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Dedicated to the 70th Birthday of Mieczystaw Borowiecki

# ON THE NON- $(\boldsymbol{p}-1)$-PARTITE $\boldsymbol{K}_{p}$-FREE GRAPHS 

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#### Abstract

We say that a graph $G$ is maximal $K_{p}$-free if $G$ does not contain $K_{p}$ but if we add any new edge $e \in E(\bar{G})$ to $G$, then the graph $G+e$ contains $K_{p}$. We study the minimum and maximum size of non- $(p-1)$-partite maximal $K_{p}$-free graphs with $n$ vertices. We also answer the interpolation question:


for which values of $n$ and $m$ are there any $n$-vertex maximal $K_{p}$-free graphs of size $m$ ?
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## 1. Introduction and Notation

We consider finite undirected graphs without loops or multiple edges. A graph $G$ has a vertex set $V(G)$ and an edge set $E(G)$. The size of a graph is the number of edges. We denote by $e(G)$ the size of the graph $G$ and by $v(G)$ the number of vertices of $G$. The set of neighbours of a vertex $v \in V(G)$ is denoted by $N_{G}(v)$, or briefly by $N(v)$. Moreover, $N_{G}[v]=N_{G}(v) \cup\{v\}$. Let $S \subseteq V(G), N_{G}[S]=\bigcup_{v \in S} N_{G}[v]$. By $G[S]$ we denote the subgraph induced by the set of vertices $S$. The degree of $v$ is denoted by $d_{G}(v)$. If $H$ is a subgraph of $G$ and $v \in V(G)$, then $d_{H}(v)=\left|N_{G}(v) \cap V(H)\right|$. For $S \subseteq V(G)$ we write $d_{S}(v)=d_{G[S]}(v)$. We also use the following notation: $S_{n}$ is the star with $n$ vertices, $K_{n}$ is the complete graph with $n$ vertices, for $k \geq 2, K_{n_{1}, \ldots, n_{k}}$ is the complete $k$-partite graph.

For undefined concepts we refer the reader to [4].
Let $n, p$ be integers and $p \geq 2$. We say that the graph $G$ is $K_{p}$-free if $G$ does not contain $K_{p}$ as a subgraph. We say that $G$ is maximal $K_{p}$-free (sometimes called $K_{p}$-saturated) if $G$ does not contain $K_{p}$ as a subgraph but if we add any new edge $e \in E(\bar{G})$ to $G$, then the graph $G+e$ contains $K_{p}$. The set of all maximal $K_{p}$-free graphs of order $n$ is denoted by $\mathrm{M}\left(n, K_{p}\right)$. A complete $k$-partite graph $K_{n_{1}, \ldots, n_{k}}$ such that $\left|n_{i}-n_{j}\right| \leq 1$ for $i, j=1, \ldots, k$ and $n_{1}+\cdots+n_{k}=n$ we call Turán's graph and denoted $T_{k}(n)$. The classical theorem of Turán [12] states that if $G$ is an $n$-vertex $K_{p}$-free graph of maximum size, then $G$ is isomorphic to $T_{p-1}(n)$. On the other hand Erdős, Hajnal and Moon [5] proved that if $G$ is maximal $K_{p}$-free with $n \geq p-1$ vertices, then $e(G) \geq(p-2) n-\frac{1}{2}(p-1)(p-2)$. However, every maximal $K_{p}$-free graph from this theorem is $(p-1)$-partite and contains a vertex of degree $n-1$. The problem of determining the minimum size of maximal $K_{p}$-free graphs with no vertex of degree $n-1$ was studied by Alon et al. [1]. The case for $p=3$ was treated by Füredi, Seress [8] and Erdős, Holzman [6]. Duffus and Hanson [7] study the minimum size of maximal $K_{p}$-free graphs with fixed minimum degree.

We will consider the maximal $K_{p}$-free graphs that are not ( $p-1$ )-partite. Let $\mathrm{s}\left(n, K_{p}\right)$ and $\mathrm{e}\left(n, K_{p}\right)$ denote minimum and maximum size of a maximal $K_{p}$-free graph with $n$ vertices that is not a $(p-1)$-partite graph, i.e.,

$$
\begin{aligned}
& \mathrm{s}\left(n, K_{p}\right)=\min \left\{e(G): G \in \mathrm{M}\left(n, K_{p}\right) \text { and } G \text { is non- }(p-1) \text {-partite }\right\} \\
& \mathrm{e}\left(n, K_{p}\right)=\max \left\{e(G): G \in \mathrm{M}\left(n, K_{p}\right) \text { and } G \text { is non- }(p-1) \text {-partite }\right\} .
\end{aligned}
$$

Let us define the following sets of graphs:
$\mathrm{S}\left(n, K_{p}\right)=\left\{G \in \mathrm{M}\left(n, K_{p}\right): e(G)=\mathrm{s}\left(n, K_{p}\right)\right.$ and $G$ is non- $(p-1)$-partite $\}$,
$\mathrm{E}\left(n, K_{p}\right)=\left\{G \in \mathrm{M}\left(n, K_{p}\right): e(G)=\mathrm{e}\left(n, K_{p}\right)\right.$ and $G$ is non- $(p-1)$-partite $\}$.
We will study possible size of the maximal $K_{p}$-free graphs with $n$ vertices. This problem for $p=3$ was solved in [11]. The same result was obtained in [3]. In these papers the minimum and maximum size of maximal $K_{3}$-free graphs was determined. Moreover, it was proved there that for every integer $m$ such that $\mathrm{s}\left(n, K_{3}\right) \leq m \leq \mathrm{e}\left(n, K_{3}\right)$ there exists a maximal $K_{3}$-free graph with size $m$ and with $n$ vertices. In Section 2 we will deal with the $K_{3}$-free graphs, we will recall some theorems and we will give the stronger result: we completely characterize the set $\mathrm{E}\left(n, K_{3}\right)$. The case for $p=4$ was studied in [2]. In Sections $3,4,5$ we will deal with the maximal $K_{p}$-free graphs for $p \geq 4$. We will determine the minimum and maximum size of $n$-vertex non- $(p-1)$-partite maximal $K_{p}$-free graphs. In Section 4 we completely determine the set $\mathrm{E}\left(n, K_{p}\right)$. In Section 5 we will solve the interpolation problem.

## 2. Maximal $K_{3}$-Free Graphs

Let $G$ be a graph with the vertex set $V(G)=\left\{v_{1}, v_{2}, \ldots, v_{k}\right\}$ and $n_{i}$ be integers for $i=1, \ldots, k$. By $G\left[n_{1}, \ldots, n_{k}\right]$ we denote the graph of order $n_{1}+\cdots+n_{k}$ obtained from $G$ in the following way: each vertex $v_{i}$ we replaced by the set $V_{i}$ of $n_{i}$ independent vertices for $i=1, \ldots, k$. We join each vertex of $V_{i}$ with each vertex of $V_{j}$ whenever vertices $v_{i}$ and $v_{j}$ are adjacent in the graph $G$.

Murty [10] characterized 2-connected graphs with diameter 2 with the minimum number of edges. Let $P$ be the Petersen graph and $G_{7}$ be the graph in Figure 1.


Figure 1. The graph $G_{7}$.
Theorem 1 [10]. Let $G$ be a 2-connected graph of order $n$ such that $\operatorname{diam}(G)=2$ with the minimum size. Then $e(G)=2 n-5$ and $G \in\left\{C_{5}[t, 1, n-t-3,1,1]: 1 \leq\right.$ $t \leq n-4\} \cup\left\{G_{7}\left[1, t_{1}, t_{2}, n-t_{1}-t_{2}-4,1,1,1\right]: t_{1}, t_{2} \geq 1, t_{1}+t_{2} \leq n-5\right\} \cup\{P\}$.

Lemma 2. Let $G$ be a non-bipartite maximal $K_{3}$-free graph. Then $G$ is 2connected and $\operatorname{diam}(G)=2$.

Proof. Suppose that there are two vertices $u, v \in V(G)$ such that $d_{G}(u, v)>2$, where $d_{G}(u, v)$ denotes the distance between $u$ and $v$. Thus, $G+u v$ does not contain $K_{3}$, so $G$ is not maximal. This yields that $\operatorname{diam}(G)=2$. Since $G$ is not bipartite and $\operatorname{diam}(G)=2$, we have that $G$ is 2 -connected.

From Theorem 1 and Lemma 2 it immediately follows
Theorem 3. Let $n \geq 5$. Then
(a) $\mathrm{s}\left(n, K_{3}\right)=2 n-5$,
(b) $\mathrm{S}\left(n, K_{3}\right)=\left\{C_{5}[t, 1, n-t-3,1,1]: 1 \leq t \leq n-4\right\} \cup\{P\}$.

For $n \geq 5$ let us denote $C_{5}^{*}[n]=\left\{C_{5}\left[n_{1}, \ldots, n_{5}\right]: n_{1}+\cdots+n_{5}=n\right\}$ and $C_{5}^{*}=\left\{C_{5}^{*}[n]: n \geq 5\right\}$. From Theorem 3 it follows that non-bipartite maximal $K_{3}$-free graphs of minimum size belong to $C_{5}^{*}$. In [7] it was proved that maximal $K_{3}$-free graphs with minimum degree 2 having minimum size belong to $C_{5}^{*}$. In the next theorem we will show that also non-bipartite $K_{3}$-free graphs with a maximum size belong to $C_{5}^{*}$. First we will show how to distribute the vertices in any graph from $C_{5}^{*}[n]$ to obtain the maximum size. Let us define the subclasses of $C_{5}^{*}[n]$ :
for $n$ even

$$
\begin{gathered}
A\left(n, K_{3}\right)=\left\{C_{5}\left[\frac{n}{2}-2, k, 1,1, \frac{n}{2}-k\right]: 1 \leq k \leq \frac{n}{2}-1\right\} \\
B\left(n, K_{3}\right)=\left\{C_{5}\left[\frac{n}{2}-1, k, 1,1, \frac{n}{2}-k-1\right]: 1 \leq k \leq \frac{n}{2}-2\right\}
\end{gathered}
$$

for $n$ odd

$$
C\left(n, K_{3}\right)=\left\{C_{5}\left[\frac{n-1}{2}-1, k, 1,1, \frac{n-1}{2}-k\right]: 1 \leq k \leq \frac{n-1}{2}-1\right\}
$$

Lemma 4. Let $n \geq 5$ and $G \in C_{5}^{*}[n]$ with the maximum size. Then

$$
G \in \begin{cases}A\left(n, K_{3}\right) \cup B\left(n, K_{3}\right) & \text { for } n \text { even } \\ C\left(n, K_{3}\right) & \text { for } n \text { odd }\end{cases}
$$

Proof. Let $G=C_{5}\left[n_{1}, n_{2}, n_{3}, n_{4}, n_{5}\right]$. Let $V_{i}(i=1, \ldots, 5)$ be independent sets of $G$ such that $\left|V_{i}\right|=n_{i}(i=1, \ldots, 5)$. First we will show that in $G$ there are two consecutive independent sets with exactly one vertex each. Let us consider two cases.

Case 1. There are two consecutive independent sets with distinct number of vertices. Without loss of generality we assume that $V_{1}$ and $V_{2}$ have distinct number of vertices and $n_{1}>n_{2}$. We show that $n_{3}=1$. If this is not true (i.e., $n_{3} \geq 2$ ), then we delete one vertex from $V_{3}$ and add one vertex to $V_{5}$, so we obtain the graph $C_{5}\left[n_{1}, n_{2}, n_{3}-1, n_{4}, n_{5}+1\right]$ having more edges than $G$, a contradiction. Now, we show that also $n_{4}=1$ or $n_{2}=2$. If $n_{4} \geq 2$, we delete one vertex from $V_{4}$ and add one vertex to $V_{1}$. Hence we obtain $C_{5}\left[n_{1}+1, n_{2}, n_{3}, n_{4}-1, n_{5}\right]$
that has more edges than $G$ if $n_{2} \neq 1$. Thus, if $G$ has a maximum size and two consecutive independent sets with distinct number of vertices, then it also has two consecutive independent sets with exactly one vertex each.

Case 2. All independent sets have the same number of vertices. Thus, $n_{1}=$ $n_{2}=n_{3}=n_{4}=n_{5}=p$. Suppose that $p \geq 2$. If we delete one vertex from $V_{2}$ and add one vertex to $V_{1}$ and we delete one vertex from $V_{3}$ and add one vertex to $V_{5}$, then we obtain a graph $C_{5}[p+1, p-1, p-1, p, p+1]$ with more edges.

Hence we may assume that $n_{3}=n_{4}=1$. Then $e(G)=n_{1}\left(n-n_{1}-2\right)+$ $n-n_{1}-2+1$. When $n$ is even, $e(G)$ achieves the maximum for $n_{1}=\frac{n}{2}-1$ or $n_{1}=\frac{n}{2}-2$. When $n$ is odd, $e(G)$ achieves the maximum for $n_{1}=\frac{n-1}{2}-1$. Thus, $G \in A\left(n, K_{3}\right) \cup B\left(n, K_{3}\right)$ for $n$ even or $G \in C\left(n, K_{3}\right)$ for $n$ odd.

Theorem 5. Let $n, q, r$ be integers such that $n \geq 5, n=2 q+r, r=0,1$. Then
(a) $\mathrm{e}\left(n, K_{3}\right)=\frac{n^{2}}{4}-\frac{n}{2}+\frac{r}{4}+1$,
(b) $\mathrm{E}\left(n, K_{3}\right)= \begin{cases}A\left(n, K_{3}\right) \cup B\left(n, K_{3}\right) & \text { for } n \text { even, } \\ C\left(n, K_{3}\right) & \text { for } n \text { odd } .\end{cases}$

Proof. Since (b) implies (a), we prove only the part (b). Let $G$ be a graph with $n$ vertices such that $G$ is a non-bipartite $K_{3}$-free of maximum size, i.e., $G \in \mathrm{E}\left(n, K_{3}\right)$. First we show that $G \in C_{5}^{*}[n]$. Next we use Lemma 4 to obtain (b). Let $v$ be the vertex of maximum degree $d(v)=\Delta(G)=\Delta$. Since $G$ is trianglefree, $N(v)$ is an independent set and since $G$ is not a bipartite $G$ contains an odd cycle of order at least 5 . Hence $G-N[v]$ contains at least one edge. Suppose that $G-N[v]$ contains two vertex-disjoint edges $x y$ and $x^{\prime} y^{\prime}$. Consider deleting all edges adjacent to $x^{\prime}$ and all edges adjacent to $y^{\prime}$ and next we join vertices $x^{\prime}$ and $y^{\prime}$ with all vertices of $N(v)$. Since $|N(v)|=\Delta(G)$, this new graph has more edges than $G$ and it is a $K_{3}$-free graph, a contradiction. Hence $G-N[v]$ does not contain two vertex-disjoint edges, so $G-N[v]=S_{t+1} \cup \bar{K}_{n-\Delta-t-2}$.

First suppose that the graph $G-N[v]$ has exactly one edge $x y$. Let $X$ and $Y$ be the sets of neighbours in $N(v)$ of $x$ and $y$, respectively. The set $X \cap Y=\emptyset$ because $G$ is $K_{3}$-free and $X \cup Y=N(v)$ because $G$ is maximal. Also, neither $X$ nor $Y$ can be empty. For any vertex $z \in(V(G) \backslash N[v]) \backslash\{x, y\}$ we have $N(z)=N(v)$. This implies that we can divide $V(G)$ into five independent sets $V_{1}=\{x\}, V_{2}=\{y\}, V_{3}=Y, V_{4}=(V(G) \backslash N(v)) \backslash\{x, y\}, V_{5}=X$ such that the sets $V_{i} \cup V_{j}, j=i+1(\bmod 5)$, induce a complete bipartite graph. Thus, $G \in C_{5}^{*}[n]$.

Now suppose that $t \geq 2$. Let us denote by $x, x_{1}, x_{2}, \ldots, x_{t}$ vertices of $S_{t+1}$ such that $x$ is a central vertex of the star. Since $N(v)=\Delta(G)$, each vertex $x_{i}(i=1, \ldots, t)$ is nonadjacent to at least one vertex of $N(v)$. Suppose that there is $j$ such that $x_{j}$ is nonadjacent to more than one vertex in $N(v)$. We can delete the edge $x x_{j}$ and join $x_{j}$ with all vertices of $N(v)$. The new graph has more
edges than $G$ and is $K_{3}$-free, a contradiction. Thus, each vertex $x_{i}(i=1, \ldots, t)$ is nonadjacent to exactly one vertex in $N(v)$. By Lemma $2 \operatorname{diam}(G)=2$ and hence $x$ has a neighbour $w$ in $N(v)$. Since $G$ is $K_{3}$-free, $w$ is nonadjacent to all neighbours of $x$. Thus, all vertices $x_{i}(i=1, \ldots, t)$ are nonadjacent to the vertex $w$. Therefore, we can divide $V(G)$ into the following independent sets: $V_{1}=$ $\{x\}, V_{2}=\left\{x_{1}, \ldots, x_{t}\right\}, V_{3}=N(v) \backslash\{w\}, V_{4}=V(G) \backslash\left(N(v) \cup V\left(S_{t+1}\right)\right), V_{5}=$ $\{w\}$. Thus, $G \in C_{5}^{*}[n]$, so by Lemma 4 we obtain (b).
For convenience we repeat the following result given in [3, 11].
Theorem 6. Let $n, q, r$ be integers such that $n \geq 5, n=2 q+r, r=0,1$. Then for any integer $m$ such that $2 n-5 \leq m \leq \frac{n^{2}}{4}-\frac{n}{2}+\frac{r}{4}+1$ there is a maximal $K_{3}$-free graph of size $m$ with $n$ vertices.
Proof. If $n=5$ then $m=2 n-5=\frac{n^{2}}{4}-\frac{n}{2}+\frac{r}{4}+1=5$ and $C_{5}$ is the only graph in $\mathrm{M}\left(5, K_{3}\right)$. For $n \geq 6$, let $G_{t}^{x}(n)=C_{5}\left[t, 1, x-t, \frac{n-x}{2}-1, \frac{n-x}{2}\right]$ where $2 \leq x \leq n-4,1 \leq t \leq x-1$ and $x, n$ are the same parity. It is easy to see that $G_{t}^{x}(n) \in \mathrm{M}\left(n, K_{3}\right)$ and $e\left(G_{t}^{x}(n)\right)=\frac{n^{2}-x^{2}}{4}+\frac{x-n}{2}+t$. Moreover, $e\left(G_{1}^{n-4}\right)=2 n-5$ and $e\left(G_{1}^{2}(n)\right)=\frac{n^{2}}{4}-\frac{n}{2}+1$ for $n=2 q, e\left(G_{1}^{3}(n)\right)=\frac{n^{2}}{4}-\frac{n}{2}+\frac{1}{4}+1$ for $n=2 q+1$.

Let $x=n-4$ and $t=1$. If we increase $t$ by 1 , then we obtain the graph with one extra edge. If we decrease $x$ by 2 , then we obtain the graph with $x-2$ extra edges, i.e, $e\left(G_{t+1}^{x}(n)\right)=e\left(G_{t}^{x}(n)\right)+1$ and $e\left(G_{t}^{x-2}(n)\right)=e\left(G_{t}^{x}(n)\right)+x-2$.

Thus, if we fix $x$ and increase $t$ by 1 from $t=1$ to $t=x-1$, then we obtain the sequence of graphs whose sizes are all integers from the interval $\left[\frac{n^{2}-x^{2}}{4}+\right.$ $\left.\frac{x-n}{2}+1, \frac{n^{2}-x^{2}}{4}+\frac{x-n}{2}+t\right]$. Next, if we decrease the value of $x$ by 2 from $x=n-4$ to $x=2$ for $n$ even and to $n=3$ for $n$ odd, then we obtain all integers $m$ from the interval $\left[2 n-5, \frac{n^{2}}{4}-\frac{n}{2}+\frac{r}{4}+1\right]$.

## 3. Minimum Size of $\operatorname{Non-}(p-1)$-partite Maximal $K_{p}$-free Graphs

The theorem of Erdős, Hajnal and Moon [5] states that if the graph $G$ is maximal $K_{p}$-free, then $e(G) \geq(p-2) n-\frac{1}{2}(p-1)(p-2)$ and the bound is realized by the complete ( $p-1$ )-partite graph $K_{1,1, \ldots, 1, n-p+2}$. The next complete ( $p-1$ )-partite graph $K_{1,1, \ldots, 2, n-p+1}$ has $(p-1) n-\frac{1}{2}(p-1) p-1$ edges. We will show that the minimum size of non- $(p-1)$-partite maximal $K_{p}$-free graphs with $n$ vertices is $(p-1) n-\frac{1}{2}(p-1) p-2$ if $n$ is large enough.

We need the following results.
Theorem 7 [9]. If $G \in \mathrm{M}\left(n, K_{p}\right)$ and $G$ contains no vertex of degree $n-1$, then $\delta \geq 2(p-2)$
Theorem 8 [1]. Let $G \in \mathrm{M}\left(n, K_{4}\right)$ and $\delta(G)=4$. If $G$ contains no vertex of degree $n-1$, then $e(G) \geq 4 n-15$.

Theorem 9. Let $p, n$ be integers such that $p \geq 3$ and $n \geq 3(p+4)$. Then

$$
\mathrm{s}\left(n, K_{p}\right)=(p-1) n-\frac{1}{2}(p-1) p-2
$$

Proof. Let $G=F+K_{p-3}, F \in \mathrm{~S}\left(n-(p-3), K_{3}\right)$. Thus, the graph $G$ is $K_{p}$-maximal non- $(p-1)$-partite and $e(G)=(p-1) n-\frac{1}{2}(p-1) p-2$. Hence $\mathrm{s}\left(n, K_{p}\right) \leq(p-1) n-\frac{1}{2}(p-1) p-2$.

Now we show that $\mathrm{s}\left(n, K_{p}\right) \geq(p-1) n-\frac{1}{2}(p-1) p-2$. We prove by induction on $p$. By Theorem 3, the result holds for $p=3$. Assume that the result holds for $p-1$, i.e. $\mathrm{s}\left(n, K_{p-1}\right) \geq(p-2) n-\frac{1}{2}(p-2)(p-1)-2$. Let $G \in \mathrm{~S}\left(n, K_{p}\right)$. Suppose that $\Delta(G)=n-1$. Let $v$ be the vertex of degree $n-1$. Since $G$ is maximal $K_{p}$-free, $G-v$ is maximal $K_{p-1}$-free. The assumption that $G$ is not ( $p-1$ )-partite implies that $G-v$ is not $(p-2)$-partite. Thus, by the induction hypothesis

$$
e(G-v) \geq(p-2) n-\frac{1}{2}(p-2)(p-1)-2
$$

hence

$$
e(G)=e(G-v)+n-1 \geq(p-1) n-\frac{1}{2}(p-1) p-2
$$

Thus, we may assume that $\Delta(G) \leq n-2$. Then by Theorem 7 we have $\delta(G) \geq$ $2(p-2)$. If $\delta(G) \geq 2(p-1)$, then $e(G) \geq(p-1) n$. Thus, to complete the proof we consider $\delta(G)=2(p-2)$ and $\delta(G)=2 p-3$.

Let $v$ be a vertex with minimum degree and let $H=V(G) \backslash N[v]$. Since $G$ is maximal, for any vertex $x \in H$ the subgraph $G[N(x) \cap N(v)]$ contains $K_{p-2}$. Let

$$
T=\{y \in N(v): y \text { is in a }(p-2)-\text { clique of } G[N(v)]\}
$$

Let $|T|=t$. Each vertex of $H$ has at least $p-2$ neighbours in $T$ and each vertex of $T$ has at least $p-3$ neighbours in $T$. Thus, $e(G[T \cup H]) \geq \frac{1}{2} t(p-3)+\sum_{x \in H} d_{T}(x)$.
Moreover,
$|E(G-v) \backslash E(G[T \cup H])| \geq \sum_{x \in N(v) \backslash T} d_{T \cup H}(x)+\frac{1}{2} \sum_{x \in N(v) \backslash T}\left(d_{G}(x)-1\right.$
$\left.-d_{T \cup H}(x)\right)+\frac{1}{2} \sum_{x \in H}\left(d_{G}(x)-d_{T}(x)\right)$
$=\frac{1}{2} \sum_{x \in N(v) \backslash T}\left(d_{G}(x)-1+d_{T \cup H}(x)\right)+\frac{1}{2} \sum_{x \in H}\left(d_{G}(x)-d_{T}(x)\right)$
$\geq \frac{1}{2} \sum_{x \in N(v) \backslash T}\left(d_{G}(x)-1\right)+\frac{1}{2} \sum_{x \in H}\left(d_{G}(x)-d_{T}(x)\right)$.
Now we can calculate the lower bound for $e(G)$. Let $\delta(G)=\delta$.
$e(G)=e(G[T \cup H])+|N(v)|+|E(G-v) \backslash E(G[T \cup H])|$
$\geq \frac{1}{2} t(p-3)+\sum_{x \in H} d_{T}(x)+\delta+\frac{1}{2} \sum_{x \in N(v) \backslash T}\left(d_{G}(x)-1\right)+\frac{1}{2} \sum_{x \in H}\left(d_{G}(x)-d_{T}(x)\right)$
$=\frac{1}{2} t(p-3)+\delta+\frac{1}{2} \sum_{x \in N(v) \backslash T}\left(d_{G}(x)-1\right)+\frac{1}{2} \sum_{x \in H}\left(d_{G}(x)+d_{T}(x)\right)$
$\geq \frac{1}{2} t(p-3)+\delta+\frac{1}{2}(\delta-t)(\delta-1)+\frac{1}{2}|H|(\delta+p-2)$
$=\frac{1}{2} t(p-2-\delta)+\frac{1}{2} \delta(\delta-1)+\frac{1}{2}(n-1-\delta)(\delta+p-2)$.
Since $\delta(G)=2(p-2)$ or $\delta(G)=2 p-3$, this expression has the smallest value when $t$ is as large as possible. Since $t \leq \delta$, we have $e(G) \geq-\frac{1}{2} \delta^{2}-\delta+\frac{1}{2} \delta n+\frac{1}{2}(n$ $-1)(p-2)$. When $\delta(G)=2(p-2)$, we have $-\frac{1}{2} \delta^{2}-\delta+\frac{1}{2} \delta n+\frac{1}{2}(n-1)(p-2) \geq$
$(p-1) n-\frac{1}{2}(p-1) p-2$ for $p \geq 5$ and $n \geq 3(n+4)$. When $\delta(G)=2 p-3$, we have $-\frac{1}{2} \delta^{2}-\delta+\frac{1}{2} \delta n+\frac{1}{2}(n-1)(p-2) \geq(p-1) n-\frac{1}{2}(p-1) p-2$ for $p \geq 4$ and $n \geq 3(n+4)$. To complete the proof note that by Theorem $8 \mathrm{~s}\left(n, K_{p}\right) \geq(p-1) n-\frac{1}{2}(p-1) p-2$ for $p=4$ and $\delta(G)=4$.

## 4. Maximum Size of $\operatorname{Non}-(p-1)$-partite $K_{p}$-Free Graphs

In this section we will give a maximum size of the non- $(p-1)$-partite $K_{p}$-free graphs for $p \geq 4$. We will also determine the set $E\left(n, K_{p}\right)$ for $p \geq 4$. We will prove this in the following way. First we will show that the non- $(p-1)$-partite $K_{p}$-free graph $G$ of maximum size is the join of the non-bipartite $K_{3}$-free graph of maximum size with the $(p-3)$-partite graph, i.e., $G=G_{1}+G_{2}$, where $G_{1} \in C_{5}^{*}$ and $G_{2}$ is complete $(p-3)$-partite. Next we will show how to distribute the vertices of $G$ between $G_{1}$ and $G_{2}$ to obtain a maximum size.

We need the following lemma.
Lemma 10. Let $G$ be a maximal $K_{p}$-free graph and $v \in V(G)$. Let $x y$ be such an edge that, $x, y \in V(G) \backslash N[v]$. Then the vertices $N(v) \cap N(x) \cap N(y)$ induce the $K_{p-2}$-free graph and $|N(v) \backslash(N(x) \cap N(y))| \geq 2$.

Proof. If the subgraph induced by $N(v) \cap N(x) \cap N(y)$ had a clique $K_{p-2}$, this clique together with $x$ and $y$ would form $K_{p}$. Since $G$ is the maximal $K_{p}$-free graph, the subgraph $N(v) \cap N(x)$ contains a clique $K^{\prime}$ on $p-2$ vertices and also the subgraph $N(v) \cap N(y)$ contains a clique $K^{\prime \prime}$ on $p-2$ vertices. If $K^{\prime}=K^{\prime \prime}$, then this clique together with $x, y$ form $K_{p}$, a contradiction. Thus, at least one vertex of $K^{\prime}$ is not adjacent to $y$ and at least one vertex of $K^{\prime \prime}$ is not adjacent to $x$.

Let us introduce the following notations. For $S \subseteq V(G), e(S)$ denotes the number of edges incident with vertices of $S$, i.e., $e(S)=e(G[N[S]])$. For $S_{1}, S_{2} \in V(G)$, by the symbol $E\left(S_{1}, S_{2}\right)$ we denote the set of all edges linking a vertex from the set $S_{1}$ with a vertex from the set $S_{2}$, i.e., $E\left(S_{1}, S_{2}\right)=\left\{u v \in E(G): u \in S_{1}, v \in S_{2}\right\}$. Let $e\left(S_{1}, S_{2}\right)=\left|E\left(S_{1}, S_{2}\right)\right|$. Let $T_{p}^{*}=\left\{T_{p}(n): n \geq p\right\}$.

Theorem 11. Let $p \geq 3$ and $n \geq p+2$. If $G \in \mathrm{E}\left(n, K_{p}\right)$, then $G=G_{1}+G_{2}$ where $G_{1} \in C_{5}^{*}$ and $G_{2}$ is complete $(p-3)$-partite.

Proof. Let $v$ be the vertex of maximum degree and $\Delta(G)=\Delta$. We consider two cases.

Case 1. $G[N(v)]$ is not $(p-2)$-partite. We prove by induction on $p$. For $p=3$ the proof follows from Theorem 5 . Suppose that the subgraph induced by $V(G) \backslash N[v]$ contains an edge. Since $|N(v)|=\Delta$, if we delete all the edges in
$G[V(G) \backslash N[v]]$ and join each vertex of $V(G) \backslash N[v]$ to all vertices of $N(v)$, then we obtain a non- $(p-1)$-partite $K_{p}$-free graph with more edges, a contradiction. Thus, $V(G) \backslash N[v]$ is the independent set of vertices. Since $G[N(v)]$ is $K_{p-1}$-free and is not $(p-2)$-partite, we have by the induction hypothesis that $G[N(v)]=$ $G_{1}+G_{2}^{\prime}$, where $G_{1} \in C_{5}^{*}$ and $G_{2}^{\prime}$ is complete ( $p-4$ )-partite. This implies that $G_{2}^{\prime}$ together with $V(G) \backslash N(v)$ form the complete ( $p-3$ )-partite graph $G_{2}$. Therefore, $G=G_{1}+G_{2}$ where $G_{1} \in C_{5}^{*}$ and $G_{2}$ is complete $(p-3)$-partite.

Case 2. $G[N(v)]$ is $(p-2)$-partite. Let $H=V(G) \backslash N[v]$. Since the graph $G$ is not $(p-1)$-partite, there is an edge in the subgraph induced by $H$. Let $x, y \in H$ and $x y \in E(G)$. Let $S$ be the maximum $K_{p-2}$-free subgraph of $G[N(v)]$ (i.e. $K_{p-2}$-free with maximum number of vertices) and $|S|=s$. Since $G[N(v)]$ contains $K_{p-2}$, we have $\Delta-s \geq 1$. Let us consider two cases.

Subcase 2.1. $\Delta-s \geq 2$. Let $F=G_{1}+G_{2}$, where $G_{1}=C_{5}[1,1,1, \Delta-s-$ $1, n-\Delta-2]$ and $G_{2} \in T_{p-3}(s)$. Note that $e(F)=n \Delta-\Delta^{2}-s^{2}+\Delta s-\Delta+$ $s+1+e\left(T_{p-3}(s)\right)$ and $F$ is non- $(p-1)$-partite $K_{p}$-free. Since $G \in E\left(n, K_{p}\right)$, it follows that $e(G) \geq n \Delta-\Delta^{2}-s^{2}+\Delta s-\Delta+s+1+e\left(T_{p-3}(s)\right)$.

On the other hand we can calculate the size of $G$ in the following way:
$e(G)=d(v)+e(H \backslash\{x, y\})+e(\{x, y\}, N(v))+e(G[N(v) \backslash S])+e(N(v) \backslash S, S)+$ $1+e(G[S])$.

Note that $e(H \backslash\{x, y\}) \leq(|H|-2) \Delta$. The subgraph induced by $N(v) \cap N(x) \cap N(y)$ is $K_{p-2}$-free, this yields that $|N(v) \cap N(x) \cap N(y)| \leq s$, since $s$ is order of the maximum $K_{p-2}$-free subgraph of $G[N(v)]$. Thus, $e(\{x, y\}, N(v)) \leq \Delta+s$. The subgraph induced by $N(u) \cap N(v)$ for any $u \in N(v) \backslash S$ is $K_{p-2}$-free, since otherwise the subgraph induced by $N[v]$ would contain $K_{p}$. Thus, $e(G[N(v) \backslash$ $S])+e(N(v) \backslash S, S) \leq(\Delta-s) s$. Therefore, $e(G) \leq \Delta+(|H|-2) \Delta+\Delta+s+$ $(\Delta-s) s+1+e(G[S]) \leq n \Delta-\Delta^{2}-s^{2}+\Delta s-\Delta+s+1+e\left(T_{p-3}(s)\right)$.
We conclude that we obtain the graph of maximum size if the equality holds. This implies the following:
(1) Each vertex of $H \backslash\{x, y\}$ has maximum degree.
(2) The set $H \backslash\{x, y\}$ is independent.
(3) The vertices $N(v) \cap N(x) \cap N(y)$ induce the maximum $K_{p-2}$-free subgraph of $G[N(v)]$.
(4) $N(v) \subseteq N(x) \cup N(y)$.
(5) Each vertex of $N(v) \backslash S$ is adjacent to all vertices of $S$.
(6) The vertices of $S$ induce a graph from $T_{p-3}^{*}$.

From (5) and (6) it immediately follows
Claim 1. $G[N(v)]$ is the complete $(p-2)$-partite graph.

Since $G[N(v)]$ is the complete ( $p-2$ )-partite graph and $N(v) \cap N(x) \cap N(y)$ induces the maximum $K_{p-2}$-free subgraph of $G[N(v)]$ (by (3)), we have the following
Claim 2. The vertices of $N(v) \cap N(x) \cap N(y)$ induce the complete ( $p-3$ )-partite graph.
Let $G_{2}$ be the subgraph of $G$ induced by $N(v) \cap N(x) \cap N(y)$, so $G_{2}$ is complete $(p-3)$-partite by Claim 2 .

Claim 3. Each vertex of $V(G) \backslash V\left(G_{2}\right)$ is adjacent to all vertices of $V\left(G_{2}\right)$.
Proof. It is easy to see that each vertex of $\left(V(G) \backslash V\left(G_{2}\right)\right) \backslash(H \backslash\{x, y\})$ is adjacent to all vertices of $V\left(G_{2}\right)$. Now we show that it holds also for each vertex of $H \backslash\{x, y\}$. First note that each vertex of $z \in H \backslash\{x, y\}$ is nonadjacent to at most two vertices of $N(v)$, since $d_{G}(z)=\Delta$ and $H \backslash\{x, y\}$ is independent (by (1) and (2)). Suppose that there is a vertex $z \in H \backslash\{x, y\}$ that is nonadjacent to a vertex of $V\left(G_{2}\right)$. First assume that $z$ is nonadjacent to exactly one vertex of $N(v)$ (i.e., a vertex of $G_{2}$ ). Thus, $z$ is adjacent either to $x$ or to $y$. Since $G$ is maximal $K_{p}$-free, $N(v) \cap N(z)$ must contain a clique on $p-2$ vertices. Since $G[N(v)]$ is complete ( $p-2$ )-partite, both $N(z) \cap N(x) \cap N(v)$ and $N(z) \cap N(y) \cap N(v)$ contains a ( $p-2$ )-clique. This implies that this clique either with $z, x$ or $z, y$ form $K_{p}$, a contradiction. Now assume that $z$ is nonadjacent to exactly two vertices of $N(v)$ (at least one of them is in $V\left(G_{2}\right)$ ). Thus, $z$ is adjacent to both $x$ and $y$. Thus, either $N(z) \cap N(x)$ or $N(z) \cap N(y)$ contains $K_{p-2}$, so $G$ contains $K_{p}$, a contradiction.

To finish the proof of this case it is enough to see that vertices of $G \backslash V\left(G_{2}\right)$ must induce the $K_{3}$-free graph that is not bipartite. Moreover, since $G$ has a maximum size $G_{1}=G \backslash V\left(G_{2}\right) \in C_{5}^{*}$. Hence $G=G_{1}+G_{2}$, where $G_{1} \in C_{5}^{*}$ and $G_{2}$ is $(p-3)$-partite.

Subcase 2.2. $\Delta-s=1$. Let $F=G_{1}+G_{2}$, where $G_{1}=C_{5}[1,1,1,1, n-\Delta-2]$ and $G_{2} \in T_{p-3}(\Delta-2)$. Note that $e(F)=(n-\Delta) \Delta+3(\Delta-2)+e\left(T_{p-3}(\Delta-2)\right)$. Thus,
(*) $\quad e(G) \geq n \Delta-\Delta^{2}+2 \Delta-5+e\left(T_{p-3}(\Delta-2)\right)$.
Let $w=N(v) \backslash S$. Since $S$ is $K_{p-2}$-free, every ( $p-2$ )-clique of $G[N(v)]$ contains $w$. From fact that $N(x) \cap N(v)$ and $N(y) \cap N(v)$ contain $K_{p-2}$, we have $w x \in E(G)$ and $w y \in E(G)$. Since $d_{G}(w) \leq \Delta$, we have $d_{S}(w) \leq s-2$. Let $u \in S$ such that $w u \notin E(G)$. Let $S^{\prime}=S \backslash\{u\}$. We can calculate the size of $G$ in the following way $e(G)=d(v)+e(H \backslash\{x, y\})+e(\{x, y\}, N(v))+e\left(\{w, u\}, S^{\prime}\right)+1+e\left(G\left[S^{\prime}\right]\right)$.

Since $\Delta(G)=\Delta, e(H \backslash\{x, y\}) \leq(|H|-2) \Delta$. By Lemma 10, $e(\{x, y\}, N(v)) \leq$ $2 \Delta-2$. Since $w$ is nonadjacent to two vertices of $S, e\left(\{w, u\}, S^{\prime}\right) \leq 2 \Delta-5$. Thus, $e(G) \leq \Delta+(n-\Delta-3) \Delta+2 \Delta-2+2 \Delta-5+1+e\left(T_{p-3}(\Delta-2)\right)=n \Delta-\Delta^{2}+$ $2 \Delta-6+e\left(T_{p-3}(\Delta-2)\right)$. But this contradicts (*).

In the next lemma we show how to distribute the edges in the graph $G=G_{1}+G_{2}$ such that $G_{1} \in C_{5}^{*}$ and $G_{2}$ is a complete ( $p-3$ )-partite graph to obtain the maximum size.

Lemma 12. Let $p \geq 4$ and $n \geq p+2, n=(p-1) q+r,(r=0,1, \ldots, p-2)$. Let $G=G_{1}+G_{2}$ be the n-vertex graph such that $G_{1} \in C_{5}^{*}$ and $G_{2}$ is a complete $(p-3)$-partite graph. If the graph $G$ has the maximum size, then the following conditions hold:
(1) $\begin{cases}\text { for } q=1,2, \quad v\left(G_{1}\right)=5, \\ \text { for } q \geq 3, & v\left(G_{1}\right) \in \begin{cases}\{2 q-1,2 q\} & \text { for } r=0, \\ \{2 q-1,2 q, 2 q+1\} & \text { for } r=1,2, \ldots, p-4, \\ \{2 q, 2 q+1\} & \text { for } r=p-3, \\ \{2 q+1\} & \text { for } r=p-2,\end{cases} \end{cases}$
(2) $G_{1}$ is the graph of maximum size in $C_{5}^{*}$ and $G_{2} \in T_{p-3}^{*}$.

Proof. Because the number of edges in $E\left(V\left(G_{1}\right), V\left(G_{2}\right)\right)$ depends neither on the structure of $G_{1}$ nor on the structure of $G_{2}$, it is easy to see that the condition (2) is satisfied for a graph of maximum size. We show that the condition (1) also holds. We prove this in the following way: if $G_{1}$ has less vertices than in thesis, then we delete a vertex from $G_{2}$ and add it to a proper set of $G_{1}$ and we show that the resulting graph has more edges; if $G_{1}$ has more vertices than in thesis, then we delete a vertex from $G_{1}$ and add it to a proper set of $G_{2}$ and we show that the new graph has more edges.

Let $v\left(G_{1}\right)=t$ and $A\left(t, K_{3}\right), B\left(t, K_{3}\right), C\left(t, K_{3}\right)$ be the families of graphs defined in Section 2. From Lemma 4 it immediately follows

Claim 4. If $t$ is odd, then $G_{1} \in C\left(t, K_{3}\right)$ and in $G_{1}$ there is a vertex $x$ such that $x$ is nonadjacent to $\frac{t-1}{2}+1$ vertices of $G_{1}$.
Claim 5. If $t$ is even, then $G_{1} \in A\left(t, K_{3}\right) \cup B\left(t, K_{3}\right)$ and in $G_{1}$ there is a vertex $x$ such that $x$ is nonadjacent to $\frac{t}{2}+1$ vertices.
Claim 6. If $t$ is odd, then $G_{1} \in C\left(t, K_{3}\right)$ and we can add a new vertex $x$ and join $x$ with $\frac{t-1}{2}$ vertices of $G_{1}$ in such a way that the resulting graph is in $C_{5}^{*}$.
Claim 7. If $t$ is even, then $G_{1} \in A\left(t, K_{3}\right) \cup B\left(t, K_{3}\right)$ and we can add a new vertex $x$ and join $x$ with $\frac{t}{2}$ vertices of $G_{1}$ in such a way that the resulting graph is in $C_{5}^{*}$.
For $q=1,2$, we have $p+2 \leq n \leq 3(p-1)-1$. It is easy to see that the result holds for $n=p+2$. Assume that $n \geq p+3$ and $v\left(G_{1}\right) \geq 6$. Let $V_{1}, V_{2}, \ldots, V_{5}$ be the independent sets that replace the vertices of $C_{5}$ in $G_{1}$. Since $v\left(G_{1}\right) \geq 6$, at least one set of $V_{1}, V_{2}, \ldots, V_{5}$ has at least two vertices. Let $x$ be the vertex in this set. Thus, $d(x) \leq n-4$. Since $n \leq 3 p-4$ and $v\left(G_{1}\right) \geq 6$, we have $v\left(G_{2}\right) \leq 3 n-10$. Hence, there is a partite set of $G_{2}$ that has less than three vertices. If we shift
the vertex $x$ to this set, then we obtain a graph with more edges, because now $d(x) \geq n-3$.

Now suppose that $q \geq 3, r=0$ and $v\left(G_{1}\right) \leq 2 q-2$. Thus, $v\left(G_{2}\right) \geq(p-3) q+2$ and there is a partite set of $G_{2}$ having more than $q$ vertices. Let $x$ be the vertex in this set, so $d(x) \leq n-q-1$. Claim 6 and Claim 7 imply that if we shift the vertex $x$ to $G_{1}$, then we obtain a graph with more edges, because now $d(x) \geq n-q$.

Suppose that for $q \geq 3, r=0$ and $v\left(G_{1}\right) \geq 2 q+1$. By Claim 4 and Claim 5 there is a vertex $x$ such that $d(x) \leq n-q-1$. Because $v\left(G_{2}\right) \leq(p-3) q-1$, the graph $G_{2}$ contains a partite set with at most $q-1$ vertices. If we shift the vertex $x$ to this set, then we obtain a graph with more edges, now $d(x) \geq n-q$.

Assume that $q \geq 3$ and $r=1,2, \ldots, p-4$. If $v\left(G_{1}\right) \leq 2 q-2$, then $v\left(G_{2}\right) \geq$ $(p-3) q+r+2$. Thus, there is a partite set of $G_{2}$ having at least $q+1$ vertices. Let $x$ be a vertex in this set, so $d(x) \leq n-q-1$. By Claim 6 and Claim 7 if we shift the vertex $x$ to $G_{1}$, then we obtain a graph with more edges, because now the vertex $x$ is adjacent to at least $n-q$ edges. If $v\left(G_{1}\right) \geq 2 q+2$, then by Claim 4 and Claim 5 there is a vertex $x$ such that $d(x) \leq n-q-2$. Since $v\left(G_{2}\right) \leq(p-3) q+r-2$, the graph $G_{2}$ contains a partite set with at most $q$ vertices. If we shift the vertex $x$ to this set, then we obtain a graph with more edges, now $d(x) \geq n-q-1$.

Assume that $q \geq 3$ and $r=p-3$. If $v\left(G_{1}\right) \leq 2 q-1$ then $v\left(G_{2}\right) \geq(p-3)(q+$ $1)+1$. Thus, there is a partite set of $G_{2}$ having at least $q+1$ vertices. Let $x$ be a vertex in this set, so $d(x) \leq n-q-1$. By Claim 6 and Claim 7 we can shift the vertex $x$ to $G_{1}$ to obtain a graph with more edges, because then the vertex $x$ is adjacent to at least $n-q$ edges. If $v\left(G_{1}\right) \geq 2 q+2$, then by Claim 4 and Claim 5 there is a vertex $x$ such that $d(x) \leq n-q-2$. Since $v\left(G_{2}\right) \leq(p-3)(q+1)-2$, the graph $G_{2}$ contains a partite set with at most $q$ vertices. If we shift the vertex $x$ to this set, then we obtain a graph with more edges, now $d(x) \geq n-q-1$.

Assume that $q \geq 3$ and $r=p-2$. If $v\left(G_{1}\right) \leq 2 q$, then $v\left(G_{2}\right) \geq(p-3)(q+1)+1$. Thus, there is a partite set of $G_{2}$ with at least $q+1$ vertices. Let $x$ be a vertex in this set, so $d(x) \leq n-q-1$. By Claim 6 and Claim 7 if we shift the vertex $x$ to $G_{1}$, then we obtain a graph with more edges, because now the vertex $x$ is adjacent to at least $n-q$ edges. If $v\left(G_{1}\right) \geq 2 q+2$, then by Claim 4 and Claim 5 there is a vertex $x$ such that $d(x) \leq n-q-2$. Since $v\left(G_{2}\right) \leq(p-3)(q+1)-1$, the graph $G_{2}$ contains a partite set with at most $q$ vertices. If we shift the vertex $x$ to this set, then we obtain a graph with more edges, now $d(x) \geq n-q-1$.

Let us denote by $G_{i}$ the graphs from Lemma 12 achieving the maximum size. Let $G_{i}(i=1, \ldots, 4)$ be the graph of order $n=(p-1) q+r, 0 \leq r \leq p-2, p \geq 4$ such that $G_{i}=G_{i 1}+G_{i 2}$, where

$$
\begin{array}{ll}
G_{11}=C_{5}, & G_{12}=T_{p-3}(n-5) \\
G_{21} \in \mathrm{E}\left(2 q-1, K_{3}\right), & G_{22}=T_{p-3}(q(p-3)+r+1),
\end{array}
$$

$$
\begin{array}{ll}
G_{31} \in \mathrm{E}\left(2 q, K_{3}\right), & G_{32}=T_{p-3}(q(p-3)+r) \\
G_{41} \in \mathrm{E}\left(2 q+1, K_{3}\right), & G_{42}=T_{p-3}(q(p-3)+r-1)
\end{array}
$$

Then, from Theorem 11 and Lemma 12 it immediately follows
Theorem 13. Let $p, n, q, r$ be integers such that $p \geq 4, n \geq p+2, n=(p-1) q+$ $r, 0 \leq r \leq p-2$. Then

$$
\mathrm{E}\left(n, K_{p}\right)= \begin{cases}\left\{G_{1}\right\} & \text { for } q=1,2, \\ \left\{G_{2}, G_{3}\right\} & \text { for } q \geq 3 \text { and } r=0, \\ \left\{G_{2}, G_{3}, G_{4}\right\} & \text { for } q \geq 3 \text { and } r=1,2, \ldots, p-4, \\ \left\{G_{3}, G_{4}\right\} & \text { for } q \geq 3 \text { and } r=p-3, \\ \left\{G_{4}\right\} & \text { for } q \geq 3 \text { and } r=p-2 .\end{cases}
$$

Theorem 13 implies the following
Theorem 14. Let $p, n, q, r$ be integers such that $p \geq 3, n \geq p+2, n=(p-1) q+$ $r, 0 \leq r \leq p-2$. Then

$$
\mathrm{e}\left(n, K_{p}\right)=\frac{p-2}{2(p-1)} n^{2}-\frac{1}{p-1} n+\frac{r(r+2)}{2(p-1)}-\frac{r}{2}+1
$$

## 5. Size of Maximal $K_{p}$-Free Graphs

Note that $e\left(K_{1,1, \ldots, 1, n-p+2}\right)=\operatorname{sat}\left(n, K_{p}\right)$ (the minimum size of the maximal $K_{p^{-}}$ free graph with $n$ vertices) and $e\left(T_{p-1}(n)\right)=t_{p-1}(n)$. Since $e\left(K_{1,1, \ldots, 2, n-p+1}\right)>$ $(p-1) n-\frac{1}{2}(p-1) p-2$, Theorem 9 implies that for large $n$ there is no maximal $K_{p^{-}}$ free graph with $n$ vertices and size $m$ such that $\operatorname{sat}\left(n, K_{p}\right)<m<\mathrm{s}\left(n, K_{p}\right)$. From Theorem 14 we have that for any pair $n, m$ such that $\mathrm{e}\left(n, K_{p}\right)<m \leq t_{p-1}(n)$ each $n$-vertex maximal $K_{p}$-free graph with $n$ edges is complete $(p-1)$-partite.

Theorem 15. Let $p, n$ be integers such that $p \geq 3, n \geq 3 p+4$. Then for any integer $m$ such that $\mathrm{s}\left(n, K_{p}\right) \leq m \leq \mathrm{e}\left(n, K_{p}\right)$ there is a maximal $K_{p}$-free graph with $n$ vertices and size $m$.

Proof. Let us consider the family of $n$-vertex graphs $\alpha(n)=\{H+Q: H \in$ $\left.C_{5}^{*}, ~ Q \in T_{p-3}^{*}, v(H)+v(Q)=n\right\}$. Observe that every graph from $\alpha(n)$ is non- $(p-1)$-partite maximal $K_{p}$-free. Let $n=q(p-1)+r, 0 \leq r \leq p-2$. If $v(Q)=p-3$ and $H \in S\left((q-1)(p-1)+r-2, K_{3}\right)$, then $e(H+Q)=\mathrm{s}\left(n, K_{p}\right)$. If $v(Q)=(p-3) q+r$ and $H \in E\left(2 q, K_{p}\right)$ or $v(Q)=(p-3) q+r-1$ and $H \in E\left(2 q+1, K_{p}\right)$, then $e(H+Q)=\mathrm{e}\left(n, K_{p}\right)$. Let $\alpha_{b}(n) \subseteq \alpha(n)$ such that $\alpha_{b}(n)=\left\{H+Q: H \in C_{5}^{*}, Q \in T_{p-3}^{*}, v(Q)=b, v(H)=n-b\right\}$. Note that for any graph from $\alpha_{b}(n)$ the number of edges adjacent to vertices of $Q$ is constant. Let $e_{b}$ be the number of edges adjacent to vertices of $Q$ in the graph from $\alpha_{b}(n)$.

From Theorem 6 it follows that for any integer $e$ such that

$$
e \in\left[e_{b}+2(n-b)-5, e_{b}+\frac{1}{4}(n-b)^{2}-\frac{1}{2}(n-b)+\frac{1}{4} r^{2}+1\right]
$$

where $r \equiv n-b(\bmod 2)$
there is a graph in $\alpha_{b}(n)$ with size $e$. To complete the proof we show that for $b$ such that $p-3 \leq b \leq q(p-3)+r$ the inequality

$$
e_{b}+\frac{1}{4}(n-b)^{2}-\frac{1}{2}(n-b)+\frac{1}{4} r^{2}+2 \geq e_{b+1}+2(n-b-1)-5
$$

holds. Or, equivalently,

$$
e_{b+1}-e_{b} \leq \frac{1}{4}(n-b)^{2}-\frac{5}{2}(n-b)+\frac{1}{4} r^{2}+9
$$

To prove this observe the following: if in $H+Q \in \alpha_{b}(n)$ we shift a vertex $v$ from $H$ to $Q$ (to the independent set $V_{1}$ with the smallest number of vertices), then we must delete all edges joining $v$ with $V_{1}$ and add all edges joining $v$ with $H$ to obtain a graph from $\alpha_{b+1}(n)$. Thus, $e_{b+1}-e_{b}=n-b-1-\left|V_{1}\right|$. To finish the proof we conclude that $n-b-1-\left|V_{1}\right| \leq \frac{1}{4}(n-b)^{2}-\frac{5}{2}(n-b)+\frac{1}{4} r^{2}+9$. Indeed, when $b \geq 3(p-3)$, we have $\left|V_{1}\right| \geq 3$, so $n-b-1-\left|V_{1}\right| \leq n-b-4 \leq$ $\frac{1}{4}(n-b)^{2}-\frac{5}{2}(n-b)+\frac{1}{4} r^{2}+9$. When $p-3 \leq b \leq 3(p-3)-1$, we have $n-b \geq 14$. Thus, $n-b-1-\left|V_{1}\right| \leq n-b-2 \leq \frac{1}{4}(n-b)^{2}-\frac{5}{2}(n-b)+\frac{1}{4} r^{2}+9$, for $n-b \geq 14$.

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