have opposite orientations. The above remarks imply that an arc $[\alpha,\beta]$ of $A!$, viewed as a permutation of the arcs of A, maps alternating walks to alternating walks. Figure 3 illustrates this. We will use this observation frequently.

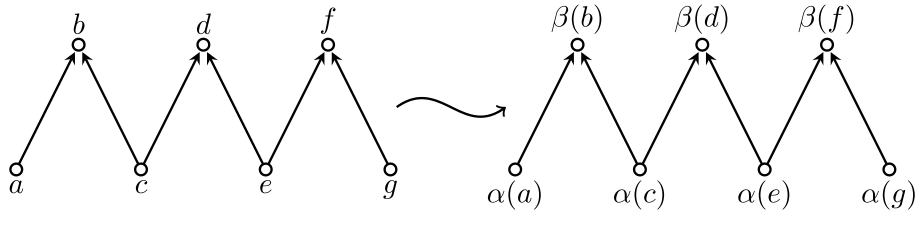

Figure 3. An arc $[\alpha,\beta]\in E(A!)$ sends alternating walks to alternating walks.

With the above remarks in mind, we can easily compute other factorials. Consider the transitive tournament T_3 on three vertices. Its in-degrees are distinct, as are its out-degrees, so $[\alpha, \beta] \in E(T_3!)$ if and only if $\alpha = \mathrm{id} = \beta$. Thus $E(T_3!) = \{[\mathrm{id}, \mathrm{id}]\}$ is the trivial group.

This is illustrated in Figure 4 (using cycle notation for the permutations), along with three other examples. The second line in the figure shows K_2 and its factorial $K_2!$, which has two loops, forming a group that is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The third part of the figure shows a graph A and its factorial; here the four arcs form a group that is isomorphic to the Klein 4-group. For the directed cycle $\overrightarrow{C_3}$, the reader can verify that its factorial is as shown, and that $E(\overrightarrow{C_3}!) \cong S_3$.

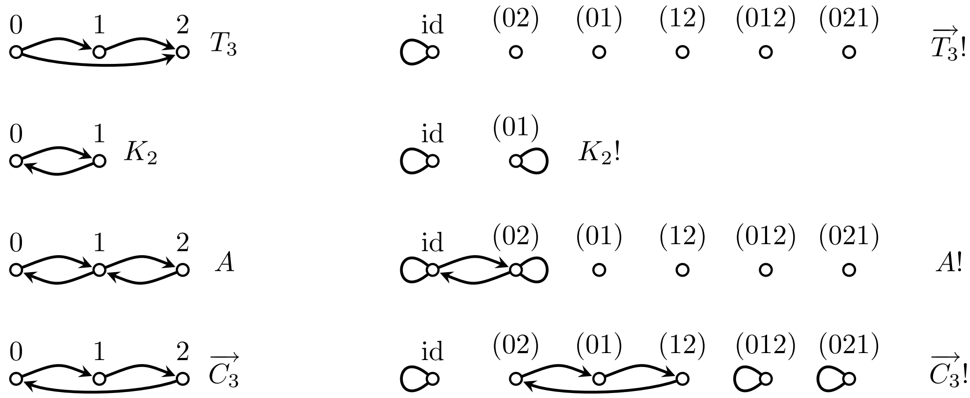

Figure 4. Examples of digraphs (left) and their factorials (right).

We remark that if A is a graph (i.e. symmetric digraph), then $A!$ is also a graph. However, this fact is not necessary here, so the simple proof is omited.

We close this section with the briefest mention of one use of the factorial operation. The *cancellation problem* for the direct product asks under what conditions $A \times C \cong B \times C$ implies $A \cong B$. Lovász [8] gives conditions on C that guarantee this: it is necessary and sufficient that C does not admit a homomorphism into a disjoint union of directed cycles. But it is also meaningful to ask for conditions on A that guarantee cancellation. As proved in [6] (for graphs) and [4] (for digraphs), the answer involves certain structural properties of the factorial $A!$.

4. An Analogue of Frucht's Theorem

We are now ready to prove our main result, an analogue of Frucht's theorem for the factorial. The proof makes frequent appeals to the remarks in Section 3, namely that an arc $[\alpha, \beta]$ of $A!$ preserves alternating walks in A, that z and $\alpha(z)$ have the same out-degree, and that z and $\beta(z)$ have the same in-degree.

Theorem 1. *For any finite group G, there is a graph A for which $E(A!) \cong G$.*

Proof. If G is trivial, then $E(K_1!) \cong G$; and if $|G| = 2$, then $E(K_2!) \cong G$. Thus assume that $|G| > 2$. By Frucht's theorem, there is a graph B for which $\mathrm{Aut}(B) \cong G$. By our construction in Section 2 (combined with the fact $|G| > 2$), we may assume that all vertices of B have degree greater than 1. Extend B to a graph A as illustrated in Figure 5. Specifically, for each vertex $x \in V(B)$, add three new vertices x_1, x_2 and x_3, as well as new arcs as indicated in Figure 5. Observe that each x_i has degree i in A, whereas each x has degree $d > 3$.

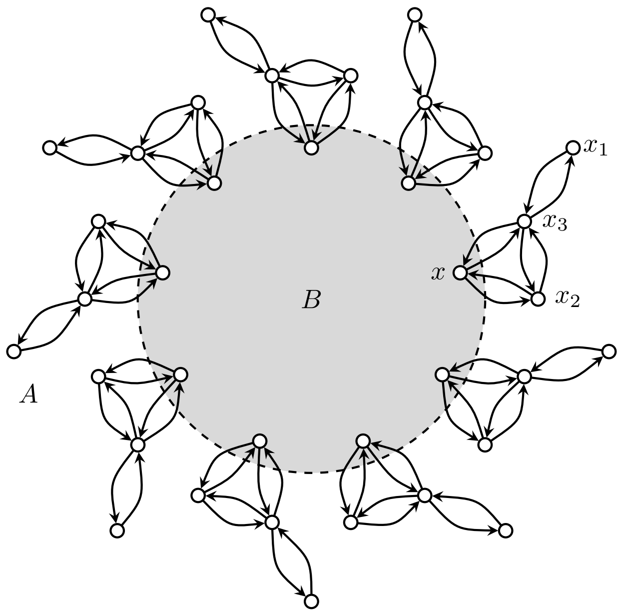

Figure 5. Construction of A. Here $G \cong \mathrm{Aut}(B) \cong \mathrm{Aut}(A) \cong E(A!)$.

So $G \cong \mathrm{Aut}(B)$. We prove the theorem by first arguing that $\mathrm{Aut}(B) \cong \mathrm{Aut}(A)$ and then that $\mathrm{Aut}(A) \cong E(A!)$.

Now, every vertex of B has degree greater than 3, but this is not so with the added vertices. Thus any automorphism of A is stable on $V(A) - V(B)$ and therefore restricts to an automorphism of B. Thus, to prove $\mathrm{Aut}(B) \cong \mathrm{Aut}(A)$, we just need to show that any automorphism of B extends to a unique automorphism of A. Suppose $\varphi \in \mathrm{Aut}(B)$. Clearly we can extend φ to an automorphism of A by declaring $\varphi(x_i) = z_i$ for each $i \in \{1, 2, 3\}$, whenever $\varphi(x) = z$. Let us write this as $\varphi(x_i) = \varphi(x)_i$. It should be equally clear from our construction (see Figure 5) that any extension of φ to A must satisfy $\varphi(x_i) = \varphi(x)_i$. It follows that $\mathrm{Aut}(B) \cong \mathrm{Aut}(A)$.

Next we confirm that $\mathrm{Aut}(A) \cong E(A!)$. Define a map $\mathrm{Aut}(A) \to E(A!)$ as $\alpha \mapsto [\alpha, \alpha]$. It is straightforward that this is an injective (group) homomorphism; showing surjectivity will complete the proof. We just need to show that any arc $[\alpha, \beta] \in E(A!)$ satisfies $\alpha = \beta$. (That is, that $E(A!)$ consists wholly of loops.)

Suppose $[\alpha, \beta] \in E(A!)$. We first show $\alpha(x) = \beta(x)$ for any $x \in V(B)$. As a map on arcs, $[\alpha, \beta]$ sends the alternating closed walk $W = [x, x_2], [x_3, x_2], [x_3, x]$ to an alternating walk $W' = [\alpha(x), \beta(x_2)], [\alpha(x_3), \beta(x_2)], [\alpha(x_3), \beta(x)]$, whose internal vertices have degrees 2 and 3, and whose end vertices have degrees greater than 3. The construction of A dictates that W' must begin and end at the same vertex of B, hence $\alpha(x) = \beta(x)$.

Next, for any $x \in V(B)$, the vertex x_1 has degree $(1, 1)$ and is joined to x by an alternating walk $[x_1, x_3], [x, x_3]$. Then $\alpha(x_1)$ must have out-degree 1, and it is on an alternating walk $[\alpha(x_1), \beta(x_3)][\alpha(x), \beta(x_3)]$ joining $\alpha(x_1)$ to $\alpha(x)$. By construction of A, there is only one vertex that meets these conditions; it must be that $\alpha(x_1) = \alpha(x)_1$. Similarly, $\beta(x_1)$ has in-degree 1 and is the first vertex of an alternating walk $[\alpha(x_3), \beta(x_1)], [\alpha(x_3), \beta(x)]$ terminating at $\beta(x) = \alpha(x)$. As above, we infer that $\beta(x_1) = \beta(x)_1 = \alpha(x)_1 = \alpha(x_1)$, so $\alpha(x_1) = \beta(x_1)$ for all x.

Repeating this argument for x_2 (but using out- and in-degrees 2, and walks of length 1) we get $\alpha(x_2) = \beta(x_2)$ for all x; and for the same reason $\alpha(x_3) = \beta(x_3)$. Thus we have shown $\alpha = \beta$, and the proof is complete. ∎

Let $\overrightarrow{P_2}$ denote the directed path on two vertices, that is, the digraph with vertex set $\{0, 1\}$ and arc set $\{[0, 1]\}$. We close with an auxiliary result that relates $E(A!)$ to the automorphism group of the direct product $A \times \overrightarrow{P_2}$.

Proposition 2. *If all vertices of a digraph A have positive in- and out-degrees, then $E(A!) \cong \mathrm{Aut}(A \times \overrightarrow{P_2})$.*

Proof. Consider the vertices of $A \times \overrightarrow{P_2}$. By the definition of the direct product, those of form $(x, 0)$ have positive out-degree and zero in-degree; and those of form $(x, 1)$ have zero out-degree and positive in-degree. Thus any automorphism of $A \times \overrightarrow{P_2}$ permutes the vertices of the first kind, and permutes the vertices of the second kind; but it sends no vertex of the first kind to the second, nor vice versa. Thus any automorphism φ can be regarded as a pair $\varphi = (\alpha, \beta)$ of permutations of $V(A)$, where $\varphi(x, 0) = (\alpha(x), 0)$ and $\varphi(x, 1) = (\beta(x), 1)$. Straightforward applications of the definitions show that the map $[\alpha, \beta] \mapsto (\alpha, \beta)$ is the desired isomorphism. ∎

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