Discussiones Mathematicae
Graph Theory 33 (2013) 289–306
doi:10.7151/dmgt.1659

VERTEX-DISTINGUISHING IE-TOTAL COLORINGS OF COMPLETE BIPARTITE GRAPHS $K_{m,n}(m < n)$[1]

Xiang'en Chen, Yuping Gao and Bing Yao

College of Mathematics and Information Science
Northwest Normal University, Lanzhou 730070, P. R. China

e-mail: chenxe@nwnu.edu.cn

Abstract

Let G be a simple graph. An IE-total coloring f of G is a coloring of the vertices and edges of G so that no two adjacent vertices receive the same color. Let $C(u)$ be the set of colors of vertex u and edges incident to u under f. For an IE-total coloring f of G using k colors, if $C(u) \neq C(v)$ for any two different vertices u and v of G, then f is called a k-vertex-distinguishing IE-total-coloring of G, or a k-VDIET coloring of G for short. The minimum number of colors required for a VDIET coloring of G is denoted by $\chi_{vt}^{ie}(G)$, and is called vertex-distinguishing IE-total chromatic number or the VDIET chromatic number of G for short. VDIET colorings of complete bipartite graphs $K_{m,n}(m < n)$ are discussed in this paper. Particularly, the VDIET chromatic numbers of $K_{m,n}(1 \leq m \leq 7, m < n)$ as well as complete graphs K_n are obtained.

Keywords: complete bipartite graphs, IE-total coloring, vertex-distinguishing IE-total coloring, vertex-distinguishing IE-total chromatic number.

2010 Mathematics Subject Classification: 05C15.

For an edge coloring (proper or not) of a graph G and a vertex v of G, denote by $S(v)$ the set of colors used to color the edges incident to v.

A proper edge coloring of a graph G is said to be *vertex-distinguishing* if for any $u, v \in V(G), u \neq v$, $S(u) \neq S(v)$. In other words, $S(u) \neq S(v)$ whenever $u \neq v$. A graph G has a vertex-distinguishing proper edge coloring if and only if it has no more than one isolated vertex and no isolated edges. Such a graph will be referred to as a *vdec*-graph. The minimum number of colors required

[1]This research is supported by NNSFC (No.61163037, 61163054), the Scientific Research Project of Northwest Normal University (No. nwnu-kjcxgc-03-61), the Natural Foudation Project of Ningxia (No. NZ1154) and the Scientific Research Foundation Project of Ningxia University ((E):ndzr10-7)

for a vertex-distinguishing proper edge coloring of a *vdec*-graph G is denoted by $\chi'_s(G)$. Vertex-distinguishing proper edge coloring has been considered in several papers [1-5, 8-9].

A general edge coloring (not necessarily proper) of a graph G is said to be *vertex-distinguishing* if $S(u) \neq S(v)$ is required for any two distinct vertices u, v. The *point-distinguishing chromatic index* of a *vdec*-graph G, denoted by $\chi_0(G)$, refers to the minimum number of colors required for a vertex-distinguishing general edge coloring of G. This parameter was introduced by Harary and Plantholt in [7]. Although the structure of complete bipartite graph is simple, it seems that the problem of determining $\chi_0(K_{m,n})$ is not easy, especially in the case $m = n$, as documented by papers of Horňák and Soták [10, 11], Zagaglia Salvi [13, 14] and Horňák and Zagaglia Salvi [12].

A *total coloring* of a graph G is an assignment of some colors to the vertices and edges of G. It is *proper* if the following three conditions are satisfied:

Condition (v): No two adjacent vertices receive the same color;

Condition (e): No two adjacent edges receive the same color;

Condition (i): No edge receives the same color as any one of its incident vertices.

For a total coloring (proper or not) f of G and a vertex v of G, denote by $C_f(v)$, or simply $C(v)$ if no confusion arise, the set of colors used to color the vertex v as well as the edges incident to v. Let $\overline{C}(v)$ be the complementary set of $C(v)$ in the set of all colors we used. Obviously $|C(v)| \leq d_G(v) + 1$ and the equality holds if the total coloring is proper.

For a proper total coloring, if $C(u) \neq C(v)$ for any two distinct vertices u and v, then the coloring is called a *vertex-distinguishing proper total coloring* and the minimum number of colors required for a vertex-distinguishing proper total coloring is denoted by $\chi_{vt}(G)$. This concept was considered in [6, 15]. In [15], the following conjecture was given.

Conjecture 1. *Suppose G is a simple graph and n_d is the number of vertices of degree d, $\delta \leq d \leq \Delta$. Let k be the minimum positive integer such that $\binom{k}{d+1} \geq n_d$ for all d such that $\delta \leq d \leq \Delta$. Then $\chi_{vt}(G) = k$ or $k + 1$.*

From [15] we know that the above conjecture is valid for complete graphs, complete bipartite graphs, paths and cycles, etc.

In this paper we propose a kind of vertex-distinguishing general total coloring called IE-total coloring. The relationship between this coloring and vertex-distinguishing proper total coloring is similar to the relationship between vertex-distinguishing general edge coloring and vertex-distinguishing proper edge coloring.

An IE-*total coloring* of a graph G is a total coloring of G such that the Condition **(v)** is satisfied. If f is an IE-total coloring of graph G using k colors and for any $u, v \in V(G)$, $u \neq v$, we have $C(u) \neq C(v)$, then f is called a *k-vertex-distinguishing* IE-*total coloring*, or a k-VDIET coloring. The number

$$\min\{k : G \text{ has a } k\text{-VDIET coloring}\}$$

is called the *vertex-distinguishing* IE-*total chromatic number* of a graph G and is denoted by $\chi^{ie}_{vt}(G)$.

The following proposition is obviously true.

Proposition 2. $\chi^{ie}_{vt}(G) \leq \chi_{vt}(G)$.

For a graph G, let n_i denote the number of vertices of degree i, $\delta \leq i \leq \Delta$. Let $\xi(G) = \min\left\{k | \binom{k}{1} + \binom{k}{2} + \binom{k}{3} + \cdots + \binom{k}{s+1} \geq n_\delta + n_{\delta+1} + \cdots + n_s, \delta \leq s \leq \Delta\right\}$. Obviously, $\chi^{ie}_{vt}(G) \geq \xi(G)$.

In the following we will consider the VDIET colorings of complete bipartite graphs $K_{m,n}(1 \leq m < n)$ and complete graphs K_n, then we will give three conjectures.

For a complete bipartite graph $K_{m,n}(1 \leq m < n)$, $\xi(K_{m,n})$ is the minimum positive integer l such that

$$\binom{l}{1} + \binom{l}{2} + \binom{l}{3} + \cdots + \binom{l}{m+1} \geq n, \tag{1}$$

$$\binom{l}{1} + \binom{l}{2} + \binom{l}{3} + \cdots + \binom{l}{n+1} \geq n+m. \tag{2}$$

Proposition 3. (i) *If* $n = \sum_{i=1}^{m+1} \binom{m+2}{i} - m + 1$, *then* $\xi(K_{m,n}) = m + 2$;

(ii) *If* $\sum_{i=1}^{m+1} \binom{m+2}{i} - m + 2 \leq n \leq \sum_{i=1}^{m+1} \binom{m+3}{i} - m$, *then* $\xi(K_{m,n}) = m + 3$.

Proof. (i) When $l = m + 1$, (1) is not valid, because

$$\binom{m+1}{1} + \binom{m+1}{2} + \cdots + \binom{m+1}{m+1} = 2^{m+1} - 1,$$

$$n = 2^{m+2} - 2 - m + 1 = 2^{m+2} - m - 1 > 2^{m+1} - 1.$$

Therefore $\xi(K_{m,n}) \geq m + 2$. Since

$$\binom{m+2}{1} + \binom{m+2}{2} + \cdots + \binom{m+2}{m+1} = 2^{m+2} - 2 \geq 2^{m+2} - m - 1 = n,$$

$$\binom{m+2}{1} + \binom{m+2}{2} + \cdots + \binom{m+2}{n+1} = 2^{m+2} - 1 = m + n,$$

so we have $\xi(K_{m,n}) = m+2$.

(ii) When $\sum_{i=1}^{m+1}\binom{m+2}{i} - m + 2 \le n \le \sum_{i=1}^{m+1}\binom{m+3}{i} - m$, i.e., $2^{m+2} - m \le n \le 2^{m+3} - (m+3) - 2 - m = 2^{m+3} - 2m - 5$, we have

$$\binom{m+2}{1} + \binom{m+2}{2} + \cdots + \binom{m+2}{n+1} = 2^{m+2} - 1 \le m + n - 1.$$

Therefore, (2) is not valid if $l = m+2$. So, $\xi(K_{m,n}) \ge m+3$. When $l = m+3$, (1) and (2) are right, so $\xi(K_{m,n}) = m+3$. ■

Proposition 4. (i) *If* $\sum_{i=1}^{m+1}\binom{k-1}{i} - m < n \le \sum_{i=1}^{m+1}\binom{k-1}{i}$ *and* $k \ge m+4$, *then* $\xi(K_{m,n}) = k-1$;

(ii) *If* $\sum_{i=1}^{m+1}\binom{k-1}{i} < n \le \sum_{i=1}^{m+1}\binom{k}{i} - m$ *and* $k \ge m+4$, *then* $\xi(K_{m,n}) = k$.

Proof. (i) As

$$\binom{k-2}{1} + \binom{k-2}{2} + \cdots + \binom{k-2}{m+1} \le \left[\binom{k-2}{0} + \binom{k-2}{1}\right] + \left[\binom{k-2}{1} + \binom{k-2}{2}\right]$$

$$+ \cdots + \left[\binom{k-2}{m} + \binom{k-2}{m+1}\right] - m = \binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{m+1} - m < n,$$

(1) is not valid if $l = k-2$. Therefore, $\xi(K_{m,n}) \ge k-1$. Because

$$\binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{m+1} \ge n,$$

$$\binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{n+1} \ge n + \binom{k-1}{m+2} + \binom{k-1}{m+3} + \cdots + \binom{k-1}{n+1} > m+n,$$

so (1) and (2) are valid if $l = k-1$. We have $\xi(K_{m,n}) = k-1$.

(ii) When $\sum_{i=1}^{m+1}\binom{k-1}{i} < n \le \sum_{i=1}^{m+1}\binom{k}{i} - m$, (1) is not valid if $l = k-1$, whereas (1) and (2) are valid if $l = k$. Therefore $\xi(K_{m,n}) = k$. ■

Theorem 5. *Let* $m \ge 1$, $n > 2^{m+2} - m - 2$. *Then* $\chi_{vt}^{ie}(K_{m,n}) = k$ *when* $\sum_{i=1}^{m+1}\binom{k-1}{i} - m < n \le \sum_{i=1}^{m+1}\binom{k}{i} - m$.

Proof. As $n > 2^{m+2} - m - 2$, we have $k > m+2$ (otherwise, if $k \le m+2$, then $n \le \sum_{i=1}^{m+1}\binom{k}{i} - m \le \sum_{i=1}^{m+1}\binom{m+2}{i} - m = 2^{m+2} - 2 - m$, a contradiction).

We prove that $K_{m,n}$ does not have a $(k-1)$-VDIET coloring. If not, suppose g is a VDIET coloring of $K_{m,n}$ using colors $1, 2, \ldots, k-1$. Let $B_0 = \{g(u_1), g(u_2), \ldots, g(u_m)\}$, $B_i = \{1, 2, \ldots, k-1\} \setminus C_g(u_i), i = 1, 2, \ldots, m$. Note that none of $B_0, B_1, B_2, \ldots, B_m$ is the color set of any vertex v_j. Let $T = \{j : |C_g(v_j)| = 1, j = 1, 2, \ldots, n\}$ and $t = |T|$. Then $B_0 \cap \{g(v_j) | j \in T\} = \emptyset$. Without loss of generality, we assume that $C_g(v_j) = \{j\}, j = 1, 2, \ldots, t$, then we have $|C_g(v_j)| \geq 2, j = t+1, \ldots, n$ and $C_g(u_i) \supseteq \{1, 2, \ldots, t, g(u_i)\}, i = 1, 2, \ldots, m$.

Case 1. $t \geq k-m-3$. For each $i \in \{1, 2, \ldots, m\}$, we have $|C_g(u_i)| \geq t+1$ and $|B_i| \leq (k-1)-(t+1) \leq (k-1)-(k-m-3+1) = m+1$. Note that $|B_0| \leq m+1$ and none of $B_0, B_1, B_2, \ldots, B_m$ is the color set of any vertex v_j. Therefore there are at most $\binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{m+1} - m$ subsets of $\{1, 2, \ldots, k-1\}$ with cardinality between 1 and $m+1$ which may become the color sets of vertices $v_1, v_2, \ldots, v_n$. This is a contradiction.

Case 2. $t \leq k-m-4$. In this case, there are at least $(k-1)-(k-m-4) = m+3$ subsets of $\{1, 2, \ldots, k-1\}$ with cardinality 1 which are not the color sets of vertices $v_1, v_2, \ldots, v_n$. This is also a contradiction because $\binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{m+1} - (m+3) < \binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{m+1} - m < n$, and at most $\binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{m+1} - (m+3)$ subsets of $\{1, 2, \ldots, k-1\}$ with cardinality between 1 and $m+1$ cannot distinguish n vertices.

In the following we prove that $K_{m,n}$ has a k-VDIET coloring. Let $V(K_{m,n}) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$ and $E(K_{m,n}) = \{u_i v_j : i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$.

Put $D(u_i) = \{1, 2, \ldots, k\} \setminus \{i\}, i = 1, 2, \ldots, m-1$, $D(u_m) = \{1, 2, \ldots, k\}$, $D(v_j) = \{j, k\}, j = 1, 2, \ldots, m-1$, $D(v_j) = \{j\}, j = m, m+1, \ldots, k-1$.

Now distribute other subsets of $\{1, 2, \ldots, k\}$ with cardinality between 2 and $m+1$ to vertices $v_k, v_{k+1}, \ldots, v_n$. These $n-k+1$ subsets are denoted by $D(v_k), D(v_{k+1}), \ldots, D(v_n)$, respectively.

Construct a mapping f from $V(K_{m,n}) \cup E(K_{m,n})$ to $\{1, 2, \ldots, k\}$ as follows:
Put $f(u_i) = k, i = 1, 2, \ldots, m$, $f(v_j) = \min D(v_j), j = 1, 2, \ldots, n$,

$f(u_i v_i) = k$ for $i = 1, 2, \ldots, m-1$, $f(u_m v_m) = m$,

$f(u_i v_j) = j, i = 1, 2, \ldots, m, j = 1, 2, \ldots, k-1, i \neq j$.

For each $j = k, k+1, \ldots, n$, we recursively let $f(u_1 v_j) = \min \big(D(u_1) \cap (D(v_j) \setminus \{f(v_j)\}\big)$ or $f(u_1 v_j) \in D(u_1) \cap D(v_j)$ when $D(u_1) \cap (D(v_j) \setminus \{f(v_j)\} = \emptyset$.

When $2 \leq i \leq m$, $f(u_i v_j) = \min \big(D(u_i) \cap (D(v_j) \setminus \{f(v_j), f(u_1 v_j), f(u_2 v_j) \ldots, f(u_{i-1} v_j)\}\big)$ or $f(u_i v_j) \in D(u_i) \cap D(v_j)$ when $D(u_i) \cap (D(v_j) \setminus \{f(v_j), f(u_1 v_j), f(u_2 v_j), \ldots, f(u_{i-1} v_j)\}) = \emptyset$.

It is not hard to see that $C_f(u_i) = D(u_i), i = 1, 2, \ldots, m$; $C_f(v_j) = D(v_j), j = 1, 2, \ldots, n$ and moreover $f(u_i) > f(v_j)$, therefore our coloring f is a vertex distinguishing IE-total coloring and then $\chi^{ie}_{vt}(K_{m,n}) \leq k$. ■

Theorem 6. *Let* $m \geq 2$, $\binom{m+2}{1} + \binom{m+2}{2} + \cdots + \binom{m+2}{m+1} - 2m + 1 < n \leq \binom{m+2}{1} + \binom{m+2}{2} + \cdots + \binom{m+2}{m+1} - m$, *i.e.,* $2^{m+2} - 2m - 1 < n \leq 2^{m+2} - m - 2$. *Then* $\chi_{vt}^{ie}(K_{m,n}) = m + 3$.

Proof. When $2^{m+2}-2m-1 < n \leq 2^{m+2}-m-2$, we have $\chi_{vt}^{ie}(K_{m,n}) \geq \xi(K_{m,n}) = m + 2$. We first prove that $K_{m,n}$ does not have a $(m + 2)$-VDIET coloring. Otherwise, suppose g is a VDIET coloring of $K_{m,n}$ using colors $1, 2, \ldots, m + 2$.

Let $B_0 = \{g(u_1), g(u_2), \ldots, g(u_m)\}$, $B_i = \{1, 2, \ldots, m + 2\} \setminus C_g(u_i), i = 1, 2, \ldots, m$. Note that $B_0, B_1, B_2, \ldots, B_m$ are distinct and at most one of them is an empty set. $B_0, B_1, B_2, \ldots, B_m$ are not the color sets of vertices $v_1, v_2, \ldots, v_n$. We give a fact as follows.

Observation 7. $|C_g(u_i)| \geq 2, i = 1, 2, \ldots, m$. *Furthermore, there exists a vertex* $v \in \{v_1, v_2, \ldots, v_n\}$ *with* $|C_g(v)| = 1$.

Proof. Suppose that there exists a vertex $u_i \in \{u_1, u_2, \ldots, u_m\}$ with $C_g(u_i) = \{\alpha\}, \alpha \in \{1, 2, \ldots, m + 2\}$. Then $\alpha \in C_g(v_j), j = 1, 2, \ldots, n$. However, $2^{m+1} - 1 < 2^{m+2} - 2m - 1 < n$, i.e., the subsets of $\{1, 2, \ldots, m + 2\}$ which contain α cannot distinguish n vertices, this is a contradiction. Therefore, $|C_g(u_i)| \geq 2, i = 1, 2, \ldots, m$.

Suppose $|C_g(v_j)| \geq 2, j = 1, 2, \ldots, n$, i.e., all 1-subsets of $\{1, 2, \ldots, m+2\}$ are not the color sets of vertices $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$. Therefore, there are at most $2^{m+2}-1-(m+2) < 2^{m+2}-1-m < m+n$ nonempty subsets of $\{1, 2, \ldots, m+2\}$ which may become the color sets of vertices $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$. This is a contradiction. □

Using the above observation, without loss of generality, we assume $C_g(v_1) = \{1\}$, Then $1 \in C_g(u_i), i = 1, 2, \ldots, m$, $g(u_i) \neq 1, i = 1, 2, \ldots, m$.

It is obvious that $B_0, B_1, B_2, \ldots, B_m$ are not the color sets of any vertex $u_i, i = 1, 2, \ldots, m$. Therefore, there are at most $2^{m+2} - 1 - m < m + n$ nonempty subsets of $\{1, 2, \ldots, m+2\}$ which may become the color sets of vertices $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$. This is a contradiction.

So, $\chi_{vt}^{ie}(K_{m,n}) \geq m + 3$.

In the following we prove that $K_{m,n}$ has a $(m + 3)$-VDIET coloring when $2^{m+2} - 2m - 1 < n \leq 2^{m+2} - m - 2$.

By Theorem 5, we can give $K_{m,t}$ a $(m + 3)$-VDIET coloring f using colors $1, 2, \ldots, m+3$, where $2^{m+2} - 2 - m < t \leq 2^{m+3} - 2m - 5$. Now delete the vertices $v_{n+1}, v_{n+2}, \ldots, v_t$ and their colors, delete the edges $u_iv_j, i = 1, 2, \ldots, m, j = n + 1, n + 2, \ldots, t$ and their colors. It is not hard to see that under the resulting coloring the color sets of $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$ do not change, so we get a $(m + 3)$-VDIET coloring g of $K_{m,n}$ using colors $1, 2, \ldots, m + 3$. ■

Theorem 8. *Let s be the minimum positive integer such that $2^s - 1 \geq 3m$. When $2^r - 2m - 1 < n \leq 2^{r+1} - 2m - 1$, we have $\chi_{vt}^{ie}(K_{m,n}) = r + 1$, where $r = m + 1, m, m - 1$ and $r \geq s$.*

Proof. $\xi(K_{m,n}) = \begin{cases} r, & \text{when } 2^r - 2m - 1 < n \leq 2^r - m - 1; \\ r + 1, & \text{when } 2^r - m - 1 < n \leq 2^{r+1} - 2m - 1. \end{cases}$

When $2^r - 2m - 1 < n \leq 2^r - m - 1$, it is obvious that $\chi_{vt}^{ie}(K_{m,n}) \geq r$. We prove that $K_{m,n}$ does not have an r-VDIET coloring when $r = m+1, m, m-1$. If not, let g be an r-VDIET coloring of $K_{m,n}$ using colors $1, 2, \ldots, r$. First we give four claims as follows.

Claim 9. $|C(v_j)| \geq 2, j = 1, 2, \ldots, n$.

Proof. Suppose the claim is not true, without loss of generality, we assume $C(v_1) = \{1\}$. Then $1 \in C(u_i), i = 1, 2, \ldots, m$. Let $B_0 = \{g(u_1), g(u_2), \ldots, g(u_m)\}$, $B_i = \{1, 2, \ldots, r\} \setminus C(u_i), i = 1, 2, \ldots, m$. Note that $1 \notin B_0, 1 \notin B_i, i = 1, 2, \ldots, m$, we have $B_0, B_1, B_2, \ldots, B_m$ are distinct and not the color sets of vertices $u_1, u_2, \ldots, u_m$. Moreover, none of $B_0, B_1, B_2, \ldots, B_m$ is the color set of any vertex $v_j, j = 1, 2, \ldots, n$, (because $C(u_i) \cap C(v_j) = \emptyset, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, and two adjacent vertices must have different colors). At most one of $B_0, B_1, B_2, \ldots, B_m$ is an empty set, so there are at most $2^r - 1 - m$ nonempty subsets of $\{1, 2, \ldots, r\}$ which are available for the vertices $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$. However, $2^r - 1 - m < m + n$, i.e., these subsets cannot distinguish $m + n$ vertices, this is a contradiction. □

Claim 10. $|C(u_i)| \geq 2, i = 1, 2, \ldots, m$.

Proof. Suppose the claim is not true. Without loss of generality we assume $C(u_1) = \{1\}$. Then $1 \in C(v_j), j = 1, 2, \ldots, n$. Thus, $\overline{C}(v_1), \overline{C}(v_2), \ldots, \overline{C}(v_n)$ are not available for vertices $v_1, v_2, \ldots, v_n$. Moreover, $\overline{C}(v_1), \overline{C}(v_2), \ldots, \overline{C}(v_n)$ cannot be the color sets of vertices $u_1, u_2, \ldots, u_m$ because $C(u_i) \cap C(v_j) \neq \emptyset$. At most one of $\overline{C}(v_1), \overline{C}(v_2), \ldots, \overline{C}(v_n)$ is an empty set, so there are at most $2^r - 1 - (n - 1)$ nonempty subsets of $\{1, 2, \ldots, r\}$ which can be the color sets of vertices $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$. However, $2^r - 1 - (n-1) \leq 2^r - 1 - m < m + n$, these subsets cannot distinguish $m + n$ vertices, this is a contradiction. □

Claim 11. $C(u_1) \cap C(u_2) \cap \cdots \cap C(u_m) = \emptyset$.

Proof. Suppose $1 \in C(u_i), i = 1, 2, \ldots, m$. Then the $m + 1$ distinct subsets $\{1\}, \overline{C}(u_1), \overline{C}(u_2), \ldots, \overline{C}(u_m)$ are not available for any vertex, and at most one of them is an empty set. Then there are at most $2^r - 1 - m$ subsets of $\{1, 2, \ldots, r\}$ which can be the color sets of vertices $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$. However, $2^r - 1 - m < m + n$, so these subsets cannot distinguish $m + n$ vertices, this is a contradiction. □

Claim 12. $C(v_1) \cap C(v_2) \cap \cdots \cap C(v_n) = \emptyset$.

Proof. Suppose $1 \in C(v_j), j = 1, 2, \ldots, n$. Then the $n+1$ distinct subsets $\{1\}, \overline{C}(v_1), \overline{C}(v_2), \ldots, \overline{C}(v_n)$ are not available for any vertex, and at most one of them is an empty set. The remaining $2^r - 1 - n$ subsets of $\{1, 2, \ldots, r\}$ cannot distinguish $m+n$ vertices because $2^r - 1 - n \leq 2^r - 1 - m < m + n$, this is a contradiction. □

Now we consider two cases.

Case 1. $r = m, m+1$. By Claims 9 and 10, all 1-subsets of $\{1, 2, \ldots, r\}$ cannot be the color sets of any vertex. So there are at most $2^r - 1 - r \leq 2^r - m - 1 < m + n$ subsets of $\{1, 2, \ldots, r\}$ which are available for vertices $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$. This is a contradiction.

Case 2. $r = m-1$. By Claims 9 and 10, all the 1-subsets $\{1\}, \{2\}, \ldots, \{m-1\}$ cannot be the color sets of any vertex. The remaining $2^{m-1} - 1 - (m-1) = 2^{m-1} - m$ subsets of $\{1, 2, \ldots, m-1\}$ cannot distinguish $m+n$ vertices when $2^{m-1} - 2m < n \leq 2^{m-1} - m - 1$, this is a contradiction, so $K_{m,n}$ does not have an $(m-1)$-VDIET coloring when $2^{m-1} - 2m < n \leq 2^{m-1} - m - 1$.

Now we consider the case $n = 2^{m-1} - 2m$. Let $t = |\{g(u_1), g(u_2), \ldots, g(u_m)\}|$. Without loss of generality we assume $\{g(u_1), g(u_2), \ldots, g(u_m)\} = \{1, 2, \ldots, t\}$. By Claims 11 and 12 we know that $2 \leq t \leq r-2$, thus if $r \leq 3$, this is a contradiction. So $r \geq 4$. None of 2-subsets of $\{1, 2, \ldots, t\}$ is available for $v_1, v_2, \ldots, v_n$.

If $\{1, 2\} \notin \{C(u_1), C(u_2), \ldots, C(u_m)\}$, then at most $2^{m-1} - 1 - m < m + n$ subsets of $\{1, 2, \ldots, m-1\}$ are available for vertices $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$, this is a contradiction.

Therefore, $\{1, 2\} \in \{C(u_1), C(u_2), \ldots, C(u_m)\}$. Without loss of generality, assume $C(u_1) = \{1, 2\}$. By Claim 12, there are at least two colors among $v_1, v_2, \ldots, v_n$, say $t+1, t+2$. Then $\{t+1, t+2\} \notin \{C(u_1), C(u_2), \ldots, C(u_m)\}$. If $\{t+1, t+2\} \notin \{C(v_1), C(v_2), \ldots, C(v_n)\}$, then at most $2^{m-1} - 1 - m < m + n$ subsets of $\{1, 2, \ldots, m-1\}$ are available for vertices $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$, this is a contradiction. Thus $\{t+1, t+2\} \in \{C(v_1), C(v_2), \ldots, C(v_n)\}$. Then $t+1 \in C(u_i)$ or $t+2 \in C(u_i)$, $i = 1, 2, \ldots, m$. However, $C(u_1) = \{1, 2\}$, this is a contradiction.

So, $K_{m,n}$ does not have an r-VDIET coloring when $2^{m-1} - 2m \leq n \leq 2^{m-1} - m - 1$ and $r = m+1, m, m-1$.

In the following we give an $(r+1)$-VDIET coloring of $K_{m,n}$ using colors $1, 2, \ldots, r, r+1$, where $r = m-1, m, m+1$.

Let $V(K_{m,n}) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$ and $E(K_{m,n}) = \{u_i v_j : i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$.

Put $D(u_i) = \{1, 2, \ldots, r+1\} \setminus \{i\}, i = 1, 2, \ldots, m-1$, $D(u_m) = \{1, 2, \ldots, r+1\}$; $D(v_j) = \{j, r+1\}, j = 1, 2, \ldots, m-1$.

When $r = m+1$, put $D(v_{r-1}) = \{r-1\}, D(v_r) = \{r\}$. When $r = m$, put $D(v_r) = \{r\}$.

Now distribute other subsets of $\{1, 2, \ldots, r+1\}$ with cardinality between 2 and r to vertices $v_{r+1}, v_{r+2}, \ldots, v_n$. These $n-r$ subsets are denoted by $D(v_{r+1}), D(v_{r+2}), \ldots, D(v_n)$, respectively.

Construct a mapping f from $V(K_{m,n}) \cup E(K_{m,n})$ to $\{1, 2, \ldots, r+1\}$ as follows: Put $f(u_i) = r+1, i = 1, 2, \ldots, m$, $f(v_j) = \min D(v_j), j = 1, 2, \ldots, n$, $f(u_i v_i) = r+1$ for $i = 1, 2, \ldots, m-1$, $f(u_i v_j) = j, i = 1, 2, \ldots, m, j = 1, 2, \ldots, m-1, i \neq j$, $f(u_i v_j) = j, i = 1, 2, \ldots, m, j = m, \ldots, r$ (if $r = m$ or $m+1$).

For each $j = r+1, r+2, \ldots, n$, we recursively let $f(u_1 v_j) = \min\big(D(u_1) \cap (D(v_j) \setminus \{f(v_j)\})\big)$ or $f(u_1 v_j) \in D(u_1) \cap D(v_j)$ when $D(u_1) \cap (D(v_j) \setminus \{f(v_j)\}) = \emptyset$.

When $2 \leq i \leq m$, $f(u_i v_j) = \min\big(D(u_i) \cap (D(v_j) \setminus \{f(v_j), f(u_1 v_j), f(u_2 v_j), \ldots, f(u_{i-1} v_j)\})\big)$ or $f(u_i v_j) \in D(u_i) \cap D(v_j)$ when $D(u_i) \cap (D(v_j) \setminus \{f(v_j), f(u_1 v_j), f(u_2 v_j), \ldots, f(u_{i-1} v_j)\}) = \emptyset$.

It is not hard to see that $C_f(u_i) = D(u_i), i = 1, 2, \ldots, m; C_f(v_j) = D(v_j), j = 1, 2, \ldots, n$ and moreover $f(u_i) > f(v_j)$, therefore our coloring f is a vertex distinguishing IE-total coloring and then $\chi_{vt}^{ie}(K_{m,n}) \leq r+1, r = m-1, m, m+1$.

So $\chi_{vt}^{ie}(K_{m,n}) = r+1, r = m-1, m, m+1$. ∎

Theorem 13. $\chi_{vt}^{ie}(K_{1,n}) = \begin{cases} 2, & when\ n = 1; \\ 3, & when\ n = 2; \\ k, & when\ \binom{k-1}{1} + \binom{k-1}{2} - 1 < n \leq \binom{k}{1} + \binom{k}{2} - 1, \\ & \qquad k \geq 3. \end{cases}$

Proof. It is easy to prove the theorem in the case $n = 1, 2$. By Theorem 5, this theorem is valid when $\binom{k-1}{1} + \binom{k-1}{2} - 1 < n \leq \binom{k}{1} + \binom{k}{2} - 1, k \geq 3$. ∎

Theorem 14. $\chi_{vt}^{ie}(K_{2,n}) = \begin{cases} 3, & when\ n = 2, 3; \\ 4, & when\ n = 4, 5, \ldots, 11; \\ 5, & when\ n = 12; \\ k, & when\ \binom{k-1}{1} + \binom{k-1}{2} + \binom{k-1}{3} - 2 < n \\ & \qquad \leq \binom{k}{1} + \binom{k}{2} + \binom{k}{3} - 2, k \geq 5. \end{cases}$

Proof. By Theorem 5, 6, 8 respectively we know the theorem is valid in each case when $n \geq 4$. Now we consider the case $n = 2, 3$. It is obvious that $\chi_{vt}^{ie}(K_{2,n}) \geq \xi(K_{2,n}) = 3$ when $n = 2, 3$. Let $V(K_{2,n}) = \{u_1, u_2, v_1, v_2, \ldots, v_n\}$ and $E(K_{2,n}) = \{u_i v_j : 1 \leq i \leq 2, 1 \leq j \leq n\}$. We give a 3-VDIET coloring of $K_{2,n}$ using colors 1, 2, 3 when $n = 2, 3$.

Let u_1, u_2 receive color 1, v_1 and its incident edges receive color 2. We assign color 3, 3, 1 to $v_2, u_1 v_2, u_2 v_2$, respectively. And when $n = 3$, we assign color 2, 3, 2 to $v_3, u_1 v_3, u_2 v_3$, respectively.

Then under the above coloring, we have $C(u_1) = \{1,2,3\}$, $C(u_2) = \{1,2\}$, $C(v_1) = \{2\}$, $C(v_2) = \{1,3\}$ and $C(v_3) = \{2,3\}$ (when $n = 3$). Thus the above coloring is a VDIET coloring of $K_{2,n}(n = 2,3)$ using 3 colors. ■

Theorem 15. $\chi_{vt}^{ie}(K_{3,n}) = \begin{cases} 4, & \textit{when } 3 \le n \le 9; \\ 5, & \textit{when } 10 \le n \le 25; \\ 6, & \textit{when } n = 26, 27; \\ k, & \textit{when } \binom{k-1}{1} + \cdots + \binom{k-1}{4} - 3 < n \\ & \quad \le \binom{k}{1} + \cdots + \binom{k}{4} - 3, k \ge 6. \end{cases}$

Proof. By Theorem 5, 6, 8 respectively we know the theorem is valid in each case when $n \ge 10$. Now we consider the case $3 \le n \le 9$.

$\xi(K_{3,n}) = \begin{cases} 3, & \text{when } n = 3,4; \\ 4, & \text{when } 5 \le n \le 9. \end{cases}$

Let $V(K_{3,n}) = \{u_1, u_2, u_3, v_1, v_2, \ldots, v_n\}$ and $E(K_{3,n}) = \{u_iv_j : 1 \le i \le 3, 1 \le j \le n\}$. We prove $K_{3,n}$ does not have a 3-VDIET coloring when $n = 3,4$. If not, let g be a 3-VDIET coloring of $K_{3,n}$ using colors $1,2,3$. Then $|C(u_i)| \ge 2, i = 1,2,3$. (Otherwise we assume $C(u_1) = \{1\}$. Then $1 \in C(v_j), j = 1,2,\ldots,n$. Thus $\overline{C}(v_1), \overline{C}(v_2), \ldots, \overline{C}(v_n)$ are not available for any vertex and at most one of them is an empty set. Therefore there are at most $2^3 - 1 - 2 = 5$ nonempty subsets of $\{1,2,3\}$ which can be the color sets of vertices $u_1, u_2, u_3, v_1, v_2, \ldots, v_n$. Five subsets cannot distinguish $n+3$ vertices when $n = 3,4$, this is a contradiction). Furthermore, $|C(v_j)| \ge 2, j = 1,2,\ldots,n$. (Otherwise we assume $C(v_1) = \{1\}$. Then $1 \in C(u_i), i = 1,2,3$. Thus $\overline{C}(u_1), \overline{C}(u_2), \overline{C}(u_3)$ are not available for any vertex and at most one of them is an empty set. Therefore there are at most $2^3 - 1 - 2 = 5$ nonempty subsets of $\{1,2,3\}$ which can be the color sets of vertices $u_1, u_2, u_3, v_1, v_2, \ldots, v_n$. Five subsets cannot distinguish $n+3$ vertices when $n = 3,4$, this is a contradiction.) So three 1-subsets of $\{1,2,3\}$ are not available for any vertx, the remaining 4 nonempty subsets of $\{1,2,3\}$ cannot distinguish $n+3$ vertices when $n = 3,4$, this is a contradiction. Therefore, $\chi_{vt}^{ie}(K_{3,n}) \ge 4$ when $n = 3,4$.

In the following we give a 4-VDIET coloring of $K_{3,n}$ using colors $1,2,3,4$ when $3 \le n \le 9$.

Let u_1, u_2, u_3 receive color 4. Suppose $\mathcal{S}_1 = (\{3\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\})$ and let $D(v_i)$ be the i-th term of $\mathcal{S}_1$, $i = 1,2,\ldots,n$. Let v_1 and its incident edges receive color 3, let v_2, u_3v_2 receive color 1 and u_1v_2, u_2v_2 receive color 2.

For $D(v_j) = \{a,b\}$, $3 \le j \le n$, $a < b$, we assign a to u_1v_j and v_j, assign b to u_2v_j and u_3v_j.

For $D(v_j) = \{a,b,c\}$, $a < b < c$, we assign a, b, c to u_1v_j, u_2v_j, u_3v_j respectively and assign b to v_j.

Then $C(u_1) = \{1,2,3,4\}$, $C(u_2) = \{2,3,4\}$, $C(u_3) = \{1,3,4\}$ and $C(v_j) = D(v_j), j = 1,2,\ldots,n$ with respect to the above coloring. Thus the above coloring is a VDIET coloring of $K_{3,n}(3 \le n \le 9)$ using 4 colors. ■

Theorem 16. $\chi^{ie}_{vt}(K_{4,n}) = \begin{cases} 4, & when\ 4 \le n \le 7; \\ 5, & when\ 8 \le n \le 23; \\ 6, & when\ 24 \le n \le 55; \\ 7, & when\ 56 \le n \le 58; \\ k, & when\ \binom{k-1}{1} + \cdots + \binom{k-1}{5} - 4 < n \\ & \qquad \le \binom{k}{1} + \cdots + \binom{k}{5} - 4, k \ge 7. \end{cases}$

Proof. It is easy to verify the theorem is valid in each case when $n \ge 8$ by Theorem 5, 6, 8 respectively. Now we consider the case $4 \le n \le 7$.

It is obvious $\chi^{ie}_{vt}(K_{4,n}) \ge \xi(K_{4,n}) = 4$, when $4 \le n \le 7$.

In the following we give a 4-VDIET coloring of $K_{4,n}$ using colors $1,2,3,4$ when $4 \le n \le 7$. Let $V(K_{4,n}) = \{u_1,u_2,u_3,u_4,v_1,v_2,\ldots,v_n\}$ and $E(K_{4,n}) = \{u_iv_j : i = 1,2,3,4; j = 1,2,\ldots,n\}$.

Let u_1,u_2,u_3,u_4 receive color 4. Suppose $\mathcal{S}_2 = (\{1,4\},\{2,4\},\{3,4\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\})$ and let $D(v_i)$ be the i-th term of $\mathcal{S}_2$, $i = 1,2,\ldots,n$. Let v_i receive the minimum number of $D(v_i)$, $i = 1,2,\ldots,n$.

For $D(v_j) = \{j,4\}$, $j = 1,2,3$, we assign color 4 to u_jv_j and color j to $u_iv_j, i = 1,2,3,4, i \ne j$.

For $D(v_j) = \{a,b\}$, $4 \le j \le n$, $a < b$, we assign color b to all edges u_iv_j if $i \ne b$ and color a to its remaining incident edge u_bv_j.

For $D(v_j) = \{1,2,3\}$, we assign color 2 to u_iv_j if $i \ne 2$ and assign color 3 to u_2v_j.

Then $C(u_i) = \{1,2,3,4\} \setminus \{i\}, i = 1,2,3$, $C(u_4) = \{1,2,3,4\}$ and $C(v_j) = D(v_j), j = 1,2,\ldots,n$ with respect to the above coloring. Thus the above coloring is a 4-VDIET coloring of $K_{4,n}, 4 \le n \le 7$. ■

Theorem 17. $\chi^{ie}_{vt}(K_{5,n}) = \begin{cases} 5, & when\ 6 \le n \le 21; \\ 6, & when\ 22 \le n \le 53; \\ 7, & when\ 54 \le n \le 117; \\ 8, & when\ 118 \le n \le 121; \\ k, & when\ \binom{k-1}{1} + \cdots + \binom{k-1}{6} - 5 < n \\ & \qquad \le \binom{k}{1} + \cdots + \binom{k}{6} - 5, k \ge 8. \end{cases}$

Proof. By Theorem 5, 6, 8 respectively we know the theorem is valid in each case. ■

Theorem 18. $\chi^{ie}_{vt}(K_{6,n}) = \begin{cases} 5, & \textit{when } 6 \le n \le 19; \\ 6, & \textit{when } 20 \le n \le 51; \\ 7, & \textit{when } 52 \le n \le 115; \\ 8, & \textit{when } 116 \le n \le 243; \\ 9, & \textit{when } 244 \le n \le 248; \\ k, & \textit{when } \binom{k-1}{1} + \cdots + \binom{k-1}{7} - 6 < n \\ & \qquad \le \binom{k}{1} + \cdots + \binom{k}{7} - 6, k \ge 9. \end{cases}$

Proof. By Theorem 5, 6, 8 respectively we know the theorem is valid in each case when $n \ge 20$. Now we consider the case $6 \le n \le 19$.

$\xi(K_{6,n}) = \begin{cases} 4, & \text{when } 6 \le n \le 9; \\ 5, & \text{when } 10 \le n \le 19. \end{cases}$

Let $V(K_{6,n}) = \{u_1, u_2, \ldots, u_6, v_1, v_2, \ldots, v_n\}$ and $E(K_{6,n}) = \{u_iv_j : 1 \le i \le 6, 1 \le j \le n\}$. We prove $K_{6,n}$ does not have a 4-VDIET coloring when $6 \le n \le 9$. If not, suppose g is a 4-VDIET coloring of $K_{6,n}$ ($6 \le n \le 9$) using colors 1, 2, 3, 4. Then $|C(u_i)| \ge 2, i = 1, 2, \ldots, 6$. (Otherwise we assume $C(u_1) = \{1\}$. Then $1 \in C(v_j), j = 1, 2, \ldots, n$. Thus $\overline{C}(v_1), \overline{C}(v_2), \ldots, \overline{C}(v_n)$ are not available for any vertex and at most one of them is an empty set. Therefore there are at most $2^4 - 1 - 5 = 10$ nonempty subsets of $\{1, 2, 3, 4\}$ which can be the color sets of vertices $u_1, u_2, \ldots, u_6, v_1, v_2, \ldots, v_n$. These subsets cannot distinguish $n + 6$ vertices when $6 \le n \le 9$, this is a contradiction.)

Furthermore, $|C(v_j)| \ge 2, j = 1, 2, \ldots, n$. (Otherwise we assume $C(v_1) = \{1\}$, then $1 \in C(u_i), i = 1, 2, \ldots, 6$. Thus $\overline{C}(u_1), \overline{C}(u_2), \ldots, \overline{C}(u_6)$ are not available for any vertex and at most one of them is an empty set. Therefore there are at most $2^4 - 1 - 5 = 10$ nonempty subsets of $\{1, 2, 3, 4\}$ which can be the color sets of vertices $u_1, u_2, \ldots, u_6, v_1, v_2, \ldots, v_n$. These subsets cannot distinguish $n + 6$ vertices when $6 \le n \le 9$, this is a contradiction.) So four 1-subsets of $\{1, 2, 3, 4\}$ are not available for any vertex, the remaining 11 nonempty subsets of $\{1, 2, 3, 4\}$ cannot distinguish $n + 6$ vertices when $6 \le n \le 9$, this is a contradiction. Therefore, $\chi^{ie}_{vt}(K_{6,n}) \ge 5$ when $6 \le n \le 9$.

In the following we give a 5-VDIET coloring of $K_{6,n}$ using colors $1, 2, 3, 4, 5$ when $6 \le n \le 19$.

Let $u_1, u_2, \ldots, u_6$ receive color 5. Suppose $\mathcal{S}_3 = (\{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{1, 2, 3, 4\})$ and let $D(v_i)$ be the i-th term of $\mathcal{S}_3$, $i = 1, 2, \ldots, n$. Let $D(u_i) = \{1, 2, 3, 4, 5\} \setminus \{i\}, i = 1, 2, 3, 4$, $D(u_5) = \{1, 2, 3, 4, 5\}$ and $D(u_6) = \{1, 2, 5\}$.

Let $u_iv_i (i = 1, 2, 3, 4), u_6v_3$ and u_6v_4 receive color 5. Let v_j and the other incident edges of v_j receive color $j, j = 1, 2, 3, 4$.

For $D(v_j) = \{a, b\}$, $5 \le j \le n$, $a < b$, we assign b to u_iv_j if $b \in D(u_i)$, assign a to v_j and its remaining incident edges.

For $D(v_j)=\{a,b,c\},\{b,c\}\neq\{3,4\}$, $a<b<c$, we assign b to u_iv_j if $b\in D(u_i)$, assign c to u_iv_j if $b\notin D(u_i)$, and assign a to v_j.

For $D(v_j)=\{a,3,4\}$, $a=1,2$, we assign a to u_iv_j if $a\in D(u_i)$, assign 3 to u_iv_j if $a\notin D(u_i)$, and assign 4 to v_j.

For $D(v_j)=\{1,2,3,4\}$, we assign 3 to u_iv_j if $3\in D(u_i)$, assign 4 to u_3v_j, assign 2 to u_6v_j, and assign 1 to v_j.

Then $C(u_i)=D(u_i), 1\leq i\leq 6$ and $C(v_j)=D(v_j), 1\leq j\leq n$ with respect to the above coloring. Thus the above coloring is a 5-VDIET coloring of $K_{6,n}, 6\leq n\leq 19$. ■

Theorem 19. $\chi_{vt}^{ie}(K_{7,n})=\begin{cases}5, & \textit{when } 7\leq n\leq 17;\\ 6, & \textit{when } 18\leq n\leq 49;\\ 7, & \textit{when } 50\leq n\leq 113;\\ 8, & \textit{when } 114\leq n\leq 241;\\ 9, & \textit{when } 242\leq n\leq 497;\\ 10, & \textit{when } 498\leq n\leq 503;\\ k, & \textit{when } \binom{k-1}{1}+\cdots+\binom{k-1}{8}-7<n\\ & \qquad\leq\binom{k}{1}+\cdots+\binom{k}{8}-7, k\geq 10.\end{cases}$

Proof. By Theorem 5, 6, 8 respectively we know the theorem is valid in each case when $n\geq 50$. Now we consider the case $n\leq 49$.

$$\xi(K_{7,n})=\begin{cases}4, & \text{when } n=7,8;\\ 5, & \text{when } 9\leq n\leq 24;\\ 6, & \text{when } 25\leq n\leq 49.\end{cases}$$

Let $V(K_{7,n})=\{u_1,u_2,\ldots,u_7,v_1,v_2,\ldots,v_n\}$ and $E(K_{7,n})=\{u_iv_j:1\leq i\leq 7, 1\leq j\leq n\}$.

We prove $K_{7,n}$ does not have a 4-VDIET coloring when $n=7,8$. If not, suppose g is a 4-VDIET coloring of $K_{7,n}(n=7,8)$ using colors $1,2,3,4$. Then $|C(u_i)|\geq 2, i=1,2,\ldots,7$. Otherwise we assume $C(u_1)=\{1\}$. Then $1\in C(v_j), j=1,2,\ldots,n, n=7,8$. Thus $\overline{C}(v_1),\overline{C}(v_2),\ldots,\overline{C}(v_n)$ are not available for any vertex and at most one of them is an empty set. Therefore there are at most $2^4-1-6=9$ nonempty subsets of $\{1,2,3,4\}$ which can be the color sets of vertices $u_1,u_2,\ldots,u_7,v_1,v_2,\ldots,v_n$. These subsets cannot distinguish 14 or 15 vertices, this is a contradiction.

Furthermore, $|C(v_j)|\geq 2, j=1,2,\ldots,n, n=7,8$. Otherwise we assume $C(v_1)=\{1\}$. Then $1\in C(u_i), i=1,2,\ldots,7$. Thus $\overline{C}(u_1),\overline{C}(u_2),\ldots,\overline{C}(u_7)$ are not available for any vertex and at most one of them is an empty set. Therefore there are at most $2^4-1-6=9$ nonempty subsets of $\{1,2,3,4\}$ which can be the color sets of vertices $u_1,u_2,\ldots,u_7,v_1,v_2,\ldots,v_n$. These subsets cannot distinguish 14 or 15 vertices, this is also a contradiction.) So four 1-subsets of $\{1,2,3,4\}$ are not available for any vertex, the remaining 11 nonempty subsets of

$\{1,2,3,4\}$ cannot distinguish 14 or 15 vertices, this is a contradiction. Therefore, $\chi_{vt}^{ie}(K_{7,n}) \geq 5$ when $n = 7, 8$.

In the following we give a 5-VDIET coloring of $K_{7,n}$ using colors $1, 2, 3, 4, 5$ when $7 \leq n \leq 17$.

Let $u_1, u_2, \ldots, u_7$ receive color 5. Suppose $\mathcal{S}_4 = (\{1,5\}, \{2,5\}, \{3,5\}, \{4,5\}, \{1,2\}, \{1,4\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{3,4,5\}, \{1,2,3,4\})$ and let $D(v_i)$ be the i-th term of $\mathcal{S}_4, i = 1, 2, \ldots, n$. Let $D(u_i) = \{1,2,3,4,5\} \setminus \{i\}, i = 1,2,3,4,$ $D(u_5) = \{1,3,5\}, D(u_6) = \{2,4,5\}$ and $D(u_7) = \{1,2,3,4,5\}$.

Let u_1v_1 and u_6v_1 receive color 5, v_1 and its other incident edges receive color 1. Let u_2v_2 and u_5v_2 receive color 5, v_2 and its other incident edges receive color 2. Let u_3v_3 and u_6v_3 receive color 5, v_3 and its other incident edges receive color 3. Let u_4v_4 and u_5v_4 receive color 5, v_4 and its other incident edges receive color 4.

For $D(v_j) = \{a, b\}, 5 \leq j \leq n$, $a < b$, we assign b to u_iv_j if $b \in D(u_i)$, assign a to v_j and its remaining incident edges.

For $D(v_j) = \{a,b,c\}, \{a,b,c\} \neq \{1,2,4\}$, $a < b < c$, we assign b to u_iv_j if $b \in D(u_i)$, assign c to u_iv_j if $b \notin D(u_i)$, and assign a to v_j.

For $D(v_j) = \{1,2,4\}$, we assign 1 to u_iv_j if $1 \in D(u_i)$, assign 2 to u_iv_j if $1 \notin D(u_i)$, and assign 4 to v_j.

For $D(v_j) = \{1,2,3,4\}$, we assign 2 to u_iv_j if $2 \in D(u_i)$, assign 4, 3, 1 to u_2v_j, u_5v_j and v_j respectively.

Then $C(u_i) = D(u_i), 1 \leq i \leq 7$ and $C(v_j) = D(v_j), j = 1, 2, \ldots, n$ with respect to the above coloring. Thus the above coloring is a 5-VDIET coloring of $K_{7,n}, 7 \leq n \leq 17$.

We prove $K_{7,n}$ does not have a 5-VDIET coloring when $18 \leq n \leq 24$. If not, suppose g is a 5-VDIET coloring of $K_{7,n}(18 \leq n \leq 24)$ using colors $1, 2, 3, 4, 5$. First we give four claims as follows.

Claim 20. $|C(u_i)| \geq 2, i = 1, 2, \ldots, 7.$

Proof. Suppose the claim is not true, without loss of generality we assume $C(u_1) = \{1\}$. Then $1 \in C(v_j), j = 1, 2, \ldots, n, 18 \leq n \leq 24$. Thus $\overline{C}(v_1), \overline{C}(v_2), \ldots, \overline{C}(v_n)$ are not available for any vertex and at most one of them is an empty set. Therefore there are at most $2^5 - 1 - 17 = 14$ nonempty subsets of $\{1,2,3,4,5\}$ which can be the color sets of vertices $u_1, u_2, \ldots, u_7, v_1, v_2, \ldots, v_n$. These subsets cannot distinguish $n + 7$ vertices when $18 \leq n \leq 24$, this is a contradiction. □

Claim 21. $|C(v_j)| \geq 2, j = 1, 2, \ldots, n, 18 \leq n \leq 24.$

Proof. Suppose the claim is not true, without loss of generality we assume $C(v_1) = \{1\}$. Then $1 \in C(u_i), i = 1, 2, \ldots, 7$. Thus $\overline{C}(u_1), \overline{C}(u_2), \ldots, \overline{C}(u_7)$, $\{g(u_1), g(u_2), \ldots, g(u_7)\}$ are not available for any vertex and at most one of them is an empty set. Therefore there are at most $2^5 - 1 - 7 = 24$ nonempty subsets of $\{1, 2, 3, 4, 5\}$ which can be the color sets of vertices $u_1, u_2, \ldots, u_7, v_1, v_2, \ldots, v_n$. These subsets cannot distinguish $n + 7$ vertices when $18 \leq n \leq 24$, this is also a contradiction. □

Claim 22. *$C(u_1) \cap C(u_2) \cap \cdots \cap C(u_7) = \emptyset$.*

Claim 23. *$C(v_1) \cap C(v_2) \cap \cdots \cap C(v_n) = \emptyset, 18 \leq n \leq 24$.*

The proofs of Claim 22 and Claim 23 are analogous to the proofs of Claim 11 and Claim 12 in Theorem 8, respectively.

By Claims 20 and 21, five 1-subsets of $\{1, 2, 3, 4, 5\}$ are not available for any vertex. The remaining 26 nonempty subsets of $\{1, 2, 3, 4, 5\}$ cannot distinguish $n+7$ vertices when $20 \leq n \leq 24$, this is a contradiction. So we assume $n = 18, 19$ in the following.

Let $t = |\{g(u_1), g(u_2), \ldots, g(u_7)\}|$, and $\{g(u_1), g(u_2), \ldots, g(u_7)\} = \{1, 2, \ldots, t\}$, by Claim 22 and Claim 23, we know that $t = 2$ or $t = 3$.

Case 1. $t = 2$, $\{f(u_1), f(u_2), \ldots, f(u_7)\} = \{1, 2\}$. Of course $\{1, 2\} \notin \{C(v_1), C(v_2), \ldots, C(v_n)\}$. If $\{1, 2\} \in \{C(u_1), C(u_2), \ldots, C(u_7)\}$, then $1 \in C(v_j)$ or $2 \in C(v_j), j = 1, 2, \ldots, n$. Thus $\{3, 4\}, \{3, 5\}, \{4, 5\}$, $\{3, 4, 5\}$ cannot be the color sets of any vertices. Moreover, five 1-subsets are not available for any vertex. Then at most $2^5 - 1 - 5 - 4 = 22$ nonempty subsets of $\{1, 2, 3, 4, 5\}$ are available for the vertices $u_1, u_2, \ldots, u_7, v_1, v_2, \ldots, v_n$. This is a contradiction because 22 subsets cannot distinguish 25 (when $n = 18$) or 26 (when $n = 19$) vertices. So $\{1, 2\}$ is not available for any vertex.

If $|C(u_i)| \geq 3, i = 1, 2, \ldots, 7$, then $\overline{C}(u_1), \overline{C}(u_2), \ldots, \overline{C}(u_7)$ cannot be the color sets of any vertices because there are 5 colors in all. At most one of $\overline{C}(u_1), \overline{C}(u_2), \ldots, \overline{C}(u_7)$ is an empty set, so there are at most $2^5 - 1 - 6 - 1 = 24$ nonempty subsets of $\{1, 2, 3, 4, 5\}$ are available for the vertices $u_1, u_2, \ldots, u_7, v_1, v_2, \ldots, v_n$. This is a contradiction because 24 subsets cannot distinguish 25 (when $n = 18$) or 26 (when $n = 19$) vertices.

Therefore, there exists a vertex u_{i_0} with $|C(u_{i_0})| = 2$. Since $\{1, 2\}$ is not available for any vertex, so without loss of generality, we assume $C(u_{i_0}) = \{1, 3\}$, then $1 \in C(v_j)$ or $3 \in C(v_j), j = 1, 2, \ldots, n$. Thus $\{4, 5\}$ is not available for any vertex. Furthermore, $\{1, 2\}$ and five 1-subsets are not available for any vertex. There are at most $2^5 - 1 - 5 - 2 = 24$ nonempty subsets of $\{1, 2, 3, 4, 5\}$ are available for the vertices $u_1, u_2, \ldots, u_7, v_1, v_2, \ldots, v_n$. This is a contradiction because 24 subsets cannot distinguish 25 (when $n = 18$) or 26 (when $n = 19$) vertices.

So $K_{7,n} (n = 18, 19)$ does not have a 5-VDIET coloring in this case.

Case 2. $t = 3$, $\{f(u_1), f(u_2), \ldots, f(u_7)\} = \{1, 2, 3\}$. By Claim 23, $|\{f(v_1), f(v_2), \ldots, f(v_n)\}| \geq 2$, so $\{f(v_1), f(v_2), \ldots, f(v_n)\} = \{4, 5\}$. Then $\{4, 5\}$ is not the color set of any vertex $u_i, i = 1, 2, \ldots, 7$. If $\{4, 5\} \in \{C(v_1), C(v_2), \ldots, C(v_n)\}$, then $4 \in C(u_i)$ or $5 \in C(u_i), i = 1, 2, \ldots, 7$. Thus $\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$ cannot be the color sets of any vertex. Moreover, five 1-subsets are not available for any vertex. Then at most $2^5 - 1 - 5 - 4 = 22$ nonempty subsets of $\{1, 2, 3, 4, 5\}$ are available for the vertices $u_1, u_2, \ldots, u_7, v_1, v_2, \ldots, v_n$. This is a contradiction because 22 subsets cannot distinguish 25 (when $n = 18$) or 26 (when $n = 19$) vertices. So $\{4, 5\}$ is not available for any vertex.

If $|C(v_j)| \geq 3, j = 1, 2, \ldots, n$, then $\overline{C}(v_1), \overline{C}(v_2), \ldots, \overline{C}(v_n)$ cannot be the color sets of any vertex because there are 5 colors in all. At most one of them is an empty set, so at most $2^5 - 1 - (n - 1) \leq 14$ nonempty subsets of $\{1, 2, 3, 4, 5\}$ are available for the vertices $u_1, u_2, \ldots, u_7, v_1, v_2, \ldots, v_n$. This is a contradiction because these subsets cannot distinguish 25 (when $n = 18$) or 26 (when $n = 19$) vertices.

Therefore, there exists a vertex v_{j_0} with $|C(v_{j_0})| = 2$. Since $\{4, 5\}$ is not available for any vertex, so without loss of generality, we assume $C(v_{j_0}) = \{1, 4\}$. Then $1 \in C(u_i)$ or $4 \in C(u_i), i = 1, 2, \ldots, 7$. Thus $\{2, 3\}$ is not available for any vertex. Moreover, $\{4, 5\}$ and five 1-subsets are not available for any vertex. There are at most $2^5 - 1 - 5 - 2 = 24$ nonempty subsets are available for the vertices $u_1, u_2, \ldots, u_7, v_1, v_2, \ldots, v_n$. This is a contradiction because 24 subsets cannot distinguish 25 (when $n = 18$) or 26 (when $n = 19$) vertices.

So $K_{7,n}$ $(n = 18, 19)$ does not have a 5-VDIET coloring.

Therefore, $\chi_{vt}^{ie}(K_{7,n}) \geq 6$ when $18 \leq n \leq 49$.

In the following we give a 6-VDIET coloring of $K_{7,n}$ using colors $1, 2, 3, 4, 5, 6$ when $18 \leq n \leq 49$.

Arrange all 49 subsets of $\{1, 2, 3, 4, 5, 6\}$ except for $\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}$, $\{6\}$, $\{4, 5\}$, $\{2, 3, 4, 5, 6\}$, $\{1, 3, 4, 5, 6\}$, $\{1, 2, 4, 5, 6\}$, $\{1, 2, 3, 5, 6\}$, $\{1, 2, 3, 4, 6\}$, $\{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 6\}$ into a sequence $\mathcal{S}_5$ such that the first 5 terms are $\{1, 6\}, \{2, 6\}, \{3, 6\}, \{4, 6\}, \{5, 6\}$ respectively. Let $D(v_j)$ be the j-th term of $\mathcal{S}_5$, $j = 1, 2, \ldots, n$. Let $D(u_i) = \{1, 2, 3, 4, 5, 6\} \setminus \{i\}, i = 1, 2, 3, 4, 5$, $D(u_6) = \{1, 2, 3, 4, 5, 6\}, D(u_7) = \{1, 2, 3, 6\}$.

Let $u_1, u_2, \ldots, u_7$ receive color 6. Let v_j receive color $j, j = 1, 2, \ldots, 5$. Let $u_i v_i$ receive color 6, $i = 1, 2, \ldots, 5$. Let $u_i v_j$ receive color j, $i = 1, 2, \ldots, 6, j = 1, 2, \ldots, 5, i \neq j$. Let $u_7 v_1, u_7 v_2, u_7 v_3, u_7 v_4$ and $u_7 v_5$ receive colors 1, 2, 3, 6 and 6 respectively.

For $D(v_j) = \{a, b\}$, $6 \leq j \leq n$, $a < b$, we assign b to $u_i v_j$ if $b \in D(u_i)$, assign a to v_j and its remaining incident edges.

For $D(v_j) = \{a, 4, 5\}$, $1 \leq a \leq 3$, we assign 5 to v_j, a to $u_i v_j$ if $a \in D(u_i)$, assign 4 to $u_i v_j$ otherwise.

For $D(v_j) = \{a, b, c\}$, $a < b < c, \{b, c\} \neq \{4, 5\}$, we assign a to v_j, b to $u_i v_j$ if $b \in D(u_i)$, assign c to $u_i v_j$ otherwise.

For $D(v_j) = \{a, b, c, d\}$, $a < b < c < d$, we assign a to v_j, b to $u_i v_j$ if

$b \in D(u_i), i \neq 6$, assign c to u_iv_j if $b \notin D(u_i), c \in D(u_i), i \neq 6$, and assign d to the remaining incident edges of v_j.

For $D(v_j) = \{1,2,3,4,5\}$, we assign 1 to v_j, assign 2, 3, 4, 5 to u_3v_j, u_4v_j, u_5v_j, u_6v_j respectively and assign 3 to the remaining incident edges of v_j.

Then $C(u_i) = D(u_i), 1 \leq i \leq 7$ and $C(v_j) = D(v_j), j = 1,2,\ldots,n$ with respect to the above coloring. Thus the above coloring is a 6-VDIET coloring of $K_{7,n}, 24 \leq n \leq 49$. ∎

Theorem 24. *Let K_n be the complete graph of order $n(n \geq 3)$. Then $\chi_{vt}^{ie}(K_n) = n$.*

Proof. As any two vertices in K_n must receive different colors under an arbitrary VDIET coloring, therefore $\chi_{vt}^{ie}(K_n) \geq n$. Of course we may be able to show that $\chi_{vt}^{ie}(K_n) = n$ by giving a VDIET coloring of K_n using n colors $1,2,\ldots,n$ as follows. Assign colors $1,2,\ldots,n$ to vertices $v_1, v_2, \ldots, v_n$ of K_n respectively and then let all edges receive the same color 1. ∎

From the results obtained in this paper, we know that for any graph G discussed in this paper except $K_n(n \geq 6)$, we have $\chi_{vt}^{ie}(G) = \xi(G)$ or $\xi(G)+1$. So we propose the following conjectures.

Conjecture 25. *For a simple graph G, if its (proper vertex coloring) chromatic number $\chi(G) \leq 4$, then we have $\chi_{vt}^{ie}(G) = \xi(G)$ or $\xi(G)+1$.*

Conjecture 26. *For a simple graph G, we have $\chi_{vt}^{ie}(G) \leq \max\{\xi(G)+1, \chi(G)\}$.*

Conjecture 27. *Let s be the minimum positive integer such that $2^s - 1 \geq 3m$. When $2^r - 2m - 1 < n \leq 2^{r+1} - 2m - 1$, we have $\chi_{vt}^{ie}(K_{m,n}) = r+1$, where $r = s, s+1, \ldots, m-2, s \leq m-2$.*

Acknowledgement

The authors would like to thank the referees for their valuable comments and helpful suggestions.

References

[1] P.N. Balister, B. Bollobás and R.H. Shelp, *Vertex distinguishing colorings of graphs with $\Delta(G) = 2$*, Discrete Math. **252** (2002) 17–29. doi:10.1016/S0012-365X(01)00287-4

[2] P.N. Balister, O.M. Riordan and R.H. Schelp, *Vertex distinguishing edge colorings of graphs*, J. Graph Theory **42** (2003) 95–109. doi:10.1002/jgt.10076

[3] C. Bazgan, A. Harkat-Benhamdine, H. Li and M. Woźniak, *On the vertex-distinguishing proper edge-colorings of graphs*, J. Combin. Theory (B) **75** (1999) 288–301.
doi:10.1006/jctb.1998.1884

[4] A.C. Burris and R.H. Schelp, *Vertex-distinguishing proper edge-colorings*, J. Graph Theory **26** (1997) 73–82.
doi:10.1002/(SICI)1097-0118(199710)26:2⟨73::AID-JGT2⟩3.0.CO;2-C

[5] J. Černý, M. Horňák and R. Soták, *Observability of a graph*, Math. Slovaca **46** (1996) 21–31.

[6] X. Chen, *Asymptotic behaviour of the vertex-distinguishing total chromatic numbers of n-cube*, J. Northwest Univ. **41(5)** (2005) 1–3.

[7] F. Harary and M. Plantholt, *The point-distinguishing chromatic index*, in: F. Harary, J.S. Maybee (Eds.), Graphs and Application, New York (1985) 147–162.

[8] M. Horňák and R. Soták, *Observability of complete multipartite graphs with equipotent parts*, Ars Combin. **41** (1995) 289–301.

[9] M. Horňák and R. Soták, *Asymptotic behaviour of the observability of Q_n*, Discrete Math. **176** (1997) 139–148.
doi:10.1016/S0012-365X(96)00292-0

[10] M. Horňák and R. Soták, *The fifth jump of the point-distinguishing chromatic index of $K_{n,n}$*, Ars Combin. **42** (1996) 233–242.

[11] M. Horňák and R. Soták, *Localization jumps of the point-distinguishing chromatic index of $K_{n,n}$*, Discuss. Math. Graph Theory **17** (1997) 243–251.
doi:10.7151/dmgt.1051

[12] M. Horňák and N. Zagaglia Salvi, *On the point-distinguishing chromatic index of complete bipartite graphs*, Ars Combin. **80** (2006) 75–85.

[13] N. Zagaglia Salvi, *On the point-distinguishing chromatic index of $K_{n,n}$*, Ars Combin. **25B** (1988) 93–104.

[14] N. Zagaglia Salvi, *On the value of the point-distinguishing chromatic index of $K_{n,n}$*, Ars Combin. **29B** (1990) 235–244.

[15] Z. Zhang, P. Qiu, B. Xu, J. Li, X.Chen and B.Yao, *Vertex-distinguishing total colorings of graphs*, Ars Combin. **87** (2008) 33–45.

Received 11 October 2010
Revised 11 July 2011
Accepted 5 March 2012