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ON MINIMUM (K_q, k) STABLE GRAPHS

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Abstract

A graph G is a (K_q, k) stable graph ($q \geq 3$) if it contains a K_q after deleting any subset of k vertices ($k \geq 0$). Andrzej Żak in the paper *On* $(K_q; k)$*-stable graphs*, (doi:/10.1002/jgt.21705) has proved a conjecture of Dudek, Szymański and Zwonek stating that for sufficiently large k the number of edges of a minimum (K_q, k) stable graph is $(2q-3)(k+1)$ and that such a graph is isomorphic to $sK_{2q-2} + tK_{2q-3}$ where s and t are integers such that $s(q-1) + t(q-2) - 1 = k$. We have proved (Fouquet *et al. On* (K_q, k) *stable graphs with small* k, Elektron. J. Combin. **19** (2012) #P50) that for $q \geq 5$ and $k \leq \frac{q}{2} + 1$ the graph K_{q+k} is the unique minimum (K_q, k) stable graph. In the present paper we are interested in the $(K_q, \kappa(q))$ stable graphs of minimum size where $\kappa(q)$ is the maximum value for which for every nonnegative integer $k < \kappa(q)$ the only (K_q, k) stable graph of minimum size is K_{q+k} and by determining the exact value of $\kappa(q)$.

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1. Introduction

For terms not defined here we refer to [1]. As usual, the *order* of a graph G is the number of its vertices and the *size* of G is the number of its edges (denoted by $e(G)$). The disjoint union of two graphs G_1 and G_2 is denoted by $G_1 + G_2$. The union of p mutually disjoint copies of a graph G is denoted by pG. For any set A of vertices, we denote by $G[A]$ the subgraph induced by A and by $G - A$ the subgraph induced by $V(G) - A$. If $A = \{v\}$ we write $G - v$ for $G - \{v\}$. For any set F of edges, we denote by $G - F$ the spanning subgraph $(V(G), E(G) - F)$. If $F = \{e\}$ we write $G - e$ instead of $G - \{e\}$. A complete subgraph of order q of G is called a *q-clique* of G. The complete graph of order q is denoted by K_q. When a graph G contains a q-clique as subgraph, we say "G contains a K_q".

In [6] Horváth and Katona considered the notion of (H, k) *edge stable* graph[2]: given a simple graph H, an integer k and a graph G containing H as subgraph, G is an (H, k) *edge stable graph* whenever the deletion of any set of k edges does not lead to an H-free graph. These authors consider (P_n, k) edge stable graphs and prove a conjecture stated in [5] on the minimum size of a (P_4, k) edge stable graph. In [2], Dudek, Szymański and Zwonek investigated the vertex version of this notion and introduced the (H, k) stable graphs.

Definition 1.1 [2]**.** Given an integer $k \geq 0$ and a graph H without isolated vertices, a graph G containing a subgraph isomorphic to H is said to be an (H, k) *stable graph* if, for every subset S of k vertices, $G - S$ contains (a subgraph isomorphic to) H.

Definition 1.2. A (H, k) stable graph with minimum size is called a *minimum* (H, k) *stable graph*. The size of a minimum (H, k) stable graph shall be denoted by $stab(H, k)$.

Note that if G is an (H, k) stable graph with minimum size then the graph obtained from G by addition or deletion of some isolated vertices is also minimum (H, k) stable. Hence we shall assume that all the graphs considered in the paper have no isolated vertices. It is clear that H is the unique $(H, 0)$ stable graph with minimum size.

In this paper we consider (K_q, k) stable graphs with $q \geq 2$. Since K_{q+k} is (K_q, k) stable, note that a trivial upper bound for $stab(K_q, k)$ is $\binom{q+k}{2}$. It is an easy exercise to see that $stab(K_2, k) = k + 1$ and that the matching $(k + 1)K_2$ is the unique minimum (K_2, k) stable graph.

Dudek, Szymański and Zwonek have proved in [2] that $stab(K_3, k) = 3(k+1)$ for $k \geq 0$ and $stab(K_4, k) = 5(k + 1)$ for $k \geq 1$ and they have obtained an upper

[2] In the original paper [6] these graphs are just called (H, k) stable by the authors.

bound for $stab(K_q, k)$ for sufficiently large k. More precisely, they have obtained the following result.

Theorem 1.3 [2]. *For every $q \geq 5$, there exists an integer $k(q)$ such that for every $k \geq k(q)$, $stab(K_q, k) \leq (2q-3)(k+1)$.*

In order to obtain Theorem 1.3, the authors consider the graph $G = sK_{2q-2} + (r-s)K_{2q-3}$ with $q \geq 5$, $k \geq (q-1)(q-2)$, $r \in \{1, \cdots, k+1\}$, $s \in \{0, \ldots, r\}$ and $r(q-2) + s - 1 = k$ and note that the number of edges of G is $(2q-3)(k+1)$. A smaller bound for $k(q)$ can be obtained by the following Proposition 1.4 (a consequence of an old result of Sylvester [7]; see a proof at the end of Section 2), and more generally apart from $k \in \{0, \ldots, q-4\}$, Theorem 1.6 below gives a better upper bound than $\binom{q+k}{2}$ for $stab(K_q, k)$.

Proposition 1.4. *Let $q \geq 4$ be an integer. Set*

$$A(q) = \bigcup_{0 \leq i \leq q-4} \{i(q-1) + j \mid 0 \leq j \leq q-4-i\}$$

and

$$B(q) = \{b \in \mathbb{N} \mid 0 \leq b \leq (q-2)(q-3) - 2\} - A(q).$$

Let k be a nonnegative integer. There exist integers s and t such that $s(q-1) + t(q-2) - 1 = k$ if and only if $k \in B(q)$ or $k \geq k(q) = (q-3)(q-2) - 1$. For such a pair (s, t), $G = sK_{2q-2} + tK_{2q-3}$ is (K_q, k) stable and $e(G) = (2q-3)(k+1)$.

Note that $|A(q)| = \frac{(q-3)(q-2)}{2}$ and $|B(q)| = |A(q)| - 1$.

Lemma 1.5. *Let $q \geq 4$ and $k \geq 0$ be integers. Then $k \in A(q)$ if and only if $[\frac{k+1}{q-1}, \frac{k+1}{q-2}]$ contains no integer.*

Theorem 1.6. *Let $q \geq 3$ and $k \geq 0$ be integers. Set $A(3) = B(3) = \emptyset$, and for $q \geq 4$, $A(q)$ and $B(q)$ are the sets defined in Proposition 1.4. For every positive integer r set*

$$\phi(r) = \frac{1}{2}\left(q - 1 + \lfloor \tfrac{k+1}{r} \rfloor\right)\left(\left(q - 2 - \lfloor \tfrac{k+1}{r} \rfloor\right) r + 2(k+1)\right).$$

Then, $stab(K_q, k)$ is at most equal to

- $\phi(1) = \frac{1}{2}(q+k-1)(q+k)$ *if $k \leq q-4$ (note that k is in $A(q)$),*
- $\min\{\phi(\lfloor \frac{k+1}{q-1} \rfloor), \phi(\lfloor \frac{k+1}{q-1} \rfloor + 1)\}$ *if $k \in A(q)$ and $k \geq q-1$,*
- $(2q-3)(k+1)$ *if $k \in B(q)$ or $k \geq k(q) = (q-3)(q-2) - 1$ (note that $\phi(r) = (2q-3)(k+1)$ for every integer $r \in [\frac{k+1}{q-1}, \frac{k+1}{q-2}]$).*

We shall give a proof of Theorem 1.6 in Section 3 by considering (K_q, k) stable graphs having cliques as components and having the minimum number of edges. As a consequence, if every component of a mimimum (K_q, k) stable graph is complete (see Problem 1.15) then the upper bound given in Theorem 1.6 is the exact value for $stab(K_q, k)$.

In light of their results, Dudek, Szymański and Zwonek propose the following conjecture.

Conjecture 1.7 [2]. *There exists an integer $k(q)$ such that for every $k \geq k(q)$, $stab(K_q, k) = (2q-3)(k+1)$.*

Note that Conjecture 1.7 is true for $q \in \{3, 4\}$. In [4] we have proved that $stab(K_5, k) = 7(k+1)$ for $k \geq 5$, which confirms Conjecture 1.7 for $q = 5$. Moreover, we have characterized (K_q, k) stable graphs with minimum size for $q \in \{3, 4, 5\}$. The following theorem summarizes these results.

Theorem 1.8 [4] **.** *Let G be a minimum (K_q, k) stable graph, with $q \in \{3, 4, 5\}$ and $k \geq k(q)$ with $k(3) = 0$, $k(4) = 1$, $k(5) = 5$. Then $G = sK_{2q-2} + tK_{2q-3}$, for any choice of s and t such that $s(q-1) + t(q-2) - 1 = k$. Moreover, K_{5+k} is the unique minimum (K_5, k) stable graph for $k \in \{1, 2, 3\}$, K_9 and $K_6 + K_7$ are the only minimum $(K_5, 4)$ stable graphs.*

An important fact is that Conjecture 1.7 of Dudek, Szymański and Zwonek has been recently solved by Żak [8], who has characterized also the extremal graphs.

Theorem 1.9 [8]. *Let $q \geq 2$, $k \geq 0$ be nonnegative integers. Then $stab(K_q, k) \geq (2q-3)(k+1)$, with equality if and only if $k = s(q-1) + t(q-2) - 1$ for some nonnegative integers s and t. In particular, $stab(K_q, k) = (2q-3)(k+1)$ for $k \geq (q-3)(q-2) - 1$. Furthermore, if G is a (K_q, k) stable graph having exactly $(2q-3)(k+1)$ edges, then $G = sK_{2q-2} + tK_{2q-3}$ where s and t are nonnegative integers such that $s(q-1) + t(q-2) - 1 = k$.*

Remark 1.10. Since (K_q, k) stable graphs with minimum size for $q \in \{3, 4, 5, 6\}$ have been characterized (see Theorem 1.8 for $q \leq 5$ and [8] for $q = 6$), to close the study of minimum (K_q, k) stable graphs we have only to consider $q \geq 7$ and $k \in A(q)$ (the set defined in Proposition 1.4).

We have proved in [4] that K_{q+k} is the unique minimum (K_q, k) stable graph for $q \geq 4$ and $k \in \{1, 2\}$, that K_{q+3} is the unique minimum $(K_q, 3)$ stable graph for $q \geq 5$ and in [3] that K_{q+k} is the unique (K_q, k) stable graph for $q \geq 6$ and $k \leq \frac{q}{2} + 1$. Remark that $\binom{q+k}{2} - (2q-3)(k+1) = \frac{(q-k-3)(q-k-2)}{2}$ and that this integer is positive for $q \geq 3$ and $k \notin \{q-3, q-2\}$. Then, as a consequence of Proposition 1.4, for $q \geq 4$ and for every integer k for which $k \in B(q) - \{q-3, q-2\}$ or $k \geq (q-3)(q-2) - 1$ the graph K_{q+k} is not minimum (K_q, k) stable. Hence, the set $\{k \in \mathbb{N} \mid K_{q+k} \text{ is minimum } (K_q, k) \text{ stable}\}$ is bounded above, and we propose the following definition.

Definition 1.11. For every integer $q \geq 4$, we denote by $\kappa(q)$ the greatest integer such that for $1 \leq k < \kappa(q)$ the only minimum (K_q, k) stable graph is K_{q+k}.

We will focuse our attention on determining the exact value of $\kappa(q)$. In two previous papers we have proved the following.

Theorem 1.12 [3, 4]**.** *$\kappa(3) = 1$, $\kappa(4) = 3$, $\kappa(5) = 4$ and for $q \geq 6$, $\kappa(q) > \frac{q}{2} + 1$.*

In this paper we give an upper bound for the value of $\kappa(q)$.

Theorem 1.13. *For every $q \geq 4$, if $\kappa(q)$ is even, then $\kappa(q) < \sqrt{2(q-1)(q-2)}$ and if $\kappa(q)$ is odd, then $\kappa(q) < \sqrt{1 + 2(q-1)(q-2)}$.*

We prove that these upper bounds are reached for values of q such that there exists a minimum $(K_q, \kappa(q))$ stable disconnected graph (note that it is the case for $q = 4$ and $q = 5$).

Theorem 1.14. *Let $q \geq 4$ and suppose that there exists a disconnected minimum $(K_q, \kappa(q))$ stable graph. Set $\rho(q) = \left\lceil \sqrt{\frac{1}{2}(q-1)(q-2)} \right\rceil - 1$.*
If $\frac{1}{2}(q-1)(q-2) > \rho(q)^2 + \rho(q)$, then $\kappa(q) = 2\rho(q) + 1$.
If $\frac{1}{2}(q-1)(q-2) \leq \rho(q)^2 + \rho(q)$, then $\kappa(q) = 2\rho(q)$.

Proofs of Theorems 1.13 and 1.14 shall be given in Subsection 3.3.

Remark that, by definition of $\kappa(q)$ and by Theorem 1.9, for every integer k in $\{l \in \mathbb{N} \mid 0 \leq l < \kappa(q) \text{ or } l \geq (q-2)(q-3) - 1\} \cup B(q)$ every component of any minimum (K_q, k) stable graph is complete, but we do not know if it is true for k in $\{\, l \in \mathbb{N} \mid l \geq \kappa(q) \text{ and } l \in A(q)\}$ (where $A(q)$ and $B(q)$ are the sets defined in Proposition 1.4).

If there is no minimum disconnected $(K_q, \kappa(q))$ stable graph then, by definition of $\kappa(q)$, there exists a connected minimum $(K_q, \kappa(q))$ stable graph G_q which is not complete. We think that it never happens, so we propose the following problem.

Problem 1.15. *Is it true that if G is a minimum (K_q, k) stable graph, then every component of G is complete?*

If the answer is positive then Theorem 1.14 gives the exact value of $\kappa(q)$ for every $q \geq 4$.

2. GENERAL RESULTS

Lemma 2.1 [2]**.** *Let G be an (H, k) stable graph with $k \geq 1$. Then, for every vertex v, $G - v$ is $(H, k-1)$ stable.*

A set of vertices of G that intersects every subgraph of G isomorphic to H is called a *transversal of all the subgraphs isomorphic to* H or simply an H*-transversal* of G. An H-transversal of G having the minimum number of vertices is said to be a *minimum* H*-transversal of* G. The number of vertices of a minimum H-transversal is denoted by $\tau_H(G)$. Remark that G is (H,k) stable if and only if $\tau_H(G) \geq k+1$

Definition 2.2. Let G be an (H,k) stable graph. If G has a minimum H-transversal having exactly $k+1$ vertices, G is said to be *exactly* (H,k) *stable.*

Lemma 2.3 [2]**.** *Let G be an (H,k) stable graph with $k \geq 1$ and $e \in E(G)$ such that $G-e$ is not (H,k) stable. Then G is exactly (H,k) stable and $G-e$ is exactly $(H,k-1)$ stable.*

Definition 2.4 [2]**.** Let G be an (H,k) stable graph. If $G-e$ is not (H,k) stable for every edge $e \in E(G)$, then G is said to be *minimal* (H,k) *stable.*

Remark 2.5. In [2] "minimal (H,k) stable graphs" are called "strong (H,k) stable graphs" by the authors. Note that an (H,k) stable graph G is minimal (H,k) stable if and only if for every $e \in E(G)$ the graph $G-e$ is exactly $(H,k-1)$ stable. Moreover, a minimal (H,k) stable graph is exactly (H,k) stable.

If there exists an edge e of an (H,k) stable graph G such that there are no subgraphs isomorphic to H containing e, then $G-e$ is an (H,k) stable graph. Hence, we have the following.

Lemma 2.6 [2]**.** *Every edge of a minimal (H,k) stable graph is contained in a subgraph isomorphic to H. Consequently, every vertex of a minimal (H,k) stable graph is also contained in a subgraph isomorphic to H.*

Remark 2.7. Clearly, every minimum (H,k) stable graph is minimal (H,k) stable.

One may ask what happens for components of an (H,k) stable graph. The following theorem gives us an answer when H is connected. We shall say that a graph containing no subgraph isomorphic to H is $(H,-1)$ stable.

Theorem 2.8. *Let H be a connected graph containing at least 2 vertices, let G be an exactly (H,k) stable graph, and let G_1, $G_2, \ldots$, G_r, with $r \geq 1$, be its components. Then, there exist integers k_1, $k_2, \ldots$, k_r, with $0 \leq k_i \leq k$, such that*
(i) *for every i, with $1 \leq i \leq r$, G_i is exactly (H,k_i) stable,*
(ii) $\sum_{i=1}^{r} k_i + (r-1) = k$,
G is minimal (H,k) stable if and only if for every i, $1 \leq i \leq r$, G_i is minimal (H,k_i) stable. Moreover, if G is minimum (H,k) stable, then for every i, $1 \leq i \leq r$, G_i is minimum (H,k_i) stable.

Proof. For each i, $1 \le i \le r$, let us consider a minimum H-transversal of G_i, say T_i, and set $k_i = |T_i| - 1$. Clearly, for each i the graph G_i is exactly (H, k_i) stable and the set $T = \bigcup_{1 \le i \le r} T_i$ is a minimum H-transversal of G. Note that the number of elements of T is $|T| = \sum_{i=1}^{r} k_i + r$ and we have $|T| > k$. Let S be any set of vertices of G such that $|S| \le |T| - 1$ and for every i denote by S_i the set $S \cap V(G_i)$. Clearly, there exists $i_0 \in \{1, \dots, r\}$ such that $|S_{i_0}| \le k_{i_0} = |T_{i_0}| - 1$. Then, $G_{i_0} - S_{i_0}$ contains a subgraph isomorphic to H, that is, G is exactly $(H, |T| - 1)$ stable, and we have $\sum_{i=1}^{r} k_i + (r-1) = k$.

Let e be an edge of G and let G_i be the component containing e.

Claim. $G - e$ is (H, k) stable if and only if $G_i - e$ is (H, k_i) stable.

Proof. Suppose that $G_i - e$ is (H, k_i) stable. Let U be an H-transversal of $G - e$. Set $U_i = U \cap V(G_i - e) = U \cap V(G_i)$ and for every $j \ne i$, $U_j = U \cap V(G_j)$. Since $(G_i - e) - U_i$ and each $G_j - U_j$, $j \ne i$, contain no subgraphs of $G - e$ isomorphic to H, we have for every j, $1 \le j \le r$, $|U_j| \ge k_j + 1$. Then, $|U| = \sum_{j=1}^{r} |U_j| \ge k + 1$. Hence, for every set S of k vertices $(G - e) - S$ contains a subgraph isomorphic to H, that is, $G - e$ is (H, k) stable.

Conversely, suppose that $G_i - e$ is not (H, k_i) stable. Let T_i be an H-transversal of $(G_i - e) - T_i$ having k_i vertices. For every $j \ne i$ let T_j be an H-transversal of G_j having $k_j + 1$ vertices. The set $T = \cup_{j=1}^{r} T_j$ has k vertices and is a H-transversal of $G - e$, that is, $G - e$ is not (H, k) stable. □

Thus, G is minimal (H, k) stable if and only if for every i, $1 \le i \le r$, G_i is minimal (H, k_i) stable.
Note that, by replacing a minimal (H, k_i) stable component G_i by any minimal (H, k_i) stable graph G_i' (connected or not), we obtain again a minimal (H, k) stable graph. Thus, if G is minimum (H, k) stable then for every i, $1 \le i \le r$, G_i is minimum (H, k_i) stable. ■

Remark 2.9. Let $r \ge 2$ be an integer, $k_1, \dots, k_r$ be r non negative integers and $k = \sum_{i=1}^{r} k_i + (r-1)$. If for every i, $1 \le i \le r$, G_i is a minimum (H, k_i) stable graph then the disjoint union $G_1 + G_2 + \dots + G_r$ may not be a minimum (H, k) stable graph. For example, K_q is minimum $(K_q, 0)$ stable, $2K_q$ and K_{q+1} are minimal $(K_q, 1)$ stable, but for $q \ge 4$ since $e(2K_q) > e(K_{q+1})$, the graph $2K_q$ is not minimum $(K_q, 1)$ stable.

Given relatively prime positive integers $a_1, \dots, a_n$, let us consider the integers that can be expressed as a sum $k_1a_1 + k_2a_2 + \dots + k_na_n$, where $k_1, k_2, \dots, k_n$ are nonnegative integers. Any such integer is said to be *representable*. Recall that the *Frobenius Problem* is the following: find the largest non-representable integer (called the *Frobenius number* and denoted by $g(a_1, \dots, a_n)$). If $n = 2$, the Frobenius number is given by the formula $g(a_1, a_2) = a_1a_2 - a_1 - a_2$. This formula was discovered by Sylvester in 1884 [7], who also demonstrated that there are a total of $N(a_1, a_2) = \frac{(a_1-1)(a_2-1)}{2}$ non-representable integers. For the particular case $a_2 = a_1 - 1$ one obtains explicitely the set of non-representable integers.

Lemma 2.10 [7]**.** *Let $a \geq 3$ be an integer and the function $\alpha : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ such that $\alpha(s,t) = sa + t(a-1)$. Set $A = \bigcup_{0 \leq i \leq a-3}\{ia + j \mid 1 \leq j \leq a-2-i\}$. Every $b \in \mathbb{N} - A$ is representable (that is, there exists a pair $\{s,t\}$ of nonnegative integers such that $b = sa + t(a-1)$), and every b in A is not representable. Moreover, every representable b has a unique representation $sa + t(a-1)$ such that $0 \leq t \leq a-1$.*

We shall give a proof of Lemma 2.10 for completeness.

***Proof of Lemma* 2.10.** Note that $\max(A) = (a-1)(a-2)-1$, $|A| = \frac{(a-1)(a-2)}{2}$ and for $s \geq 0$ and $t \geq 1$, $\alpha(s,t) = \alpha(s+1, t-1) - 1$.

Consider the infinite matrix $\{\alpha(s,t)\}_{s\geq 0,\ t\geq 0}$. For any $t \geq 0$ the values of the diagonal $\{\alpha(i, t-i) \mid 0 \leq i \leq t\}$ are the consecutive integers $\{t(a-1)+i \mid 0 \leq i \leq t\}$. For $s \geq 0$, the values of the (partial) diagonal $\{\alpha(s+i, a-i-1) \mid 0 \leq i \leq a-1\}$ are the consecutive integers $sa + (a-1)^2, sa + (a-1)^2 + 1, \ldots, sa + a(a-1)$.

Since $\alpha(0, a-1) = \alpha(a-2, 0) + 1$ and for every $s \geq 0$, $\alpha(s+a-1, 0) + 1 = \alpha(s+1, a-1) = sa + a(a-1) + 1$, every integer $b \geq (a-2)(a-1)$ appears in

$$\{\alpha(i, a-2-i) \mid 0 \leq i \leq a-2\} \cup \bigcup_{s\geq 0}\{\alpha(s+i, a-i-1) \mid 0 \leq i \leq a-1\}.$$

Let $B = \bigcup_{0 \leq i \leq a-3}\{\alpha(j, i-j) \mid 0 \leq j \leq i\} = \bigcup_{0\leq i\leq a-3}\{i(a-1) + j \mid 0 \leq j \leq i\}$. Clearly $|B| = |A|$. It is easy to check that A and B are disjoint sets and that $A \cup B = \{0, 1, \ldots, (a-2)(a-1) - 1\}$. Thus, every $b \in A$ is not representable and for every integer $b \in \mathbb{N} - A$ there exists a unique pair (s,t) with $s \geq 0$ and $0 \leq t \leq a-1$ such that $b = sa + t(a-1)$. ■

Remark 2.11. It is easy to see that every representable $b < a(a-1)$ has a unique representation. For a representable $b \geq a(a-1)$, since we can choose values of $t \geq a$, it is possible that $b = \alpha(s,t) = \alpha(s', t')$ for distinct pairs (s,t) and (s', t'). Indeed, if $s \geq a-1$, then for every positive integer $r \leq \lfloor \frac{s}{a-1} \rfloor$, $\alpha(s,t) = \alpha(s - r(a-1), ra + t)$.

***Proof of Proposition* 1.4.** Let us apply Lemma 2.10 to $a = q-1$ and $b = k+1$. $B(q)$ is the set of integers $k \leq (q-3)(q-2) - 3$ such that $k+1$ is representable as $s(q-1) + t(q-2)$. More precisely, $B(q) = \bigcup_{1\leq i\leq q-4}\{i(q-2) + j - 1 \mid 0 \leq j \leq i\}$.

It is easy to see that the set of integers k such that $k+1$ is not representable as $s(q-1) + t(q-2)$ is $A(q) = \bigcup_{0\leq i\leq q-4}\{i(q-1) + j \mid 0 \leq j \leq q-4-i\}$.

A minimum K_q-transversal of $G = sK_{2q-2} + tK_{2q-3}$ contains exactly $s(q-1) + t(q-2) = k+1$ vertices, that is G is (K_q, k) stable, and it is easy to check that $e(G) = (2q-3)(k+1)$. ■

***Proof of Lemma* 1.5.** If there exist integers s and t such that $s(q-1)+t(q-2) = k+1$ then $\frac{k+1}{q-1} = s+t-\frac{t}{q-1}$ and $\frac{k+1}{q-2} = s+t+\frac{s}{q-2}$, and hence $r = s+t \in [\frac{k+1}{q-1}, \frac{k+1}{q-2}]$. Conversely, let $r \in [\frac{k+1}{q-1}, \frac{k+1}{q-2}]$. Then $q-2 \le \frac{k+1}{r} \le q-1$. If $k+1 = r(q-1)$ then we are done. If $\frac{k+1}{q-1} < r$ then $q-2 = \lfloor \frac{k+1}{r} \rfloor$ is the quotient in the division of $k+1$ by r. Hence, if s denotes the remainder, then $k+1 = r(q-2)+s = s(q-1)+(r-s)(q-2)$. We conclude by applying Proposition 1.4. ■

3. Minimum (K_q, k) Stable Graphs

In this section we are interested in (K_q, k) stable graphs with minimum size $(q \ge 3)$. Recall that $stab(K_q, k) = \min\{e(G) \mid G \text{ is } (K_q, k) \text{ stable}\}$.

3.1. Some known results

We give here some known results about this topic.

By Remark 2.5 and Lemma 2.6 we have:

Properties 3.1 [2]**.** *A minimal (K_q, k) stable graphs G has the following properties:*

(P$_1$) *G is exactly (K_q, k) stable.*

(P$_2$) *For every edge e, $G-e$ is exactly $(K_q, k-1)$ stable.*

(P$_3$) *For every vertex v, $G-v$ is exactly $(K_q, k-1)$ stable.*

(P$_4$) *Every vertex of G belongs to some q-clique of G.*

(P$_5$) *Every edge of G belongs to some q-clique of G.*

Remark 3.2. For any two integers $q \ge 3$ and $k \ge 1$, K_{q+k} is minimal (K_q, k) stable.

Proposition 3.3 [4]**.** *K_5 is the unique minimum $(K_4, 1)$ stable graph, K_6 is the unique minimum $(K_4, 2)$ stable graph and for every integer $q \ge 5$ and every integer $k \in \{1, 2, 3\}$, K_{q+k} is the unique minimum (K_q, k) stable graph.*

Dudek *et al.* [2] defined the family $\mathcal{A}_r^{(K_q,k)}$ with $k \ge 0$, $q \ge 3$, $1 \le r \le k+1$ as the family of graphs consisting of r complete graphs K_{i_j} with $i_1 \ge \cdots \ge i_r \ge q$ satisfying the condition $\sum_{i=1}^{r}(i_j - q) + (r-1) = k$ and they proved that every graph in $\mathcal{A}_r^{(K_q,k)}$ is minimal (K_q, k) stable. We observe that if a (K_q, k) stable graph G is a disjoint union of $r \ge 1$ cliques K_{i_j}, $1 \le j \le r$, then by Theorem 2.8, $G \in \mathcal{A}_r^{(K_q,k)}$. They defined a graph $G \in \mathcal{A}_r^{(K_q,k)}$ as a *balanced union* if $|i_j - i_l| \in \{0, 1\}$ for every j and l in $\{1, 2, \ldots, r\}$ and they proved that given q,

k and r there is exactly one balanced union $\mathcal{B}_r^{(K_q,k)}$ in $\mathcal{A}_r^{(K_q,k)}$, and that $\mathcal{B}_r^{(K_q,k)}$ has the minimum number of edges among the graphs in $\mathcal{A}_r^{(K_q,k)}$.

In [2] the following lemma has been given. We give its proof for completeness.

Lemma 3.4 [2]. *Let G_0 be a (K_q, k_0) stable graph ($k_0 \geq 0$) which has the minimum size among all graphs beeing a disjoint union of r cliques ($r \geq 1$), $G_j \equiv K_{q+k_j}$ with $1 \leq j \leq r$, $k_j \geq 0$. There exist nonnegative integers s and k such that $0 \leq s \leq r-1$, $G_0 = sK_{q+k+1} + (r-s)K_{q+k}$ with $r(k+1)+s = k_0+1$ and $e(G_0) = \frac{1}{2r}\left(r(q-1)+k_0+1-s\right)\left(r(q-2)+k_0+1+s\right)$.*

Proof. Suppose, without loss of generality, that $k_1 \geq k_2 \geq \cdots \geq k_r$ and that there exist two components G_i and G_j with $i < j$ such that $k_i - k_j \geq 2$. By substituting $G'_i \equiv K_{q+k_i-1}$ for G_i and $G'_j \equiv K_{q+k_j+1}$ for G_j, we obtain a new (K_q, k) stable graph G'_0 such that $e(G'_0) = e(G_0) - (k_i - k_j - 1) < e(G_0)$, which is a contradiction. Thus, for any i and any j, $0 \leq |k_i - k_j| \leq 1$. Hence, either for any i and any j, k_i and k_j have the same value k and we have $G_0 = rK_{q+k}$ with $k \geq 0$, or there exist distinct k_i and k_j and we have $G_0 = sK_{q+k+1} + (r-s)K_{q+k}$ with $k \geq 0$ and $0 \leq s \leq r-1$. Hence, a minimum K_q-transversal of G_0 has $k_0 + 1 = s(k+2) + (r-s)(k+1) = s + r(k+1)$ vertices. Note that r divides k_0+1-s. We have $2e(G_0) = s(q+k+1)(q+k) + (r-s)(q+k)(q+k-1)$. Since $k+1 = \frac{k_0+1-s}{r}$, we obtain $e(G_0) = \frac{1}{2r}(r(q-1)+k_0+1-s)(r(q-2)+k_0+1+s)$. ■

Remark 3.5. In Lemma 3.4 the integers q, k_0 and r are given. Given q and k_0, in order to obtain an upper bound for $stab(K_q, k_0)$ we will determine the values of r for which $e(G_0(r)) = \frac{1}{2r}(r(q-1)+k_0+1-s)(r(q-2)+k_0+1+s))$ is minimum. We note that if every component of a minimum (K_q, k_0) stable graph is complete then the minimum value of $e(G_0(r))$ is exactly $stab(K_q, k_0)$.

3.2. Proof of Theorem 1.6

First we give a technical lemma used for proving Theorem 1.6.

Lemma 3.6. *Let a and b be positive integers and for $x > 0$ consider the real-to-real function*

$$f(x) = \frac{1}{2}\left(a+1+\left\lfloor \frac{b}{x} \right\rfloor\right)\left(\left(a-\left\lfloor \frac{b}{x} \right\rfloor\right)x + 2b\right).$$

Then, f is continuous on $(0, +\infty)$, nonincreasing on $(0, \frac{b}{a+1}]$, constant on $[\frac{b}{a+1}, \frac{b}{a}]$ and nondecreasing on $[\frac{b}{a}, +\infty)$. Moreover $\min\{f(r) \mid r \in \mathbb{N} - \{0\}\}$ *is equal to*

- $f(1) = \frac{1}{2}(a+b+1)(a+b)$ *if* $[\frac{b}{a+1}, \frac{b}{a}]$ *contains no integer and* $b < a$,
- $\min\left\{f(\lfloor \frac{b}{a+1} \rfloor), f(\lfloor \frac{b}{a+1} \rfloor + 1)\right\}$ *if* $[\frac{b}{a+1}, \frac{b}{a}]$ *contains no integer and* $b > a+1$,

- $(2a+1)b$ *if* $[\frac{b}{a+1}, \frac{b}{a}]$ *contains at least one integer* r *(and is equal to* $f(r)$ *for every such* r*).*

Proof. For $x > b$ we have $\lfloor \frac{b}{x} \rfloor = 0$ and $f(x) = \frac{1}{2}(a+1)(ax+2b)$. For every integer $p \geq 1$ and for every $x \in [\frac{b}{p+1}, \frac{b}{p}]$ we have $\lfloor \frac{b}{x} \rfloor = p$, and hence $f(x) = \frac{1}{2}(a+1+p)((a-p)x+2b)$. It is easy to see that the function f is continuous on $(0,+\infty)$, nonincreasing on $[0, \frac{b}{a+1}]$, constant on $[\frac{b}{a+1}, \frac{b}{a}]$ and nondecreasing on $[\frac{b}{a}, +\infty)$. The minimum value for $f(x)$ (with a x positive real number) is the integer $(2a+1)b$ and is reached for every real number x in $[\frac{b}{a+1}, \frac{b}{a}]$. We note that if r is a positive integer, then $f(r)$ is a positive integer.

Now we will find the minimum value of $f(r)$ when r is a positive integer.

Case 1. $[\frac{b}{a+1}, \frac{b}{a}] \cap \mathbb{N} = \emptyset$. Note that $0 < \frac{b}{a} - \frac{b}{a+1} < 1$ (that is, $0 < b < a(a+1)$), $0 \leq \lfloor \frac{b}{a+1} \rfloor \leq a$ and $\lfloor \frac{b}{a+1} \rfloor < \frac{b}{a+1} < \frac{b}{a} < \lceil \frac{b}{a} \rceil = \lfloor \frac{b}{a+1} \rfloor + 1$.

Case 1.1. $b < a$. Since $\lceil \frac{b}{a} \rceil = 1$ and $f(r)$ is non decreasing on $[\frac{b}{a}, +\infty)$, the minimum value is $f(1) = \frac{1}{2}(a+b+1)(a+b)$.

Case 1.2. $b \geq a$. Since $b \notin \{a, a+1\}$, we have $b > a+1$ and $1 \leq \lfloor \frac{b}{a+1} \rfloor \leq a$, hence the minimum value is

$$\min \left\{ f\left(\left\lfloor \tfrac{b}{a+1} \right\rfloor \right), f\left(\left\lfloor \tfrac{b}{a+1} \right\rfloor + 1 \right) \right\}.$$

Let β be the remainder of the division of b by $a+1$. In order to obtain the value $f(\lfloor \frac{b}{a+1} \rfloor)$ we must know the integer $p_1 \geq a+1$ such that $\frac{b}{p_1+1} < \lfloor \frac{b}{a+1} \rfloor \leq \frac{b}{p_1}$. Since $\lfloor \frac{b}{a+1} \rfloor = \frac{b-\beta}{a+1}$, we have $p_1 = \lfloor \frac{b(a+1)}{b-\beta} \rfloor$, and hence

$$f\left(\left\lfloor \tfrac{b}{a+1} \right\rfloor \right) = \tfrac{1}{2}\left(a+1+p_1\right)\left((a-p_1)\left(\tfrac{b-\beta}{a+1} \right) + 2b \right).$$

In the same way we obtain

$$f\left(\left\lfloor \tfrac{b}{a+1} \right\rfloor + 1 \right) = \tfrac{1}{2}(a+1+p_2)\left((a-p_2)\left(\tfrac{b+a+1-\beta}{a+1} \right) + 2b \right)$$

with $p_2 = \left\lfloor \frac{b(a+1)}{b+a+1-\beta} \right\rfloor$.

Case 2. $[\frac{b}{a+1}, \frac{b}{a}] \cap \mathbb{N} \neq \emptyset$. For any integer r such that $\frac{b}{a+1} \leq r \leq \frac{b}{a}$, $f(r)$ is equal to the minimum value $(2a+1)b$. ■

***Proof of Theorem* 1.6.** In order to avoid confusion between "k" of the statement of Theorem 1.6 and "k" appearing in the proof of Lemma 3.4, let us replace "k" by "k_0" in the statement of Theorem 1.6. Consider the (K_q, k_0) stable graph G_0 defined in Lemma 3.4 and see Remark 3.5. We have $G_0 = sK_{q+k+1} + (r-s)K_{q+k}$ with $r(k+1)+s = k_0+1$ and $e(G_0) = \frac{1}{2r}(r(q-1)+k_0+1-s)(r(q-2)+k_0+1+s)$. Since $k+1$ is the quotient of k_0+1 divided by r

and s is the remainder, we have $s = k_0 + 1 - r\lfloor \frac{k_0+1}{r} \rfloor$. Hence,

$$e(G_0(r)) = \frac{1}{2}\left(q - 1 + \left\lfloor \frac{k_0+1}{r} \right\rfloor\right)\left(\left(q - 2 - \left\lfloor \frac{k_0+1}{r} \right\rfloor\right) r + 2(k_0+1)\right).$$

Set $a = q - 2$, $b = k_0 + 1$ and apply Lemma 3.6 and Lemma 1.5. ∎

3.3. Minimum (K_q, k) stable graph for small k

In the following, if no confusion is possible, we simply denote the integer $\kappa(q)$ by κ.

Lemma 3.7. *Suppose that $q \geq 4$. If κ is even, then $stab(K_q, \kappa-1) < e(2K_{q+\frac{\kappa}{2}-1})$ and $stab(K_q, \kappa) \leq e(K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1})$.*
If κ is odd, then $stab(K_q, \kappa - 1) < e(K_{q+\frac{\kappa-1}{2}} + K_{q+\frac{\kappa-3}{2}})$ and $stab(K_q, \kappa) \leq e(2K_{q+\frac{\kappa-1}{2}})$.

Proof. Recall that, by definition of κ, $K_{q+\kappa-1}$ is the only minimum $(K_q, \kappa - 1)$ stable. If κ is even then $2K_{q+\frac{\kappa}{2}-1}$ is exactly $(K_q, \kappa-1)$ stable and $K_{q+\frac{\kappa}{2}}+K_{q+\frac{\kappa}{2}-1}$ is exactly (K_q, κ) stable. If κ is odd then $K_{q+\frac{\kappa-1}{2}} + K_{q+\frac{\kappa-3}{2}}$ is exactly $(K_q, \kappa - 1)$ stable and $2K_{q+\frac{\kappa-1}{2}}$ is exactly (K_q, κ) stable. ∎

Lemma 3.8. *Let $q \geq 3$ and $p \geq 0$ be two integers. Then,*
$e(K_{q+2p}) < e(K_{q+p} + K_{q+p-1})$ if and only if $p^2 + p < \frac{1}{2}(q-1)(q-2)$ and
$e(K_{q+2p}) = e(K_{q+p} + K_{q+p-1})$ if and only if $p_0 = \frac{1}{2}(\sqrt{1 + 2(q-1)(q-2)} - 1)$ is an integer and $p = p_0$.
$e(K_{q+2p+1}) < e(2K_{q+p})$ if and only if $(p+1)^2 < \frac{1}{2}(q-1)(q-2)$ and
$e(K_{q+2p+1}) = e(2K_{q+p})$ if and only if $p_1 = \frac{1}{2}(\sqrt{2(q-1)(q-2)} - 1)$ is an integer and $p = p_1$.

Proof. It is easy to check that $e(K_{q+2p}) - e(K_{q+p} + K_{q+p-1}) = p^2 + p - \frac{1}{2}(q-1)(q-2)$ and $e(K_{q+2p+1}) - e(2K_{q+p}) = (p+1)^2 - \frac{1}{2}(q-1)(q-2)$. These polynomials of degree 2 in p have positive roots $p_0 = \frac{1}{2}(\sqrt{1 + 2(q-1)(q-2)} - 1)$ and $p_1 = \frac{1}{2}(\sqrt{2(q-1)(q-2)} - 1)$ respectively. ∎

***Proof of Theorem* 1.13.** If $\kappa = 2p$ then, by Lemma 3.7, $stab(K_q, \kappa - 1) < e(2K_{q+\frac{\kappa}{2}-1})$. Since $\kappa - 1 = 2(p-1) + 1$, by Lemma 3.8, $p^2 < \frac{1}{2}(q-1)(q-2)$, that is, $\kappa < \sqrt{2(q-1)(q-2)}$.

If $\kappa = 2p + 1$ then by Lemma 3.7, $stab(K_q, \kappa - 1) < e(K_{q+\frac{\kappa-1}{2}} + K_{q+\frac{\kappa-3}{2}})$. Since $\kappa - 1 = 2p$, by Lemma 3.8, $p < \frac{1}{2}(\sqrt{1 + 2(q-1)(q-2)} - 1)$, that is, $\kappa < \sqrt{1 + 2(q-1)(q-2)}$. ∎

Theorem 3.9. *Let $q \geq 4$ and suppose that there exists a minimum (K_q, κ) stable graph G_0 which is disconnected. Then G_0 is isomorphic to $K_{q+\lfloor\frac{\kappa}{2}\rfloor} + K_{q+\lfloor\frac{\kappa-1}{2}\rfloor}$.*

Proof. Let G_0 be a minimum (K_q, κ) stable disconnected graph having $r \geq 2$ connected components $G_1, G_2, \ldots, G_r$. By Theorem 2.9, there are integers $k_1 \geq k_2 \geq \cdots \geq k_r$ with $\sum_{i=1}^{r} k_i + (r-1) = \kappa$ such that for $1 \leq i \leq r$, G_i is minimum (K_q, k_i) stable. For every i, since $k_i < \kappa$, we have $G_i \equiv K_{q+k_i}$.

Let us suppose that $r \geq 3$. We have $k_r + k_{r-1} = \kappa - (k_{r-2} + k_{r-3} + \cdots + k_1) - (r-1) \leq \kappa - 2$. Hence, $e(K_{q+k_r+k_{r-1}+1}) < e(K_{q+k_r}) + e(K_{q+k_{r-1}})$ and the graph $K_{q+k_1} + K_{q+k_2} + \cdots + K_{q+k_{r-2}} + K_{q+k_{r-1}+k_r+1}$ is (K_q, κ) stable with strictly smaller size than $K_{k_1} + K_{k_2} + \cdots + K_{k_r}$, a contradiction. Hence, $r = 2$, $G_0 \in \mathcal{B}_2^{(K_q,\kappa)}$ and by Lemma 3.4 the theorem follows. ∎

Note that Theorem 3.9 implies that there exists at most one disconnected minimum (K_q, κ) stable graph and this graph, if it exists, is

- either isomorphic to $K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1}$ (if κ is even)
- or else isomorphic to $2K_{q+\frac{\kappa-1}{2}}$ (if κ is odd).

***Proof of Theorem* 1.14.** By Lemma 3.7 and Theorem 3.9,

if κ is odd, then

$$e(K_{q+\kappa-1}) < e(K_{q+\frac{\kappa-1}{2}} + K_{q+\frac{\kappa-3}{2}}) < stab(K_q, \kappa) = e(2K_{q+\frac{\kappa-1}{2}}) \leq e(K_{q+\kappa})$$

(note that, by Lemma 3.8, it may be possible that $e(2K_{q+\frac{\kappa-1}{2}}) = e(K_{q+\kappa})$ for some values of q);

if κ is even, then

$$e(K_{q+\kappa-1}) < e(2K_{q+\frac{\kappa}{2}-1}) < stab(K_q, \kappa) = e(K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1}) \leq e(K_{q+\kappa})$$

(note that, by Lemma 3.8, it may be possible that $e(K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1}) = e(K_{q+\kappa})$ for some values of q).

For $\kappa = 2p+1$ we have

$$\frac{1}{2}(q+2p)(q+2p-1) < (q+p-1)^2 < (q+p)(q+p-1) \leq \frac{1}{2}(q+2p+1)(q+2p).$$

This implies that

$$(A) \qquad p^2 + p < \tfrac{1}{2}(q-1)(q-2) \leq (p+1)^2.$$

For $\kappa = 2p$ we have

$$\frac{1}{2}(q+2p-1)(q+2p-2) < (q+p-1)(q+p-2) < (q+p-1)^2 \leq \frac{1}{2}(q+2p)(q+2p-1).$$

This implies that

$$(B) \qquad p^2 < \tfrac{1}{2}(q-1)(q-2) \le p^2 + p.$$

Combining (A) and (B) yields

$$p^2 < \frac{1}{2}(q-1)(q-2) \le (p+1)^2.$$

This implies that

$$\sqrt{\frac{1}{2}(q-1)(q-2)} - 1 \le p < \sqrt{\frac{1}{2}(q-1)(q-2)}.$$

Hence, $p = \rho(q) = \left\lceil \sqrt{\frac{1}{2}(q-1)(q-2)} \right\rceil - 1$.

By inequalities (A) and (B), position of $\frac{1}{2}(q-1)(q-2)$ in comparison to $\rho(q)^2 + \rho(q)$ determines the parity of κ. Hence, if $\frac{1}{2}(q-1)(q-2) > \rho(q)^2 + \rho(q)$, then $\kappa = 2\rho(q) + 1 = 2\left\lceil \sqrt{\frac{1}{2}(q-1)(q-2)} \right\rceil - 1$ else $\kappa = 2\rho(q) = 2\left\lceil \sqrt{\frac{1}{2}(q-1)(q-2)} \right\rceil - 2$. ■

If there is no minimum disconnected $(K_q, \kappa(q))$ stable graph then, by definition of $\kappa(q)$, there exists a connected minimum $(K_q, \kappa(q))$ stable graph G_q distinct from a clique. Note that if such a graph exists, then

$$e(G_q) < \min\{e(K_{q+\kappa}), e(K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1})\}, \text{ if } \kappa = \kappa(q) \text{ is even}$$

or

$$e(G_q) < \min\{e(K_{q+\kappa}), e(2K_{q+\frac{\kappa-1}{2}})\}, \text{ if } \kappa = \kappa(q) \text{ is odd.}$$

A positive answer to Problem 1.15 states that there is no such graph G_q.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory, **244** (Springer, Series Graduate Texts in Mathematics, 2008).

[2] A. Dudek, A. Szymański and M. Zwonek, *(H, k) stable graphs with minimum size*, Discuss. Math. Graph Theory **28** (2008) 137–149. doi:10.7151/dmgt.1397

[3] J.-L. Fouquet, H. Thuillier, J.-M. Vanherpe and A.P. Wojda, *On (K_q, k) stable graphs with small k*, Electron. J. Combin. **19** (2012) #P50.

[4] J.-L. Fouquet, H. Thuillier, J.-M. Vanherpe and A.P. Wojda, *On (K_q, k) vertex stable graphs with minimum size*, Discrete Math. **312** (2012) 2109–2118. doi:10.1016/j.disc.2011.04.017

[5] P. Frankl and G.Y. Katona, *Extremal k-edge-hamiltonian hypergraphs*, Discrete Math. **308** (2008) 1415–1424.
doi:10.1016/j.disc.2007.07.074

[6] G.Y. Katona and I. Horváth, *Extremal P_4-stable graphs*, Discrete Appl. Math. **159** (2011) 1786–1792.
doi:10.1016/j.dam.2010.11.016

[7] J.J. Sylvester, Question 7382, Mathematical Questions from the Educational Times, (1884) 41:21.

[8] A. Żak, *On $(K_q; k)$-stable graphs*, J. Graph Theory, (2012),to appear.
doi:/10.1002/jgt.21705

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