

## ON MINIMUM $(K_q, k)$ STABLE GRAPHS

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### Abstract

A graph  $G$  is a  $(K_q, k)$  stable graph ( $q \geq 3$ ) if it contains a  $K_q$  after deleting any subset of  $k$  vertices ( $k \geq 0$ ). Andrzej Żak in the paper *On  $(K_q; k)$ -stable graphs*, (doi:10.1002/jgt.21705) has proved a conjecture of Dudek, Szymański and Zwonek stating that for sufficiently large  $k$  the number of edges of a minimum  $(K_q, k)$  stable graph is  $(2q - 3)(k + 1)$  and that such a graph is isomorphic to  $sK_{2q-2} + tK_{2q-3}$  where  $s$  and  $t$  are integers such that  $s(q - 1) + t(q - 2) - 1 = k$ . We have proved (Fouquet *et al.* *On  $(K_q, k)$  stable graphs with small  $k$* , Elektron. J. Combin. **19** (2012) #P50) that for  $q \geq 5$  and  $k \leq \frac{q}{2} + 1$  the graph  $K_{q+k}$  is the unique minimum  $(K_q, k)$  stable graph. In the present paper we are interested in the  $(K_q, \kappa(q))$  stable graphs of minimum size where  $\kappa(q)$  is the maximum value for which for every nonnegative integer  $k < \kappa(q)$  the only  $(K_q, k)$  stable graph of minimum size is  $K_{q+k}$  and by determining the exact value of  $\kappa(q)$ .

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## 1. INTRODUCTION

For terms not defined here we refer to [1]. As usual, the *order* of a graph  $G$  is the number of its vertices and the *size* of  $G$  is the number of its edges (denoted by  $e(G)$ ). The disjoint union of two graphs  $G_1$  and  $G_2$  is denoted by  $G_1 + G_2$ . The union of  $p$  mutually disjoint copies of a graph  $G$  is denoted by  $pG$ . For any set  $A$  of vertices, we denote by  $G[A]$  the subgraph induced by  $A$  and by  $G - A$  the subgraph induced by  $V(G) - A$ . If  $A = \{v\}$  we write  $G - v$  for  $G - \{v\}$ . For any set  $F$  of edges, we denote by  $G - F$  the spanning subgraph  $(V(G), E(G) - F)$ . If  $F = \{e\}$  we write  $G - e$  instead of  $G - \{e\}$ . A complete subgraph of order  $q$  of  $G$  is called a  $q$ -*clique* of  $G$ . The complete graph of order  $q$  is denoted by  $K_q$ . When a graph  $G$  contains a  $q$ -clique as subgraph, we say “ $G$  contains a  $K_q$ ”.

In [6] Horváth and Katona considered the notion of  $(H, k)$  *edge stable* graph<sup>2</sup>: given a simple graph  $H$ , an integer  $k$  and a graph  $G$  containing  $H$  as subgraph,  $G$  is an  $(H, k)$  *edge stable graph* whenever the deletion of any set of  $k$  edges does not lead to an  $H$ -free graph. These authors consider  $(P_n, k)$  edge stable graphs and prove a conjecture stated in [5] on the minimum size of a  $(P_4, k)$  edge stable graph. In [2], Dudek, Szymański and Zwonek investigated the vertex version of this notion and introduced the  $(H, k)$  *stable graphs*.

**Definition 1.1** [2]. Given an integer  $k \geq 0$  and a graph  $H$  without isolated vertices, a graph  $G$  containing a subgraph isomorphic to  $H$  is said to be an  $(H, k)$  *stable graph* if, for every subset  $S$  of  $k$  vertices,  $G - S$  contains (a subgraph isomorphic to)  $H$ .

**Definition 1.2.** A  $(H, k)$  stable graph with minimum size is called a *minimum  $(H, k)$  stable graph*. The size of a minimum  $(H, k)$  stable graph shall be denoted by  $stab(H, k)$ .

Note that if  $G$  is an  $(H, k)$  stable graph with minimum size then the graph obtained from  $G$  by addition or deletion of some isolated vertices is also minimum  $(H, k)$  stable. Hence we shall assume that all the graphs considered in the paper have no isolated vertices. It is clear that  $H$  is the unique  $(H, 0)$  stable graph with minimum size.

In this paper we consider  $(K_q, k)$  stable graphs with  $q \geq 2$ . Since  $K_{q+k}$  is  $(K_q, k)$  stable, note that a trivial upper bound for  $stab(K_q, k)$  is  $\binom{q+k}{2}$ . It is an easy exercise to see that  $stab(K_2, k) = k + 1$  and that the matching  $(k + 1)K_2$  is the unique minimum  $(K_2, k)$  stable graph.

Dudek, Szymański and Zwonek have proved in [2] that  $stab(K_3, k) = 3(k + 1)$  for  $k \geq 0$  and  $stab(K_4, k) = 5(k + 1)$  for  $k \geq 1$  and they have obtained an upper

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<sup>2</sup>In the original paper [6] these graphs are just called  $(H, k)$  stable by the authors.

bound for  $\text{stab}(K_q, k)$  for sufficiently large  $k$ . More precisely, they have obtained the following result.

**Theorem 1.3** [2]. *For every  $q \geq 5$ , there exists an integer  $k(q)$  such that for every  $k \geq k(q)$ ,  $\text{stab}(K_q, k) \leq (2q - 3)(k + 1)$ .*

In order to obtain Theorem 1.3, the authors consider the graph  $G = sK_{2q-2} + (r - s)K_{2q-3}$  with  $q \geq 5$ ,  $k \geq (q - 1)(q - 2)$ ,  $r \in \{1, \dots, k + 1\}$ ,  $s \in \{0, \dots, r\}$  and  $r(q - 2) + s - 1 = k$  and note that the number of edges of  $G$  is  $(2q - 3)(k + 1)$ . A smaller bound for  $k(q)$  can be obtained by the following Proposition 1.4 (a consequence of an old result of Sylvester [7]; see a proof at the end of Section 2), and more generally apart from  $k \in \{0, \dots, q - 4\}$ , Theorem 1.6 below gives a better upper bound than  $\binom{q+k}{2}$  for  $\text{stab}(K_q, k)$ .

**Proposition 1.4.** *Let  $q \geq 4$  be an integer. Set*

$$A(q) = \bigcup_{0 \leq i \leq q-4} \{i(q-1) + j \mid 0 \leq j \leq q-4-i\}$$

and

$$B(q) = \{b \in \mathbb{N} \mid 0 \leq b \leq (q-2)(q-3) - 2\} - A(q).$$

*Let  $k$  be a nonnegative integer. There exist integers  $s$  and  $t$  such that  $s(q-1) + t(q-2) - 1 = k$  if and only if  $k \in B(q)$  or  $k \geq k(q) = (q-3)(q-2) - 1$ . For such a pair  $(s, t)$ ,  $G = sK_{2q-2} + tK_{2q-3}$  is  $(K_q, k)$  stable and  $e(G) = (2q-3)(k+1)$ .*

Note that  $|A(q)| = \frac{(q-3)(q-2)}{2}$  and  $|B(q)| = |A(q)| - 1$ .

**Lemma 1.5.** *Let  $q \geq 4$  and  $k \geq 0$  be integers. Then  $k \in A(q)$  if and only if  $\left[\frac{k+1}{q-1}, \frac{k+1}{q-2}\right]$  contains no integer.*

**Theorem 1.6.** *Let  $q \geq 3$  and  $k \geq 0$  be integers. Set  $A(3) = B(3) = \emptyset$ , and for  $q \geq 4$ ,  $A(q)$  and  $B(q)$  are the sets defined in Proposition 1.4. For every positive integer  $r$  set*

$$\phi(r) = \frac{1}{2} \left( q - 1 + \left\lfloor \frac{k+1}{r} \right\rfloor \right) \left( \left( q - 2 - \left\lfloor \frac{k+1}{r} \right\rfloor \right) r + 2(k+1) \right).$$

*Then,  $\text{stab}(K_q, k)$  is at most equal to*

- $\phi(1) = \frac{1}{2}(q+k-1)(q+k)$  if  $k \leq q-4$  (note that  $k$  is in  $A(q)$ ),
- $\min\{\phi(\lfloor \frac{k+1}{q-1} \rfloor), \phi(\lfloor \frac{k+1}{q-1} \rfloor + 1)\}$  if  $k \in A(q)$  and  $k \geq q-1$ ,
- $(2q-3)(k+1)$  if  $k \in B(q)$  or  $k \geq k(q) = (q-3)(q-2) - 1$  (note that  $\phi(r) = (2q-3)(k+1)$  for every integer  $r \in [\frac{k+1}{q-1}, \frac{k+1}{q-2}]$ ).

We shall give a proof of Theorem 1.6 in Section 3 by considering  $(K_q, k)$  stable graphs having cliques as components and having the minimum number of edges. As a consequence, if every component of a minimum  $(K_q, k)$  stable graph is

complete (see Problem 1.15) then the upper bound given in Theorem 1.6 is the exact value for  $\text{stab}(K_q, k)$ .

In light of their results, Dudek, Szymański and Zwonek propose the following conjecture.

**Conjecture 1.7** [2]. *There exists an integer  $k(q)$  such that for every  $k \geq k(q)$ ,  $\text{stab}(K_q, k) = (2q - 3)(k + 1)$ .*

Note that Conjecture 1.7 is true for  $q \in \{3, 4\}$ . In [4] we have proved that  $\text{stab}(K_5, k) = 7(k + 1)$  for  $k \geq 5$ , which confirms Conjecture 1.7 for  $q = 5$ . Moreover, we have characterized  $(K_q, k)$  stable graphs with minimum size for  $q \in \{3, 4, 5\}$ . The following theorem summarizes these results.

**Theorem 1.8** [4]. *Let  $G$  be a minimum  $(K_q, k)$  stable graph, with  $q \in \{3, 4, 5\}$  and  $k \geq k(q)$  with  $k(3) = 0$ ,  $k(4) = 1$ ,  $k(5) = 5$ . Then  $G = sK_{2q-2} + tK_{2q-3}$ , for any choice of  $s$  and  $t$  such that  $s(q - 1) + t(q - 2) - 1 = k$ . Moreover,  $K_{5+k}$  is the unique minimum  $(K_5, k)$  stable graph for  $k \in \{1, 2, 3\}$ ,  $K_9$  and  $K_6 + K_7$  are the only minimum  $(K_5, 4)$  stable graphs.*

An important fact is that Conjecture 1.7 of Dudek, Szymański and Zwonek has been recently solved by Žak [8], who has characterized also the extremal graphs.

**Theorem 1.9** [8]. *Let  $q \geq 2$ ,  $k \geq 0$  be nonnegative integers. Then  $\text{stab}(K_q, k) \geq (2q - 3)(k + 1)$ , with equality if and only if  $k = s(q - 1) + t(q - 2) - 1$  for some nonnegative integers  $s$  and  $t$ . In particular,  $\text{stab}(K_q, k) = (2q - 3)(k + 1)$  for  $k \geq (q - 3)(q - 2) - 1$ . Furthermore, if  $G$  is a  $(K_q, k)$  stable graph having exactly  $(2q - 3)(k + 1)$  edges, then  $G = sK_{2q-2} + tK_{2q-3}$  where  $s$  and  $t$  are nonnegative integers such that  $s(q - 1) + t(q - 2) - 1 = k$ .*

**Remark 1.10.** Since  $(K_q, k)$  stable graphs with minimum size for  $q \in \{3, 4, 5, 6\}$  have been characterized (see Theorem 1.8 for  $q \leq 5$  and [8] for  $q = 6$ ), to close the study of minimum  $(K_q, k)$  stable graphs we have only to consider  $q \geq 7$  and  $k \in A(q)$  (the set defined in Proposition 1.4).

We have proved in [4] that  $K_{q+k}$  is the unique minimum  $(K_q, k)$  stable graph for  $q \geq 4$  and  $k \in \{1, 2\}$ , that  $K_{q+3}$  is the unique minimum  $(K_q, 3)$  stable graph for  $q \geq 5$  and in [3] that  $K_{q+k}$  is the unique  $(K_q, k)$  stable graph for  $q \geq 6$  and  $k \leq \frac{q}{2} + 1$ . Remark that  $\binom{q+k}{2} - (2q - 3)(k + 1) = \frac{(q-k-3)(q-k-2)}{2}$  and that this integer is positive for  $q \geq 3$  and  $k \notin \{q - 3, q - 2\}$ . Then, as a consequence of Proposition 1.4, for  $q \geq 4$  and for every integer  $k$  for which  $k \in B(q) - \{q - 3, q - 2\}$  or  $k \geq (q - 3)(q - 2) - 1$  the graph  $K_{q+k}$  is not minimum  $(K_q, k)$  stable. Hence, the set  $\{k \in \mathbb{N} \mid K_{q+k} \text{ is minimum } (K_q, k) \text{ stable}\}$  is bounded above, and we propose the following definition.

**Definition 1.11.** For every integer  $q \geq 4$ , we denote by  $\kappa(q)$  the greatest integer such that for  $1 \leq k < \kappa(q)$  the only minimum  $(K_q, k)$  stable graph is  $K_{q+k}$ .

We will focus our attention on determining the exact value of  $\kappa(q)$ . In two previous papers we have proved the following.

**Theorem 1.12** [3, 4].  $\kappa(3) = 1$ ,  $\kappa(4) = 3$ ,  $\kappa(5) = 4$  and for  $q \geq 6$ ,  $\kappa(q) > \frac{q}{2} + 1$ .

In this paper we give an upper bound for the value of  $\kappa(q)$ .

**Theorem 1.13.** For every  $q \geq 4$ , if  $\kappa(q)$  is even, then  $\kappa(q) < \sqrt{2(q-1)(q-2)}$  and if  $\kappa(q)$  is odd, then  $\kappa(q) < \sqrt{1 + 2(q-1)(q-2)}$ .

We prove that these upper bounds are reached for values of  $q$  such that there exists a minimum  $(K_q, \kappa(q))$  stable disconnected graph (note that it is the case for  $q = 4$  and  $q = 5$ ).

**Theorem 1.14.** Let  $q \geq 4$  and suppose that there exists a disconnected minimum  $(K_q, \kappa(q))$  stable graph. Set  $\rho(q) = \left\lceil \sqrt{\frac{1}{2}(q-1)(q-2)} \right\rceil - 1$ .

If  $\frac{1}{2}(q-1)(q-2) > \rho(q)^2 + \rho(q)$ , then  $\kappa(q) = 2\rho(q) + 1$ .

If  $\frac{1}{2}(q-1)(q-2) \leq \rho(q)^2 + \rho(q)$ , then  $\kappa(q) = 2\rho(q)$ .

Proofs of Theorems 1.13 and 1.14 shall be given in Subsection 3.3.

Remark that, by definition of  $\kappa(q)$  and by Theorem 1.9, for every integer  $k$  in  $\{l \in \mathbb{N} \mid 0 \leq l < \kappa(q) \text{ or } l \geq (q-2)(q-3) - 1\} \cup B(q)$  every component of any minimum  $(K_q, k)$  stable graph is complete, but we do not know if it is true for  $k$  in  $\{l \in \mathbb{N} \mid l \geq \kappa(q) \text{ and } l \in A(q)\}$  (where  $A(q)$  and  $B(q)$  are the sets defined in Proposition 1.4).

If there is no minimum disconnected  $(K_q, \kappa(q))$  stable graph then, by definition of  $\kappa(q)$ , there exists a connected minimum  $(K_q, \kappa(q))$  stable graph  $G_q$  which is not complete. We think that it never happens, so we propose the following problem.

**Problem 1.15.** Is it true that if  $G$  is a minimum  $(K_q, k)$  stable graph, then every component of  $G$  is complete?

If the answer is positive then Theorem 1.14 gives the exact value of  $\kappa(q)$  for every  $q \geq 4$ .

## 2. GENERAL RESULTS

**Lemma 2.1** [2]. Let  $G$  be an  $(H, k)$  stable graph with  $k \geq 1$ . Then, for every vertex  $v$ ,  $G - v$  is  $(H, k-1)$  stable.

A set of vertices of  $G$  that intersects every subgraph of  $G$  isomorphic to  $H$  is called a *transversal of all the subgraphs isomorphic to  $H$*  or simply an  *$H$ -transversal* of  $G$ . An  $H$ -transversal of  $G$  having the minimum number of vertices is said to be a *minimum  $H$ -transversal* of  $G$ . The number of vertices of a minimum  $H$ -transversal is denoted by  $\tau_H(G)$ . Remark that  $G$  is  $(H, k)$  stable if and only if  $\tau_H(G) \geq k + 1$ .

**Definition 2.2.** Let  $G$  be an  $(H, k)$  stable graph. If  $G$  has a minimum  $H$ -transversal having exactly  $k + 1$  vertices,  $G$  is said to be *exactly  $(H, k)$  stable*.

**Lemma 2.3** [2]. Let  $G$  be an  $(H, k)$  stable graph with  $k \geq 1$  and  $e \in E(G)$  such that  $G - e$  is not  $(H, k)$  stable. Then  $G$  is exactly  $(H, k)$  stable and  $G - e$  is exactly  $(H, k - 1)$  stable.

**Definition 2.4** [2]. Let  $G$  be an  $(H, k)$  stable graph. If  $G - e$  is not  $(H, k)$  stable for every edge  $e \in E(G)$ , then  $G$  is said to be *minimal  $(H, k)$  stable*.

**Remark 2.5.** In [2] “minimal  $(H, k)$  stable graphs” are called “strong  $(H, k)$  stable graphs” by the authors. Note that an  $(H, k)$  stable graph  $G$  is minimal  $(H, k)$  stable if and only if for every  $e \in E(G)$  the graph  $G - e$  is exactly  $(H, k - 1)$  stable. Moreover, a minimal  $(H, k)$  stable graph is exactly  $(H, k)$  stable.

If there exists an edge  $e$  of an  $(H, k)$  stable graph  $G$  such that there are no subgraphs isomorphic to  $H$  containing  $e$ , then  $G - e$  is an  $(H, k)$  stable graph. Hence, we have the following.

**Lemma 2.6** [2]. Every edge of a minimal  $(H, k)$  stable graph is contained in a subgraph isomorphic to  $H$ . Consequently, every vertex of a minimal  $(H, k)$  stable graph is also contained in a subgraph isomorphic to  $H$ .

**Remark 2.7.** Clearly, every minimum  $(H, k)$  stable graph is minimal  $(H, k)$  stable.

One may ask what happens for components of an  $(H, k)$  stable graph. The following theorem gives us an answer when  $H$  is connected. We shall say that a graph containing no subgraph isomorphic to  $H$  is  $(H, -1)$  stable.

**Theorem 2.8.** Let  $H$  be a connected graph containing at least 2 vertices, let  $G$  be an exactly  $(H, k)$  stable graph, and let  $G_1, G_2, \dots, G_r$ , with  $r \geq 1$ , be its components. Then, there exist integers  $k_1, k_2, \dots, k_r$ , with  $0 \leq k_i \leq k$ , such that

(i) for every  $i$ , with  $1 \leq i \leq r$ ,  $G_i$  is exactly  $(H, k_i)$  stable,

(ii)  $\sum_{i=1}^r k_i + (r - 1) = k$ ,

$G$  is minimal  $(H, k)$  stable if and only if for every  $i$ ,  $1 \leq i \leq r$ ,  $G_i$  is minimal  $(H, k_i)$  stable. Moreover, if  $G$  is minimum  $(H, k)$  stable, then for every  $i$ ,  $1 \leq i \leq r$ ,  $G_i$  is minimum  $(H, k_i)$  stable.

**Proof.** For each  $i$ ,  $1 \leq i \leq r$ , let us consider a minimum  $H$ -transversal of  $G_i$ , say  $T_i$ , and set  $k_i = |T_i| - 1$ . Clearly, for each  $i$  the graph  $G_i$  is exactly  $(H, k_i)$  stable and the set  $T = \bigcup_{1 \leq i \leq r} T_i$  is a minimum  $H$ -transversal of  $G$ . Note that the number of elements of  $T$  is  $|T| = \sum_{i=1}^r k_i + r$  and we have  $|T| > k$ . Let  $S$  be any set of vertices of  $G$  such that  $|S| \leq |T| - 1$  and for every  $i$  denote by  $S_i$  the set  $S \cap V(G_i)$ . Clearly, there exists  $i_0 \in \{1, \dots, r\}$  such that  $|S_{i_0}| \leq k_{i_0} = |T_{i_0}| - 1$ . Then,  $G_{i_0} - S_{i_0}$  contains a subgraph isomorphic to  $H$ , that is,  $G$  is exactly  $(H, |T| - 1)$  stable, and we have  $\sum_{i=1}^r k_i + (r - 1) = k$ .

Let  $e$  be an edge of  $G$  and let  $G_i$  be the component containing  $e$ .

**Claim.**  $G - e$  is  $(H, k)$  stable if and only if  $G_i - e$  is  $(H, k_i)$  stable.

**Proof.** Suppose that  $G_i - e$  is  $(H, k_i)$  stable. Let  $U$  be an  $H$ -transversal of  $G - e$ . Set  $U_i = U \cap V(G_i - e) = U \cap V(G_i)$  and for every  $j \neq i$ ,  $U_j = U \cap V(G_j)$ . Since  $(G_i - e) - U_i$  and each  $G_j - U_j$ ,  $j \neq i$ , contain no subgraphs of  $G - e$  isomorphic to  $H$ , we have for every  $j$ ,  $1 \leq j \leq r$ ,  $|U_j| \geq k_j + 1$ . Then,  $|U| = \sum_{j=1}^r |U_j| \geq k + 1$ . Hence, for every set  $S$  of  $k$  vertices  $(G - e) - S$  contains a subgraph isomorphic to  $H$ , that is,  $G - e$  is  $(H, k)$  stable.

Conversely, suppose that  $G_i - e$  is not  $(H, k_i)$  stable. Let  $T_i$  be an  $H$ -transversal of  $(G_i - e) - T_i$  having  $k_i$  vertices. For every  $j \neq i$  let  $T_j$  be an  $H$ -transversal of  $G_j$  having  $k_j + 1$  vertices. The set  $T = \bigcup_{j=1}^r T_j$  has  $k$  vertices and is a  $H$ -transversal of  $G - e$ , that is,  $G - e$  is not  $(H, k)$  stable.  $\square$

Thus,  $G$  is minimal  $(H, k)$  stable if and only if for every  $i$ ,  $1 \leq i \leq r$ ,  $G_i$  is minimal  $(H, k_i)$  stable.

Note that, by replacing a minimal  $(H, k_i)$  stable component  $G_i$  by any minimal  $(H, k_i)$  stable graph  $G'_i$  (connected or not), we obtain again a minimal  $(H, k)$  stable graph. Thus, if  $G$  is minimum  $(H, k)$  stable then for every  $i$ ,  $1 \leq i \leq r$ ,  $G_i$  is minimum  $(H, k_i)$  stable.  $\blacksquare$

**Remark 2.9.** Let  $r \geq 2$  be an integer,  $k_1, \dots, k_r$  be  $r$  non negative integers and  $k = \sum_{i=1}^r k_i + (r - 1)$ . If for every  $i$ ,  $1 \leq i \leq r$ ,  $G_i$  is a minimum  $(H, k_i)$  stable graph then the disjoint union  $G_1 + G_2 + \dots + G_r$  may not be a minimum  $(H, k)$  stable graph. For example,  $K_q$  is minimum  $(K_q, 0)$  stable,  $2K_q$  and  $K_{q+1}$  are minimal  $(K_q, 1)$  stable, but for  $q \geq 4$  since  $e(2K_q) > e(K_{q+1})$ , the graph  $2K_q$  is not minimum  $(K_q, 1)$  stable.

Given relatively prime positive integers  $a_1, \dots, a_n$ , let us consider the integers that can be expressed as a sum  $k_1 a_1 + k_2 a_2 + \dots + k_n a_n$ , where  $k_1, k_2, \dots, k_n$  are nonnegative integers. Any such integer is said to be *representable*. Recall that the *Frobenius Problem* is the following: find the largest non-representable integer (called the *Frobenius number* and denoted by  $g(a_1, \dots, a_n)$ ). If  $n = 2$ , the Frobenius number is given by the formula  $g(a_1, a_2) = a_1 a_2 - a_1 - a_2$ . This formula



was discovered by Sylvester in 1884 [7], who also demonstrated that there are a total of  $N(a_1, a_2) = \frac{(a_1-1)(a_2-1)}{2}$  non-representable integers. For the particular case  $a_2 = a_1 - 1$  one obtains explicitly the set of non-representable integers.

**Lemma 2.10** [7]. *Let  $a \geq 3$  be an integer and the function  $\alpha : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  such that  $\alpha(s, t) = sa + t(a - 1)$ . Set  $A = \bigcup_{0 \leq i \leq a-3} \{ia + j \mid 1 \leq j \leq a - 2 - i\}$ . Every  $b \in \mathbb{N} - A$  is representable (that is, there exists a pair  $\{s, t\}$  of nonnegative integers such that  $b = sa + t(a - 1)$ ), and every  $b$  in  $A$  is not representable. Moreover, every representable  $b$  has a unique representation  $sa + t(a - 1)$  such that  $0 \leq t \leq a - 1$ .*

We shall give a proof of Lemma 2.10 for completeness.

**Proof of Lemma 2.10.** Note that  $\max(A) = (a-1)(a-2)-1$ ,  $|A| = \frac{(a-1)(a-2)}{2}$  and for  $s \geq 0$  and  $t \geq 1$ ,  $\alpha(s, t) = \alpha(s+1, t-1) - 1$ .

Consider the infinite matrix  $\{\alpha(s, t)\}_{s \geq 0, t \geq 0}$ . For any  $t \geq 0$  the values of the diagonal  $\{\alpha(i, t-i) \mid 0 \leq i \leq t\}$  are the consecutive integers  $\{t(a-1) + i \mid 0 \leq i \leq t\}$ . For  $s \geq 0$ , the values of the (partial) diagonal  $\{\alpha(s+i, a-i-1) \mid 0 \leq i \leq a-1\}$  are the consecutive integers  $sa + (a-1)^2, sa + (a-1)^2 + 1, \dots, sa + a(a-1)$ .

Since  $\alpha(0, a-1) = \alpha(a-2, 0) + 1$  and for every  $s \geq 0$ ,  $\alpha(s+a-1, 0) + 1 = \alpha(s+1, a-1) = sa + a(a-1) + 1$ , every integer  $b \geq (a-2)(a-1)$  appears in

$$\{\alpha(i, a-2-i) \mid 0 \leq i \leq a-2\} \cup \bigcup_{s \geq 0} \{\alpha(s+i, a-i-1) \mid 0 \leq i \leq a-1\}.$$

Let  $B = \bigcup_{0 \leq i \leq a-3} \{\alpha(j, i-j) \mid 0 \leq j \leq i\} = \bigcup_{0 \leq i \leq a-3} \{i(a-1) + j \mid 0 \leq j \leq i\}$ . Clearly  $|B| = |A|$ . It is easy to check that  $A$  and  $B$  are disjoint sets and that  $A \cup B = \{0, 1, \dots, (a-2)(a-1) - 1\}$ . Thus, every  $b \in A$  is not representable and for every integer  $b \in \mathbb{N} - A$  there exists a unique pair  $(s, t)$  with  $s \geq 0$  and  $0 \leq t \leq a-1$  such that  $b = sa + t(a-1)$ . ■

**Remark 2.11.** It is easy to see that every representable  $b < a(a-1)$  has a unique representation. For a representable  $b \geq a(a-1)$ , since we can choose values of  $t \geq a$ , it is possible that  $b = \alpha(s, t) = \alpha(s', t')$  for distinct pairs  $(s, t)$  and  $(s', t')$ . Indeed, if  $s \geq a-1$ , then for every positive integer  $r \leq \lfloor \frac{s}{a-1} \rfloor$ ,  $\alpha(s, t) = \alpha(s-r(a-1), ra+t)$ .

**Proof of Proposition 1.4.** Let us apply Lemma 2.10 to  $a = q-1$  and  $b = k+1$ .  $B(q)$  is the set of integers  $k \leq (q-3)(q-2)-3$  such that  $k+1$  is representable as  $s(q-1) + t(q-2)$ . More precisely,  $B(q) = \bigcup_{1 \leq i \leq q-4} \{i(q-2) + j - 1 \mid 0 \leq j \leq i\}$ .

It is easy to see that the set of integers  $k$  such that  $k+1$  is not representable as  $s(q-1) + t(q-2)$  is  $A(q) = \bigcup_{0 \leq i \leq q-4} \{i(q-1) + j \mid 0 \leq j \leq q-4-i\}$ .

A minimum  $K_q$ -transversal of  $G = sK_{2q-2} + tK_{2q-3}$  contains exactly  $s(q-1) + t(q-2) = k+1$  vertices, that is  $G$  is  $(K_q, k)$  stable, and it is easy to check that  $e(G) = (2q-3)(k+1)$ . ■



**Proof of Lemma 1.5.** If there exist integers  $s$  and  $t$  such that  $s(q-1) + t(q-2) = k+1$  then  $\frac{k+1}{q-1} = s + t - \frac{t}{q-1}$  and  $\frac{k+1}{q-2} = s + t + \frac{s}{q-2}$ , and hence  $r = s + t \in [\frac{k+1}{q-1}, \frac{k+1}{q-2}]$ . Conversely, let  $r \in [\frac{k+1}{q-1}, \frac{k+1}{q-2}]$ . Then  $q-2 \leq \frac{k+1}{r} \leq q-1$ . If  $k+1 = r(q-1)$  then we are done. If  $\frac{k+1}{q-1} < r$  then  $q-2 = \lfloor \frac{k+1}{r} \rfloor$  is the quotient in the division of  $k+1$  by  $r$ . Hence, if  $s$  denotes the remainder, then  $k+1 = r(q-2) + s = s(q-1) + (r-s)(q-2)$ . We conclude by applying Proposition 1.4. ■

### 3. MINIMUM $(K_q, k)$ STABLE GRAPHS

In this section we are interested in  $(K_q, k)$  stable graphs with minimum size ( $q \geq 3$ ). Recall that  $\text{stab}(K_q, k) = \min\{e(G) \mid G \text{ is } (K_q, k) \text{ stable}\}$ .

#### 3.1. Some known results

We give here some known results about this topic.

By Remark 2.5 and Lemma 2.6 we have:

**Properties 3.1** [2]. *A minimal  $(K_q, k)$  stable graphs  $G$  has the following properties:*

- (P<sub>1</sub>)  $G$  is exactly  $(K_q, k)$  stable.
- (P<sub>2</sub>) For every edge  $e$ ,  $G - e$  is exactly  $(K_q, k-1)$  stable.
- (P<sub>3</sub>) For every vertex  $v$ ,  $G - v$  is exactly  $(K_q, k-1)$  stable.
- (P<sub>4</sub>) Every vertex of  $G$  belongs to some  $q$ -clique of  $G$ .
- (P<sub>5</sub>) Every edge of  $G$  belongs to some  $q$ -clique of  $G$ .

**Remark 3.2.** For any two integers  $q \geq 3$  and  $k \geq 1$ ,  $K_{q+k}$  is minimal  $(K_q, k)$  stable.

**Proposition 3.3** [4].  $K_5$  is the unique minimum  $(K_4, 1)$  stable graph,  $K_6$  is the unique minimum  $(K_4, 2)$  stable graph and for every integer  $q \geq 5$  and every integer  $k \in \{1, 2, 3\}$ ,  $K_{q+k}$  is the unique minimum  $(K_q, k)$  stable graph.

Dudek *et al.* [2] defined the family  $\mathcal{A}_r^{(K_q, k)}$  with  $k \geq 0$ ,  $q \geq 3$ ,  $1 \leq r \leq k+1$  as the family of graphs consisting of  $r$  complete graphs  $K_{i_j}$  with  $i_1 \geq \dots \geq i_r \geq q$  satisfying the condition  $\sum_{i=1}^r (i_j - q) + (r-1) = k$  and they proved that every graph in  $\mathcal{A}_r^{(K_q, k)}$  is minimal  $(K_q, k)$  stable. We observe that if a  $(K_q, k)$  stable graph  $G$  is a disjoint union of  $r \geq 1$  cliques  $K_{i_j}$ ,  $1 \leq j \leq r$ , then by Theorem 2.8,  $G \in \mathcal{A}_r^{(K_q, k)}$ . They defined a graph  $G \in \mathcal{A}_r^{(K_q, k)}$  as a *balanced union* if  $|i_j - i_l| \in \{0, 1\}$  for every  $j$  and  $l$  in  $\{1, 2, \dots, r\}$  and they proved that given  $q$ ,

$k$  and  $r$  there is exactly one balanced union  $\mathcal{B}_r^{(K_q, k)}$  in  $\mathcal{A}_r^{(K_q, k)}$ , and that  $\mathcal{B}_r^{(K_q, k)}$  has the minimum number of edges among the graphs in  $\mathcal{A}_r^{(K_q, k)}$ .

In [2] the following lemma has been given. We give its proof for completeness.

**Lemma 3.4** [2]. *Let  $G_0$  be a  $(K_q, k_0)$  stable graph ( $k_0 \geq 0$ ) which has the minimum size among all graphs beeing a disjoint union of  $r$  cliques ( $r \geq 1$ ),  $G_j \equiv K_{q+k_j}$  with  $1 \leq j \leq r$ ,  $k_j \geq 0$ . There exist nonnegative integers  $s$  and  $k$  such that  $0 \leq s \leq r-1$ ,  $G_0 = sK_{q+k+1} + (r-s)K_{q+k}$  with  $r(k+1) + s = k_0 + 1$  and  $e(G_0) = \frac{1}{2r}(r(q-1) + k_0 + 1 - s)(r(q-2) + k_0 + 1 + s)$ .*

**Proof.** Suppose, without loss of generality, that  $k_1 \geq k_2 \geq \dots \geq k_r$  and that there exist two components  $G_i$  and  $G_j$  with  $i < j$  such that  $k_i - k_j \geq 2$ . By substituting  $G'_i \equiv K_{q+k_i-1}$  for  $G_i$  and  $G'_j \equiv K_{q+k_j+1}$  for  $G_j$ , we obtain a new  $(K_q, k)$  stable graph  $G'_0$  such that  $e(G'_0) = e(G_0) - (k_i - k_j - 1) < e(G_0)$ , which is a contradiction. Thus, for any  $i$  and any  $j$ ,  $0 \leq |k_i - k_j| \leq 1$ . Hence, either for any  $i$  and any  $j$ ,  $k_i$  and  $k_j$  have the same value  $k$  and we have  $G_0 = rK_{q+k}$  with  $k \geq 0$ , or there exist distinct  $k_i$  and  $k_j$  and we have  $G_0 = sK_{q+k+1} + (r-s)K_{q+k}$  with  $k \geq 0$  and  $0 \leq s \leq r-1$ . Hence, a minimum  $K_q$ -transversal of  $G_0$  has  $k_0 + 1 = s(k+2) + (r-s)(k+1) = s + r(k+1)$  vertices. Note that  $r$  divides  $k_0 + 1 - s$ . We have  $2e(G_0) = s(q+k+1)(q+k) + (r-s)(q+k)(q+k-1)$ . Since  $k+1 = \frac{k_0+1-s}{r}$ , we obtain  $e(G_0) = \frac{1}{2r}(r(q-1) + k_0 + 1 - s)(r(q-2) + k_0 + 1 + s)$ . ■

**Remark 3.5.** In Lemma 3.4 the integers  $q$ ,  $k_0$  and  $r$  are given. Given  $q$  and  $k_0$ , in order to obtain an upper bound for  $stab(K_q, k_0)$  we will determine the values of  $r$  for which  $e(G_0(r)) = \frac{1}{2r}(r(q-1) + k_0 + 1 - s)(r(q-2) + k_0 + 1 + s)$  is minimum. We note that if every component of a minimum  $(K_q, k_0)$  stable graph is complete then the minimum value of  $e(G_0(r))$  is exactly  $stab(K_q, k_0)$ .

### 3.2. Proof of Theorem 1.6

First we give a technical lemma used for proving Theorem 1.6.

**Lemma 3.6.** *Let  $a$  and  $b$  be positive integers and for  $x > 0$  consider the real-to-real function*

$$f(x) = \frac{1}{2} \left( a + 1 + \left\lfloor \frac{b}{x} \right\rfloor \right) \left( \left( a - \left\lfloor \frac{b}{x} \right\rfloor \right) x + 2b \right).$$

*Then,  $f$  is continuous on  $(0, +\infty)$ , nonincreasing on  $(0, \frac{b}{a+1}]$ , constant on  $[\frac{b}{a+1}, \frac{b}{a}]$  and nondecreasing on  $[\frac{b}{a}, +\infty)$ . Moreover  $\min\{f(r) \mid r \in \mathbb{N} - \{0\}\}$  is equal to*

- $f(1) = \frac{1}{2}(a+b+1)(a+b)$  if  $[\frac{b}{a+1}, \frac{b}{a}]$  contains no integer and  $b < a$ ,
- $\min\{f(\lfloor \frac{b}{a+1} \rfloor), f(\lfloor \frac{b}{a+1} \rfloor + 1)\}$  if  $[\frac{b}{a+1}, \frac{b}{a}]$  contains no integer and  $b > a+1$ ,

- $(2a+1)b$  if  $[\frac{b}{a+1}, \frac{b}{a}]$  contains at least one integer  $r$  (and is equal to  $f(r)$  for every such  $r$ ).

**Proof.** For  $x > b$  we have  $\lfloor \frac{b}{x} \rfloor = 0$  and  $f(x) = \frac{1}{2}(a+1)(ax+2b)$ . For every integer  $p \geq 1$  and for every  $x \in [\frac{b}{p+1}, \frac{b}{p}]$  we have  $\lfloor \frac{b}{x} \rfloor = p$ , and hence  $f(x) = \frac{1}{2}(a+1+p)((a-p)x+2b)$ . It is easy to see that the function  $f$  is continuous on  $(0, +\infty)$ , nonincreasing on  $[0, \frac{b}{a+1}]$ , constant on  $[\frac{b}{a+1}, \frac{b}{a}]$  and nondecreasing on  $[\frac{b}{a}, +\infty)$ . The minimum value for  $f(x)$  (with a  $x$  positive real number) is the integer  $(2a+1)b$  and is reached for every real number  $x$  in  $[\frac{b}{a+1}, \frac{b}{a}]$ . We note that if  $r$  is a positive integer, then  $f(r)$  is a positive integer.

Now we will find the minimum value of  $f(r)$  when  $r$  is a positive integer.

*Case 1.*  $[\frac{b}{a+1}, \frac{b}{a}] \cap \mathbb{N} = \emptyset$ . Note that  $0 < \frac{b}{a} - \frac{b}{a+1} < 1$  (that is,  $0 < b < a(a+1)$ ),  $0 \leq \lfloor \frac{b}{a+1} \rfloor \leq a$  and  $\lfloor \frac{b}{a+1} \rfloor < \frac{b}{a+1} < \frac{b}{a} < \lceil \frac{b}{a} \rceil = \lfloor \frac{b}{a+1} \rfloor + 1$ .

*Case 1.1.*  $b < a$ . Since  $\lceil \frac{b}{a} \rceil = 1$  and  $f(r)$  is non decreasing on  $[\frac{b}{a}, +\infty)$ , the minimum value is  $f(1) = \frac{1}{2}(a+b+1)(a+b)$ .

*Case 1.2.*  $b \geq a$ . Since  $b \notin \{a, a+1\}$ , we have  $b > a+1$  and  $1 \leq \lfloor \frac{b}{a+1} \rfloor \leq a$ , hence the minimum value is

$$\min \left\{ f\left(\left\lfloor \frac{b}{a+1} \right\rfloor\right), f\left(\left\lfloor \frac{b}{a+1} \right\rfloor + 1\right) \right\}.$$

Let  $\beta$  be the remainder of the division of  $b$  by  $a+1$ . In order to obtain the value  $f(\lfloor \frac{b}{a+1} \rfloor)$  we must know the integer  $p_1 \geq a+1$  such that  $\frac{b}{p_1+1} < \lfloor \frac{b}{a+1} \rfloor \leq \frac{b}{p_1}$ . Since  $\lfloor \frac{b}{a+1} \rfloor = \frac{b-\beta}{a+1}$ , we have  $p_1 = \lfloor \frac{b(a+1)}{b-\beta} \rfloor$ , and hence

$$f\left(\left\lfloor \frac{b}{a+1} \right\rfloor\right) = \frac{1}{2}(a+1+p_1)\left((a-p_1)\left(\frac{b-\beta}{a+1}\right)+2b\right).$$

In the same way we obtain

$$f\left(\left\lfloor \frac{b}{a+1} \right\rfloor + 1\right) = \frac{1}{2}(a+1+p_2)\left((a-p_2)\left(\frac{b+a+1-\beta}{a+1}\right)+2b\right)$$

with  $p_2 = \left\lfloor \frac{b(a+1)}{b+a+1-\beta} \right\rfloor$ .

*Case 2.*  $[\frac{b}{a+1}, \frac{b}{a}] \cap \mathbb{N} \neq \emptyset$ . For any integer  $r$  such that  $\frac{b}{a+1} \leq r \leq \frac{b}{a}$ ,  $f(r)$  is equal to the minimum value  $(2a+1)b$ . ■

**Proof of Theorem 1.6.** In order to avoid confusion between “ $k$ ” of the statement of Theorem 1.6 and “ $k$ ” appearing in the proof of Lemma 3.4, let us replace “ $k$ ” by “ $k_0$ ” in the statement of Theorem 1.6. Consider the  $(K_q, k_0)$  stable graph  $G_0$  defined in Lemma 3.4 and see Remark 3.5. We have  $G_0 = sK_{q+k+1} + (r-s)K_{q+k}$  with  $r(k+1)+s = k_0+1$  and  $e(G_0) = \frac{1}{2r}(r(q-1)+k_0+1-s)(r(q-2)+k_0+1+s)$ . Since  $k+1$  is the quotient of  $k_0+1$  divided by  $r$

and  $s$  is the remainder, we have  $s = k_0 + 1 - r \lfloor \frac{k_0+1}{r} \rfloor$ . Hence,

$$e(G_0(r)) = \frac{1}{2} \left( q - 1 + \left\lfloor \frac{k_0 + 1}{r} \right\rfloor \right) \left( \left( q - 2 - \left\lfloor \frac{k_0 + 1}{r} \right\rfloor \right) r + 2(k_0 + 1) \right).$$

Set  $a = q - 2$ ,  $b = k_0 + 1$  and apply Lemma 3.6 and Lemma 1.5. ■

### 3.3. Minimum $(K_q, k)$ stable graph for small $k$

In the following, if no confusion is possible, we simply denote the integer  $\kappa(q)$  by  $\kappa$ .

**Lemma 3.7.** *Suppose that  $q \geq 4$ . If  $\kappa$  is even, then  $\text{stab}(K_q, \kappa - 1) < e(2K_{q+\frac{\kappa}{2}-1})$  and  $\text{stab}(K_q, \kappa) \leq e(K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1})$ . If  $\kappa$  is odd, then  $\text{stab}(K_q, \kappa - 1) < e(K_{q+\frac{\kappa-1}{2}} + K_{q+\frac{\kappa-3}{2}})$  and  $\text{stab}(K_q, \kappa) \leq e(2K_{q+\frac{\kappa-1}{2}})$ .*

**Proof.** Recall that, by definition of  $\kappa$ ,  $K_{q+\kappa-1}$  is the only minimum  $(K_q, \kappa - 1)$  stable. If  $\kappa$  is even then  $2K_{q+\frac{\kappa}{2}-1}$  is exactly  $(K_q, \kappa - 1)$  stable and  $K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1}$  is exactly  $(K_q, \kappa)$  stable. If  $\kappa$  is odd then  $K_{q+\frac{\kappa-1}{2}} + K_{q+\frac{\kappa-3}{2}}$  is exactly  $(K_q, \kappa - 1)$  stable and  $2K_{q+\frac{\kappa-1}{2}}$  is exactly  $(K_q, \kappa)$  stable. ■

**Lemma 3.8.** *Let  $q \geq 3$  and  $p \geq 0$  be two integers. Then,  $e(K_{q+2p}) < e(K_{q+p} + K_{q+p-1})$  if and only if  $p^2 + p < \frac{1}{2}(q-1)(q-2)$  and  $e(K_{q+2p}) = e(K_{q+p} + K_{q+p-1})$  if and only if  $p_0 = \frac{1}{2}(\sqrt{1+2(q-1)(q-2)} - 1)$  is an integer and  $p = p_0$ .  $e(K_{q+2p+1}) < e(2K_{q+p})$  if and only if  $(p+1)^2 < \frac{1}{2}(q-1)(q-2)$  and  $e(K_{q+2p+1}) = e(2K_{q+p})$  if and only if  $p_1 = \frac{1}{2}(\sqrt{2(q-1)(q-2)} - 1)$  is an integer and  $p = p_1$ .*

**Proof.** It is easy to check that  $e(K_{q+2p}) - e(K_{q+p} + K_{q+p-1}) = p^2 + p - \frac{1}{2}(q-1)(q-2)$  and  $e(K_{q+2p+1}) - e(2K_{q+p}) = (p+1)^2 - \frac{1}{2}(q-1)(q-2)$ . These polynomials of degree 2 in  $p$  have positive roots  $p_0 = \frac{1}{2}(\sqrt{1+2(q-1)(q-2)} - 1)$  and  $p_1 = \frac{1}{2}(\sqrt{2(q-1)(q-2)} - 1)$  respectively. ■

**Proof of Theorem 1.13.** If  $\kappa = 2p$  then, by Lemma 3.7,  $\text{stab}(K_q, \kappa - 1) < e(2K_{q+\frac{\kappa}{2}-1})$ . Since  $\kappa - 1 = 2(p-1) + 1$ , by Lemma 3.8,  $p^2 < \frac{1}{2}(q-1)(q-2)$ , that is,  $\kappa < \sqrt{2(q-1)(q-2)}$ .

If  $\kappa = 2p + 1$  then by Lemma 3.7,  $\text{stab}(K_q, \kappa - 1) < e(K_{q+\frac{\kappa-1}{2}} + K_{q+\frac{\kappa-3}{2}})$ . Since  $\kappa - 1 = 2p$ , by Lemma 3.8,  $p < \frac{1}{2}(\sqrt{1+2(q-1)(q-2)} - 1)$ , that is,  $\kappa < \sqrt{1+2(q-1)(q-2)}$ . ■

**Theorem 3.9.** *Let  $q \geq 4$  and suppose that there exists a minimum  $(K_q, \kappa)$  stable graph  $G_0$  which is disconnected. Then  $G_0$  is isomorphic to  $K_{q+\lfloor \frac{\kappa}{2} \rfloor} + K_{q+\lfloor \frac{\kappa-1}{2} \rfloor}$ .*

**Proof.** Let  $G_0$  be a minimum  $(K_q, \kappa)$  stable disconnected graph having  $r \geq 2$  connected components  $G_1, G_2, \dots, G_r$ . By Theorem 2.9, there are integers  $k_1 \geq k_2 \geq \dots \geq k_r$  with  $\sum_{i=1}^r k_i + (r-1) = \kappa$  such that for  $1 \leq i \leq r$ ,  $G_i$  is minimum  $(K_q, k_i)$  stable. For every  $i$ , since  $k_i < \kappa$ , we have  $G_i \equiv K_{q+k_i}$ .

Let us suppose that  $r \geq 3$ . We have  $k_r + k_{r-1} = \kappa - (k_{r-2} + k_{r-3} + \dots + k_1) - (r-1) \leq \kappa - 2$ . Hence,  $e(K_{q+k_r+k_{r-1}+1}) < e(K_{q+k_r}) + e(K_{q+k_{r-1}})$  and the graph  $K_{q+k_1} + K_{q+k_2} + \dots + K_{q+k_{r-2}} + K_{q+k_{r-1}+k_r+1}$  is  $(K_q, \kappa)$  stable with strictly smaller size than  $K_{k_1} + K_{k_2} + \dots + K_{k_r}$ , a contradiction. Hence,  $r = 2$ ,  $G_0 \in \mathcal{B}_2^{(K_q, \kappa)}$  and by Lemma 3.4 the theorem follows. ■

Note that Theorem 3.9 implies that there exists at most one disconnected minimum  $(K_q, \kappa)$  stable graph and this graph, if it exists, is

- either isomorphic to  $K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1}$  (if  $\kappa$  is even)
- or else isomorphic to  $2K_{q+\frac{\kappa-1}{2}}$  (if  $\kappa$  is odd).

**Proof of Theorem 1.14.** By Lemma 3.7 and Theorem 3.9,

if  $\kappa$  is odd, then

$$e(K_{q+\kappa-1}) < e(K_{q+\frac{\kappa-1}{2}} + K_{q+\frac{\kappa-3}{2}}) < \text{stab}(K_q, \kappa) = e(2K_{q+\frac{\kappa-1}{2}}) \leq e(K_{q+\kappa})$$

(note that, by Lemma 3.8, it may be possible that  $e(2K_{q+\frac{\kappa-1}{2}}) = e(K_{q+\kappa})$  for some values of  $q$ );

if  $\kappa$  is even, then

$$e(K_{q+\kappa-1}) < e(2K_{q+\frac{\kappa}{2}-1}) < \text{stab}(K_q, \kappa) = e(K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1}) \leq e(K_{q+\kappa})$$

(note that, by Lemma 3.8, it may be possible that  $e(K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1}) = e(K_{q+\kappa})$  for some values of  $q$ ).

For  $\kappa = 2p + 1$  we have

$$\frac{1}{2}(q+2p)(q+2p-1) < (q+p-1)^2 < (q+p)(q+p-1) \leq \frac{1}{2}(q+2p+1)(q+2p).$$

This implies that

$$(A) \quad p^2 + p < \frac{1}{2}(q-1)(q-2) \leq (p+1)^2.$$

For  $\kappa = 2p$  we have

$$\frac{1}{2}(q+2p-1)(q+2p-2) < (q+p-1)(q+p-2) < (q+p-1)^2 \leq \frac{1}{2}(q+2p)(q+2p-1).$$

This implies that

$$(B) \quad p^2 < \frac{1}{2}(q-1)(q-2) \leq p^2 + p.$$

Combining (A) and (B) yields

$$p^2 < \frac{1}{2}(q-1)(q-2) \leq (p+1)^2.$$

This implies that

$$\sqrt{\frac{1}{2}(q-1)(q-2)} - 1 \leq p < \sqrt{\frac{1}{2}(q-1)(q-2)}.$$

Hence,  $p = \rho(q) = \left\lceil \sqrt{\frac{1}{2}(q-1)(q-2)} \right\rceil - 1$ .

By inequalities (A) and (B), position of  $\frac{1}{2}(q-1)(q-2)$  in comparison to  $\rho(q)^2 + \rho(q)$  determines the parity of  $\kappa$ . Hence, if  $\frac{1}{2}(q-1)(q-2) > \rho(q)^2 + \rho(q)$ , then  $\kappa = 2\rho(q) + 1 = 2 \left\lceil \sqrt{\frac{1}{2}(q-1)(q-2)} \right\rceil - 1$  else  $\kappa = 2\rho(q) = 2 \left\lceil \sqrt{\frac{1}{2}(q-1)(q-2)} \right\rceil - 2$ . ■

If there is no minimum disconnected  $(K_q, \kappa(q))$  stable graph then, by definition of  $\kappa(q)$ , there exists a connected minimum  $(K_q, \kappa(q))$  stable graph  $G_q$  distinct from a clique. Note that if such a graph exists, then

$$e(G_q) < \min\{e(K_{q+\kappa}), e(K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2}-1})\}, \text{ if } \kappa = \kappa(q) \text{ is even}$$

or

$$e(G_q) < \min\{e(K_{q+\kappa}), e(2K_{q+\frac{\kappa-1}{2}})\}, \text{ if } \kappa = \kappa(q) \text{ is odd.}$$

A positive answer to Problem 1.15 states that there is no such graph  $G_q$ .

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