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Dedicated to Mietek Borowiecki on the occasion of his seventieth birthday

CHOICE-PERFECT GRAPHS

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Abstract

Given a graph $G = (V, E)$ and a set L_v of admissible colors for each vertex $v \in V$ (termed the *list* at v), a *list coloring* of G is a (proper) vertex coloring $\varphi : V \to \bigcup_{v \in V} L_v$ such that $\varphi(v) \in L_v$ for all $v \in V$ and $\varphi(u) \neq \varphi(v)$ for all $uv \in E$. If such a φ exists, G is said to be *list colorable*. The *choice number* of G is the smallest natural number k for which G is list colorable whenever each list contains at least k colors.

In this note we initiate the study of graphs in which the choice number equals the clique number or the chromatic number in every induced subgraph. We call them *choice-ω-perfect* and *choice-χ-perfect* graphs, respectively. The main result of the paper states that the square of every cycle is choice-χ-perfect.

Keywords: graph coloring, list coloring, choice-perfect graph.

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1. Introduction

Based on the long-ago manuscript [15], in this note we initiate the study of some hereditary classes of graphs that are defined by their list colorings in relation to chromatic number and clique number. In flavor, these questions have strongly been motivated by the theory of perfect graphs.

We begin with some basic definitions on list coloring. Let a graph $G = (V, E)$ be given together with a set L_v of admissible colors for each vertex $v \in V$. We call L_v the *list* at v. A *list coloring* of G is a (proper) vertex coloring $\varphi : V \to \bigcup_{v \in V} L_v$ such that

- $\varphi(v) \in L_v$ for all $v \in V$,
- $\varphi(u) \neq \varphi(v)$ for all $uv \in E$.

If such a φ exists, G is said to be *list colorable.* A *k-assignment* is a collection of lists such that $|L_v| = k$ for all $v \in V$. If G is list colorable for all k-assignments, it is said to be *k-choosable.* The *choice number* of G, also called the *list chromatic number* of G, is the smallest natural number k for which G is k-choosable. We denote it by $\chi_\ell(G)$ here[1].

We also recall the following standard notation from the literature:

- $\omega(G)$ is the *clique number* (the largest number of mutually adjacent vertices in G),
- $\chi(G)$ is the *chromatic number* (the smallest number of colors in a coloring in which no two adjacent vertices have the same color),
- $\Delta(G)$ is the maximum vertex degree,
- $\mathrm{col}(G)$ is the *coloring number*[2] (1 plus the largest minimum vertex degree taken over all induced subgraphs of G).

In the theory of graph coloring, the following chain of inequalities is basic:

$$\omega(G) \leq \chi(G) \leq \chi_\ell(G) \leq \mathrm{col}(G) \leq \Delta(G) + 1. \tag{1}$$

Requiring equality at some points in (1) does not mean much structural restriction. It is a substantial change, however, if equality is required for all induced subgraphs of G, too. In this way, perfect graphs are defined in terms of the equality $\chi = \omega$. Here we initiate analogous study in connection with $\chi_\ell = \chi$ and $\chi_\ell = \omega$.

Definition 1. We say that a graph $G = (V, E)$ is

- choice-ω-perfect if $\chi_\ell(G') = \omega(G')$ holds for all induced subgraphs $G' \subseteq G$;
- choice-χ-perfect if $\chi_\ell(G') = \chi(G')$ holds for all induced subgraphs $G' \subseteq G$.

Problem 2. (i) Characterize choice-ω-perfect graphs.
(ii) Characterize choice-χ-perfect graphs.

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[1] A further standard notation is $\mathrm{ch}(G)$.

[2] There are several equivalent definitions of $\mathrm{col}(G)$. If $v_1, \dots, v_n$ is any ordering of the vertices of G (where $n = |V(G)|$), let $d^-(v_i)$ denote the number of vertices adjacent to v_i in the set $\{v_j \mid 1 \leq j < i\}$. Then $\mathrm{col}(G) = \min\big(\max_{1\leq i\leq n} d^-(v_i) + 1\big)$, where the minimum is taken over all orderings of the vertex set. With another terminology, if G admits a vertex order such that $\max_{1\leq i\leq n} d^-(v_i) \leq d$, then it is called *$d$-degenerate.*

At the current state of graph coloring theory, part (ii) of Problem 2 looks hopelessly difficult. It is not known whether all line graphs, or all claw-free graphs, or all total graphs satisfy the equality $\chi_\ell = \chi$ (see, e.g., [14] for the numerous related references); and since these classes are induced-hereditary, in case of an affirmative answer they would also be choice-χ-perfect. But perhaps part (i) will be easier to solve, due to the structure of perfect graphs behind it.

Remark 3. Every choice-ω-perfect graph is perfect and also choice-χ-perfect. Moreover, a perfect graph is choice-χ-perfect if and only if it is choice-ω-perfect.

1.1. Some known or 'almost known' results

Some theorems on list colorings of graphs, which were stated in the original papers just in the form $\chi_\ell(G) = \chi(G)$, can be directly translated to choice-perfectness, due to the fact that the graph classes in question are induced-hereditary[3]. We mention such results first. When the graphs involved in them are perfect, too, then not only choice-χ-perfectness but also choice-ω-perfectness follows.

Theorem 4 (Rubin [4])**.** *A connected bipartite graph is choice-χ-perfect, and also choice-ω-perfect, if and only if the sequential removal of vertices of degree* 1 *yields a single vertex or an even cycle or a $\theta_{2,2,2s}$-graph, which means two vertices joined by three internally disjoint paths of lengths* 2, 2, *and* $2s$, *respectively* (*for some integer* $s \geq 1$)*.*

Theorem 5. *Let $G = L(H)$ be the line graph of a graph or multigraph H.*

- (Galvin [6]). *If H is a bipartite multigraph, then G is choice-ω-perfect.*
- (Peterson, Woodall [13]). *If H is a multigraph and G is perfect, then G is choice-ω-perfect.*
- (Woodall [20]). *If H is a multicircuit, then G is choice-χ-perfect.*
- (Juvan, Mohar, Thomas [12]). *If H is a series-parallel graph, then G is choice-χ-perfect.*

Theorem 6 (Gravier, Maffray [8])**.** *If G is perfect, claw-free* (*i.e., contains no induced* $K_{1,3}$)*, and has* $\omega(G) \leq 3$*, then G is choice-ω-perfect.*

We describe some further choice-perfect classes, which can be deduced shortly from known results.

A graph is *chordal* (also called *triangulated*) if it contains no induced cycles longer than 3.

[3] A graphical property $\mathcal{P}$, or the class $\mathcal{G}$ of graphs satisfying $\mathcal{P}$, is called *induced-hereditary* if $G \in \mathcal{G}$ implies $G' \in \mathcal{G}$ for all induced subgraphs $G' \subset G$. For an overview on the theory of hereditary graph properties we refer to the survey [2].

Proposition 7. *Every chordal graph is choice-χ-perfect and also choice-ω-perfect.*

Proof. Every chordal graph G satisfies the equality $\omega(G) = \mathrm{col}(G)$, and induced subgraphs of chordal graphs are chordal. Hence, the assertion follows by inequality (1). ■

Given a graph F, another graph G is called *F-free* if G contains no *induced* subgraphs isomorphic to F. Complete multipartite graphs belong to the perfect class of P_4-free graphs.

Theorem 8. *A complete multipartite graph is choice-χ-perfect, and equivalently it is choice-ω-perfect, if and only if it is $K_{3,3}$-free and $K_{2,4}$-free.*

Proof. Both $K_{3,3}$ and $K_{2,4}$ have choice number 3, hence they have to be excluded. Then, if some vertex class of (an induced subgraph of) a complete multipartite graph G has more than three vertices, all the other classes must be singletons. Selecting a color for a vertex in a singleton class and deleting the color from all the other lists, the assertion follows by induction for such graphs. Otherwise, if all classes have at least two vertices, then either $G \cong K_{2,2,\ldots,2}$ or $G \cong K_{3,2,\ldots,2}$. The equality $\chi_\ell = \chi = \omega$ holds in both of them; this was proved by Erdős, Rubin and Taylor in [4] for the former and by Gravier and Maffray in [7] for the latter. ■

We close this part with an observation in which not all graphs are perfect.

Proposition 9. *Every unicyclic graph is choice-χ-perfect. In particular, all cycles and all graphs of maximum degree 2 are choice-χ-perfect.*

Proof. Let G be unicyclic and G' an induced subgraph of G. Theorem 4 settles all cases with $\chi(G') = 2$. For $\chi(G') = 3$ the successive removal vertices of degree 1 keeps both χ and χ_ℓ unchanged, and reduces G' to an odd cycle. The latter has $\chi_\ell = \chi = 3$ by inequality (1). ■

1.2. Powers of cycles

The k^{th} power of a graph $G = (V, E)$ is obtained from G by inserting an edge between any two vertices $u, v \in V$ that are at distance at most k apart. (Multiple edges are replaced with single ones.) Powers of paths are interval graphs (and hence they are chordal), therefore it is easy to see that they are choice-χ-perfect and also choice-ω-perfect. For powers of cycles, however, the analogous problem looks quite hard. The main result of this note is the solution for $k = 2$, i.e. squares of cycles, which we state in the following theorem and prove in Section 2.

Theorem 10. *The graph C_n^2 is choice-χ-perfect for all $n \geq 3$.*

Corollary 11. *The graph C_n^2 is choice-ω-perfect if and only if it is perfect; that is, precisely if $n \leq 6$.*

Proof. For $n < 6$ we have $C_n^2 \cong K_n$, moreover $C_6^2 \cong \overline{3K_2}$; these graphs are perfect. On the other hand, $C_7^2 \cong \overline{C_7}$, and for $n \geq 8$ one of the induced cycles $v_1v_3v_5v_7 \dots v_{2\lfloor \frac{n+1}{2} \rfloor -1}$ and $v_1v_3v_4v_6v_7 \dots v_{2\lfloor \frac{n+1}{2} \rfloor -1}$ is odd, showing that these graphs are not perfect. ∎

The total graph $T(G)$ of a graph $G = (V, E)$ has vertex set $V \cup E$, and two of its vertices are adjacent if the corresponding vertices/edges in G are adjacent or incident. That is, G and its line graph $L(G)$ are vertex-disjoint induced subgraphs of $T(G)$, and the bipartite subgraph between them represents vertex-edge incidences. The following result was proved not only in [15] but also independently and simultaneously in the published paper [11] by Juvan, Mohar and Škrekovski.

Corollary 12. *The total graph $T(C_{3k})$ of C_{3k} is 3-choosable.*

Proof. Assuming $C_n = v_1v_2 \dots v_n$ with edges $e_i = v_iv_{i+1}$ (where $v_{n+1} = v_1$), we see that $T(C_n)$ is isomorphic to the square C_{2n}^2 of the cycle $v_1e_1v_2e_2 \dots v_ne_n$. ∎

The past of the present paper. In the preliminary version of Hilton and Johnsnon's paper [10], several questions were raised which we answered in the unpublished manuscript [15]. It remained a rough draft only, being sufficiently detailed to make sure that its results cited in [10] are valid, but not polished for a wider distribution. As its title indicates, it dealt with two aspects of list colorings: the rather irregular behavior of Hall number and the equalities $\chi_\ell = \chi$ and $\chi_\ell = \omega$. Later we published its theorems and further related results on Hall number in an elaborated way in the papers [16] and [17]. During the past one and a half decades, however, the other part of [15] that dealt with choice-perfect graphs remained unpublished.

2. Choice-perfectness of Squares of Cycles

In this section we prove Theorem 10. In order to create a more transparent structure, we split the proof into subsections. We note that the proof techniques for the different cases are quite different.

2.1. The small cases $3 \leq n \leq 5$

In these cases $C_n^2 \cong K_n$, and also every induced subgraph of C_n^2 is a complete graph. Since every K_m satisfies $\chi(K_m) = \chi_\ell(K_m) = m$, the graph C_n^2 is choice-χ-perfect for every $n \leq 5$.

Hence, in the rest of the proof we assume that $n > 5$.

2.2. Triangle-free subgraphs

Let H be a triangle-free induced subgraph of C_n^2. Since the four neighbors of each vertex of C_n^2 induce P_4, we obtain that if $v \in V(H)$ then at most two neighbors of v are present in H. In other words, H has maximum degree 2. Thus, all components of H are paths and/or cycles. Actually, if there is a cycle component (and not a triangle, by assumption) then H itself is a cycle, but this is unimportant in the present proof.

If H contains a component which is an odd cycle, then $\chi(H) = 3$. Since H has maximum degree 2, we have $\chi_\ell(H) \leq 3$ by inequality (1). On the other hand, if all components of H are even cycles and paths, then $\chi_\ell(H) = 2$ by [4]. Hence, $\chi_\ell = \chi$ holds in either case and therefore choice-χ-perfectness follows.

2.3. Cycles divisible by 3

Here we consider the square of the graph $C_{3k} = v_1 v_2 \dots v_{3k}$, and interpret subscripts modulo $3k$. That is, for any $j > 0$, vertex v_{3k+j} is the same as v_j.

We first prove $\chi_\ell(C_{3k}^2) = 3$. This will imply $\chi_\ell(H) = \chi(H)$ for all $H \subseteq C_{3k}^2$ containing at least one triangle, whereas the choice-χ-perfectness of triangle-free induced subgraphs $H \subset C_{3k}^2$ holds by the previous subsection. Those two complementary facts will imply that C_{3k}^2 is choice-χ-perfect. As we cited above, in the context of total colorings the equality $\chi_\ell(C_{6k}^2) = 3$ appeared also in the paper by Juvan *et al.* [11]. Their argument is different; although, digging one level deeper, the two proofs have a common root in the results of Alon and Tarsi [1] that we shall apply further in a later subsection.

Consider the $k = n/3$ triangles $v_i v_{i+1} v_{i+2}$ for $i = 3j + 1$ with $j = 0, 1, \dots, k - 1$. They cover the vertex set and omitting their edges from C_{3k}^2 we obtain the Hamiltonian cycle $v_1 v_{3k} v_2 v_4 v_3 v_5 v_7 v_6 v_8 v_{10} \dots v_{3k-3} v_{3k-1} v_1$ composed by the concatenation of the segments $v_{3j} v_{3j+2} v_{3j+4}$ with $j = 0, 1, \dots, k-1$ (where $v_0 = v_{3k}$). Consequently, $\chi_\ell(C_{3k}^2) = 3$ follows by the 'Cycle-Plus-Triangles Theorem' of Fleischner and Stiebitz [5].

2.4. The graph itself for $n = 3k + 1$ and $n = 3k + 2$

If $n > 6$ is not a multiple of 3, it is easily seen that $\chi(C_n^2) = 4$. Indeed, trying to construct a proper vertex 3-coloring that starts with $\varphi(v_i) = i$ for $i = 1, 2, 3$ we obtain that $\varphi_{3k-1} = 2$, $\varphi(v_{3k}) = 3$, and $\varphi(v_{3k+1}) = 1$ should hold. For $n = 3k+1$ this creates the monochromatic edge $v_1 v_{3k+1}$ in color 1; and for $n = 3k + 2$ it forces $\varphi(v_{3k+2}) = 2 = \varphi(v_2)$, that is an edge in color 2. In either case we obtain $\chi(C_n^2) = 4$.

The proof will be done if we show that $\chi_\ell(C_n^2) = 4$ holds for all $n > 5$ (which now means $n \geq 7$). One of the several possibilities is to consider the subgraph $H = C_n^2 - v_1 - v_5$. The vertex order $v_4, v_6, v_7, v_8, \dots, v_{n-1}, v_n, v_2, v_3$ has the property that every vertex is preceded by at most two of its neighbors. Hence, $\chi_\ell(H) \le \mathrm{col}(H) \le 3$ holds by inequality (1).

Let $\{L_{v_i} \mid 1 \le i \le n\}$ be any list assignment with lists of size 4 on the vertices of C_n^2. If v_1 and v_5 admit a common color, say $c \in L_{v_1} \cap L_{v_5}$, we color v_1 and v_5 with c, and remove c from the lists of all neighbors of v_1 and v_5. On the other hand, if $L_{v_1} \cap L_{v_5} = \emptyset$, then there exists a color $c \in L_{v_1} \setminus (L_{v_3} \cup L_{v_5})$ or $c \in L_{v_5} \setminus (L_{v_3} \cup L_{v_1})$. We color one of v_1 and v_5 with c, choose any color from the list of the other one, and remove these two colors from the lists of the neighbors of v_1 and v_5. In either case, the removal of colors leaves H with at least three colors in each list, because we never assign more than one color from L_{v_3} to $\{v_1, v_5\}$. Thus, C_n^2 is colorable from the lists L_{v_i} $(i = 1, \ldots, n)$.

Remark 13. The removal of $\{v_1, v_5\}$ is a particular case of the 'List Reduction Lemma' proved by Voigt and the present author in [18]. It ensures that if $X \subset V$ is a set such that the edges meeting X form a 2-choosable graph, then from any lists of size k one can choose colors for the vertices of X in such a way that, for each $v \in V \setminus X$, at most one color of L_v occurs on the neighbors of v in X. Hence, if $\chi_\ell(G-X) < k$, then $\chi_\ell(G) \le k$. An alternative way to prove the upper bound $\chi_\ell(C_n^2) \le 4$ for $n > 5$ is to observe that C_n^2 is 4-regular, and then apply the list coloring analogue of Brooks's theorem [4, 19] by which $\chi_\ell(G) \le \Delta(G)$ holds for every connected graph G other than an odd cycle or a complete graph. Moreover, if n is even, then a stronger result is valid. Translating a theorem of Juvan, Mohar and Škrekovski [11, Lemma 2.1] from the total graph of $C_{n/2}$ to the square of C_n, it follows that C_n^2 is list colorable whenever the vertices $v_1, v_3, \ldots, v_{n-1}$ have lists of size 3 and $v_2, v_4, \ldots, v_n$ have lists of size 4.

The preceding comment leads to the question—or better to say, to the group of questions—how many lists can be shorter than χ_ℓ under various side conditions, in such a way that they still admit a list coloring. An extreme case is the class of odd cycles where all lists can be allowed to have size 2, assuming that not all are the same.

2.5. Proper induced subgraphs for $n = 3k + 1$

We assume that $C_n = v_1 v_2 \ldots v_{3k+1}$ and consider the subgraph $H = C_n^2 - v_{3k+1}$. Now H is a chain of triangles $v_i v_{i+1} v_{i+2}$ $(i = 1, \ldots, k-2)$ closed to a 'cycle' with the edge $v_{3k} v_1$.

Let us orient the edges of H as follows: $v_i \to v_{i+1}$ $(i = 1, \ldots, 3k-1)$, $v_i \to v_{i+2}$ $(i = 1, \ldots, 3k-2)$, $v_{3k} \to v_1$. We denote by $\vec{H}$ the digraph obtained. We are going to compare the numbers of even and odd directed paths from v_{3k} to v_i in $\vec{H}$ $(i = 1, \ldots, 3k\,;$ the last case $i = 3k$ means a directed cycle, that is oriented cyclically). By induction on i, the following facts can be observed:

- For all $i \equiv 1 \pmod 3$ there is precisely one more odd path from v_{3k} to v_i than the number of even paths.

- For all $i \equiv 2 \pmod 3$ there is precisely one more even path from v_{3k} to v_i than the number of odd paths.

- For all $i \equiv 0 \pmod 3$ the number of even and odd paths from v_{3k} to v_i is exactly the same.

Indeed, there is just one odd path $v_{3k} \to v_1$, just one even path $v_{3k} \to v_2$, and for $i > 2$ the vertex v_i can be reached directly either from v_{i-2} or from v_{i-1} while the parity of each path switches to opposite in one step. For subscripts divisible by 3, the 'one more odd path' at v_{i-2} and the 'one more even path' at v_{i-1} cancel each other, yielding equal number for odd and even paths; and every other v_i is preceded by two neighbors one of which counts the same number for odd and even paths while the 1 surplus at the other neighbor switches to the opposite parity of path length. (The exact numbers can be expressed from the Fibonacci sequence as $\lfloor f_i/2 \rfloor$ and $\lceil f_i/2 \rceil$, where $f_1 = f_2 = 1$ and $f_i = f_{i-2} + f_{i-1}$ for $i > 2$.)

Applying the third case for $i = 3k$, and taking into account that $\vec{H} - v_{3k}v_1$ contains no directed cycles, we see that the number of even and odd directed cycles in $\vec{H}$ is the same.

With the terminology of [1], by an *Eulerian subgraph* of a digraph we mean a subgraph such that each vertex has the same in- and out-degree. Note that connectivity is not assumed here.

If an Eulerian subgraph of $\vec{H}$ does not contain the edge $v_{3k}v_1$, then v_1 must have out-degree 0 in it; this implies that v_2, too, has out-degree 0; and so on, we obtain that the subgraph is edgeless, and there is precisely one subgraph of this kind.

On the other hand, if an Eulerian subgraph contains the edge $v_{3k}v_1$, then v_1 must have out-degree 1. This means the deletion of one edge, v_1v_2 or v_1v_3. Should we delete v_1v_2, vertex v_2 must have out-degree 0. Should we keep v_1v_2 and delete v_1v_3, vertex v_2 must have out-degree 1. In either case, v_3 has in-degree at most 1, which yields its out-degree to be 0 or 1. This property propagates through all vertices and we obtain that in every Eulerian subgraph of $\vec{H}$, each vertex has in- and out-degree 0 or 1. It follows that these subgraphs containing the edge $v_{3k}v_1$ are precisely the directed cycles of $\vec{H}$. Thus, counting also the edgeless subgraph, the number of Eulerian subgraphs of $\vec{H}$ with an even number of edges is precisely one larger than the number of those with an odd number of edges.

In [1], Alon and Tarsi proved the following theorem: If $\vec{G}$ is an orientation of a graph G such that the numbers of even and odd Eulerian subgraphs in $\vec{G}$ are not equal, then G is list colorable for every list assignment in which the list L_v of

each vertex v is larger than the out-degree of v in $\vec{G}$. By the observations above, this theorem can be applied to $\vec{H}$, which has maximum out-degree 2. Thus, we obtain that $\chi_\ell(H) \leq 3$ holds.
Consequently, we have $\chi_\ell = \chi = 3$ for all proper induced subgraphs containing at least one triangle. We have also seen that the equality $\chi_\ell = \chi$ holds for all triangle-free induced subgraphs. Thus, choice-χ-perfectness follows for $n = 3k+1$.

2.6. The case of $n = 3k + 2$

In this case it is not true that all proper induced subgraphs are 3-colorable. Indeed, if we remove vertex v_{3k+2} and edge v_1v_{3k+1}, we obtain a uniquely 3-colorable graph in which the 3-coloring requires to have the set $\{v_{3i+1} \mid 1 \leq i \leq k\}$ monochromatic. Thus, putting back the edge v_1v_{3k+1} we obtain that $\chi(C^2_{3k+2} - v_{3k+2}) = 4$ holds while the graph is of course 4-choosable.

As above, we also have $\chi_\ell(H) = \chi(H)$ for all *triangle-free* induced subgraphs $H \subset C^2_{3k+2}$. Hence, the proof will be done if we prove that every subgraph obtained by deleting two vertices is 3-choosable.

Suppose that one of the deleted vertices is v_4. Observe that deleting further any one of the vertices $v_1, v_2, v_3, v_5, v_6, v_7$ we obtain a 2-degenerate graph. For example, omitting v_1 or v_2, the degree of vertex v_3 decreases to 2 and then in the sequence $v_3, v_4, v_5, \ldots$ each vertex is followed by at most two of its neighbors. Similarly, the sequence $v_4, v_5, v_6, \ldots$ has the same property if v_3 is removed. Since every 2-degenerate graph is 3-choosable by (1), the proof is done in these cases.

Suppose that v_4 is deleted with some further vertex v_j, where $8 \leq j \leq 3k+2$. Then the graph H obtained consists of two chains of triangles connected by the two disjoint edges v_3v_5 and $v_{j-1}v_{j+1}$. (For $j = 3k+2$, the latter edge is $v_{3k+1}v_1$.) To simplify work with subscripts we assume that the two chains are induced by $V' := \{v_a, v_{a+1}, \ldots, v_b\}$ and $V'' := \{v_c, v_{c+1}, \ldots, v_d\}$. Here $a = 5$, $b = j - 1$, $c = j + 1$, and $d = 3k + 5$; the latter means the same as $d = 3$. (Possibly also $c = 3k + 3$ holds, which means $c = 1$.)

Consider any 3-assignment on the vertices of H. We first investigate colorings of the subgraph H' induced by V'. Assume that the list of v_a and v_b is $L_a = \{a_1, a_2, a_3\}$ and $L_b = \{b_1, b_2, b_3\}$, respectively. We construct the bipartite graph B' on the vertex set $L_a \cup L_b$ in which a_ib_j is an edge if and only if H' admits a list coloring with $\varphi(v_a) = a_i$ and $\varphi(v_b) = b_j$. Both sequences $v_a, v_{a+1}, \ldots, v_b$ and $v_b, v_{b-1}, \ldots, v_a$ have the property that each vertex is preceded by at most two of its neighbors. Hence, the vertices of H' can be colored from their lists in either of these orders. It follows that B' has no isolated vertices.

Then, either B' has three mutually disjoint edges, or some vertex of L_b has degree at least two in B'. The latter can be seen by taking a maximum matching $M \subset B'$ and then choosing an edge e that covers a vertex of L_a not contained in the edges of M (assuming that $|M| \leq 2$). The degree of the vertex $e \cap L_b$ is greater than one in B'.

In the same way, we can construct the bipartite graph B'' on the vertex set $L_c \cup L_d$, whose edges represent the possible ordered pairs $(\varphi(v_c), \varphi(v_d))$ of colors in the list colorings of H''. Also here, either B'' contains three disjoint edges or some vertex of L_d has degree at least two in B''.

Suppose that B' contains a matching M of three edges. Then take any list coloring φ'' of H''. From the three edges of M, select one $a_i b_j$ such that $b_j \neq \varphi''(v_c)$ and $a_i \neq \varphi''(v_d)$. By the definition of B', there exists a list coloring φ' of H' such that $\varphi(v_a) = a_i$ and $\varphi(v_b) = b_j$. Then (φ', φ'') is a list coloring of H. Analogously, a list coloring of H can be obtained if B'' has a matching of size three.

If both B' and B'' have matching number at most two, we begin with a vertex of degree greater than one in L_b. We may assume (by renumbering of colors if necessary) that $a_1 b_1$ and $a_2 b_1$ are edges of B'. Since also some vertex of L_d has degree greater than one, there is an edge $c_i d_j$ of B'' such that $c_i \neq b_1$. Then one of a_1 and a_2 is surely different from d_j. Assuming that $a_1 \neq d_j$, we can take list colorings φ' of H' and φ'' of H'' such that $\varphi'(v_a) = a_1$, $\varphi'(v_b) = b_1$, $\varphi''(v_c) = c_i$, $\varphi''(v_d) = d_j$. Then (φ', φ'') is a list coloring of H.

Hence, the proof of the theorem is completed.

3. Some Further Aspects of 'Perfectness'

If μ and ξ are any quantitative characteristics defined for all graphs (or for the members of a reasonably rich graph class), such that $\mu(G) \leq \xi(G)$ is valid for all graphs G, then it is natural to raise the problem of characterizing those G for which $\mu(G') = \xi(G')$ holds for every induced subgraph $G' \subseteq G$. The theory of perfect graphs is the success story concerning $\mu = \omega$ and $\xi = \chi$.

As a game version of $\chi_\ell = \chi$, it would be interesting to study the relation between game chromatic number and its list coloring analogue introduced by Borowiecki *et al.* in [3]. Similar questions arise in connection with pairs of graph coloring invariants μ and ξ provided that $\mu(G) = \xi(G)$ hereditarily holds for an infinite family of graphs G.

We note, however, that some pairs of functions do not really seem to offer comparisons of much interest. An example is the so-called Hall number, which is interesting on its own right as a lower bound on χ_ℓ. Let $G = (V, E)$ be a graph, together with lists L_v of allowed colors for its vertices $v \in V$. For any $X \subseteq V$ and any color $c \in \bigcup_{v \in V} L_v$ we consider the *independence number* $\alpha(X, c)$ of the subgraph of G induced by the set $\{v \in X \mid c \in L_v\}$. A necessary condition for the list colorability of G is that

$$\sum\nolimits_c \alpha(X, c) \geq |X| \qquad \text{for all } X \subseteq V \tag{2}$$

should hold. The *Hall number* of G, denoted by $h(G)$, is defined as the smallest positive integer k such that G is list colorable whenever (2) is satisfied and $|L_v| \geq k$ holds for all $v \in V$.

The term 'Hall number' originates from the fact that if G is a complete graph then (2) is equivalent to the classical Hall Condition. This implies $h(K_n) = 1$ for all $n \geq 1$, therefore the Hall number can be much smaller than χ_ℓ, even on choice-ω-perfect graphs. The characterization theorem by Hilton and Johnson in [9] states that $h(G) = 1$ holds if and only if each block of G is a complete graph. In particular, $h(C_{2k}) = 2 = \chi_\ell(C_{2k})$ holds for all even cycles, but $h(G) = 1 < 2 = \chi_\ell(G)$ for all $G \subsetneqq C_{2k}$ with at least one edge. Hence, the equality $\chi_\ell = h$ is very far from being hereditary.

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