ON VERTICES ENFORCING A HAMILTONIAN CYCLE

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Abstract

A nonempty vertex set $X \subseteq V(G)$ of a hamiltonian graph $G$ is called an $H$-force set of $G$ if every $X$-cycle of $G$ (i.e. a cycle of $G$ containing all vertices of $X$) is hamiltonian. The $H$-force number $h(G)$ of a graph $G$ is defined to be the smallest cardinality of an $H$-force set of $G$. In the paper the study of this parameter is introduced and its value or a lower bound for outerplanar graphs, planar graphs, $k$-connected graphs and prisms over graphs is determined.

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1. Introduction

One of the most intensively studied areas in graph theory deals with questions concerning cycles. The development of this area has undergone a natural growth and evolution in the questions studied and results obtained. One particular subarea involves questions about cycles containing specific sets of vertices of a graph, see e.g. a recent survey paper [13].

This paper is intended to contribute to this area. Throughout this article we consider finite simple hamiltonian graphs. We shall try to answer the question how small the cardinality of a subset of the vertex set of a given hamiltonian graph can be that the only cycles containing this subset are hamiltonian ones.

We shall use a standard terminology according to [7] except for some terms defined throughout this paper.

For a graph $G$ and a set $X \subseteq V(G)$, an $X$-cycle of $G$ is a cycle containing all vertices of $X$. Let $G$ be a hamiltonian graph. A nonempty vertex set $X \subseteq V(G)$ is called a hamiltonian cycle enforcing set (in short an H-force set) of $G$ if every $X$-cycle of $G$ is hamiltonian. For the graph $G$ we define $h(G)$ to be the smallest cardinality of an H-force set of $G$ and call it the H-force number of $G$.

In this paper we study the H-force number for several families of graphs. First we survey known results on this parameter for some families of graphs originally stated in different terms.

The following is obvious

**Proposition 1.** If $X$ is an H-force set of a graph $G$ and $X \subseteq Y \subseteq V(G)$, then $Y$ is an H-force set of $G$ too.

**Proposition 2.** If $H$ is a hamiltonian spanning subgraph of $G$, then $h(H) \leq h(G)$.

**Proposition 3.** If $C$ is a nonhamiltonian cycle of $G$, then any H-force set of $G$ contains a vertex of $V(G) \setminus V(C)$.

The following example demonstrates that the task to determine the H-force number of a graph is not easy in general.

**Example 4.** Let $G$ be the dodecahedral graph. Then $h(G) = 15$.

**Proof.** Let $X \subseteq V(G)$ be an H-force set of $G$ and let $\bar{X} = V(G) \setminus X$.

It is easy to see that $\bar{X}$ does not contain any of the following configurations (because the subgraph induced on the remaining vertices is hamiltonian, see Figure 1).

(a) two vertices with distance 3 (e.g. 1, 7),
(b) two vertices with distance 5 (e.g. 1, 19),
(c) three vertices inducing a 3-path (e.g. 1, 2, 3).
Suppose that there is an H-force set $X \subseteq V(G)$ with $|X| \leq 14$.

Case 1. If there are two adjacent vertices in $\bar{X}$. Then, without loss of
generality, let $4, 5 \in \bar{X}$, then $9, 10, 11, 12, 13, 14, 15, 16, 19, 20 \in X$ by (a), $17, 18 \in X$ by (b), and $1, 3, 6, 8 \in X$ by (c), thus $|X| \geq 16$, a contradiction.

Case 2. If any two vertices of $\bar{X}$ are nonadjacent, then there are two vertices
in $\bar{X}$ incident with the same face of $G$; without loss of
generality, let $6, 8 \in \bar{X}$.

Hence, $4, 5, 7 \in X$ (Case 1), $1, 2, 3, 9, 11, 13, 15, 16, 17, 18, 19 \in X$ by (a), and
$10 \in X$ or $20 \in X$ by (a), as well. Finally, $|X| \geq 15$, a contradiction.

It is still necessary to show that there is an H-force set of size 15 in $G$.
Let $X_1 = \{1, 2, 3, 4, 5\}$, $X_2 = \{7, 9, 11, 13, 15, 16, 17, 18, 19, 20\}$, $X = X_1 \cup X_2$, $\bar{X} = V(G) \setminus X$ and let $C$ be an X-cycle of $G$. The subgraph $G[X]$ of $G$ induced
by $X$ consists of two components $G[X_1]$ and $G[X_2]$, the second of them contains
five vertices of degree 1. Therefore, $C$ contains at least two edges between $X_1$ and $\bar{X}$ and at least five edges between $X_2$ and $\bar{X}$. Hence, $C$ contains at least
seven edges between $X$ and $\bar{X}$, thus at least four vertices of $\bar{X}$. Because $G$ does
not contain any cycle of length 19, $C$ must be a hamiltonian cycle of $G$ and $X$ is
an H-force set of $G$. 

2. 1-hamiltonian Graphs

Through this paper, the number of vertices (the order) of a graph will be denoted
by $n$. A graph $G$ is $k$-hamiltonian ($1 \leq k \leq n - 3$) if $G - U$ is hamiltonian for
every $U \subseteq V(G)$ with $0 \leq |U| \leq k$. In particular, $G$ is 1-hamiltonian, if it is
hamiltonian and for any vertex $u \in V(G)$ the graph $G - u$ is hamiltonian too, i.e.
any $n - 1$ vertices lie on a common nonhamiltonian cycle of $G$ and thus there is
no H-force set of cardinality $n - 1$ in $G$. 

Proposition 5. The 1-hamiltonian graphs are exactly the graphs with H-force number equal to their order.

Several sufficient conditions for graphs to be 1-hamiltonian have been obtained by various authors. The following two conditions in terms of vertex degrees are of Dirac-type and of Ore-type, respectively.

Theorem 6 (Chartrand, Kapoor, Link [6]). Let $G$ be a graph of order $n \geq 4$. If $\delta(G) \geq \lceil \frac{n}{2} \rceil + 1$, then $G$ is 1-hamiltonian.

Theorem 7 (Chartrand, Kapoor, Link [6]). Let $G$ be a graph of order $n \geq 4$. If for every pair of non-adjacent vertices $x, y \in V(G)$, $\deg_G(x) + \deg_G(y) \geq n + 1$, then $G$ is 1-hamiltonian.

The connectivity and the independence number of a graph $G$ will be denoted by $\kappa(G)$ and $\alpha(G)$, respectively. A simple relationship linking the connectivity, the independence number and hamiltonian properties was discovered by Chvátal and Erdős [9], namely, that a graph $G$ is hamiltonian if $\alpha(G) \leq \kappa(G)$, and, moreover

Theorem 8 (Chvátal, Erdős [9]). If $G$ is a graph with $\kappa(G) \geq 3$ and $\alpha(G) < \kappa(G)$, then $G$ is 1-hamiltonian.

A major theorem of Tutte [21] states that every 4-connected planar graph $G$ is hamiltonian. The following strengthening was obtained by the same proof technique.

Theorem 9 (Nelson [17]). Every 4-connected planar graph $G$ is 1-hamiltonian.

A Halin graph is a union of a tree $T \neq K_2$ without vertices of degree 2 and a cycle $C$ connecting the leaves of $T$ in the cyclic order determined by a plane embedding of $T$. Bondy [2] showed that every Halin graph is hamiltonian and improved this statement to the following (unpublished) result (see [16]).

Theorem 10 (Bondy). Every Halin graph $G$ is 1-hamiltonian.

A graph $G$ is claw-free if it has no induced subgraph isomorphic to $K_{1,3}$ (the claw), and it is locally connected (locally $k$-connected) if, for each vertex $u \in V(G)$, the neighbourhood $N(u)$ of $u$ induces a connected ($k$-connected) subgraph. Oberly and Sumner [18] have shown that every connected, locally connected, claw-free graph of order $\geq 3$ is hamiltonian.

Theorem 11 (Broersma, Veldman [4]). If $G$ is a connected, locally 2-connected, claw-free graph of order $\geq 4$, then $G$ is 1-hamiltonian.

The $k$-th power $G^k$ of a graph $G$ is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if their distance in $G$ is $\leq k$. The famous result of Fleischner [12] states that the square $G^2$ of any 2-connected graph $G$ is hamiltonian.
Theorem 12 (Chartrand et al. [5]). The square $G^2$ of a 2-connected graph $G$ is 1-hamiltonian.

All conditions of Theorems 6–12 are also sufficient for the mentioned graphs to be hamiltonian connected (Erdős, Gallai [11]; Ore [19]; Chvátal, Erdős [9]; Thomassen [20] and Chiba, Nishizeki [8]; Barefoot [1]; Kanetkar, Rao [14]; Chartrand et al. [5]). Recall, that a graph $G$ is hamiltonian connected if any two vertices of $G$ are connected by a hamiltonian path. Nevertheless, there exist graphs that are either 1-hamiltonian or hamiltonian connected. The graph $G_1$ (Figure 2, see Zamfirescu [22]) is 1-hamiltonian, but not hamiltonian connected and the graph $G_c$ (Figure 2) is hamiltonian connected, but not 1-hamiltonian. Both are very probably the smallest graphs of its type.

\begin{figure}
\centering
\includegraphics[width=0.7\textwidth]{figure2}
\caption{Figure 2}
\end{figure}

There are a lot of results concerning $k$-hamiltonian graphs, however, in this paper we start to study the H-force number with the aim to find a decomposition of the class of hamiltonian graphs in which the 1-hamiltonian graphs (including $k$-hamiltonian graphs, $k \geq 2$, as subsets) form an extremal subclass.

3. Graphs with Given H-force Number

Now, we will answer the question for which pairs of integers $k$ and $n$ with $n \geq 3$ and $1 \leq k \leq n$ there exists a hamiltonian graph $G$ of order $n$ such that $h(G) = k$.

For the cycle $C_n$ and the wheel $W_n$ of order $n$ it is obvious that $h(C_n) = 1$ and $h(W_n) = n$. But what can we say for $k$ with $2 \leq k \leq n - 1$?

Theorem 13. For all integers $k$ and $n$ where $2 \leq k \leq n - 2$ there exists a (planar) hamiltonian graph $G$ of order $n$ with $h(G) = k$.

Proof. Consider the cycle $C_n = [v_1, v_2, \ldots, v_n]$. Let $G$ be the graph with the vertex set $V = V(C_n)$ and the edge set $E = E(C_n) \cup \{v_2v_n\} \cup \{v_iv_{i+1} \mid 3 \leq i \leq k\} \cup \{v_kv_n\}$. Note that the graph induced by $\{v_1, v_2, \ldots, v_k, v_n\}$ in $G$ is the wheel $W_k$ (or the cycle $C_3$, if $k = 2$). The graph $G$ is hamiltonian and even planar. It
is not difficult to see that \( \{v_1, \ldots, v_{k-1}\} \cup \{u\} \), for any \( u \in \{v_{k+1}, \ldots, v_{n-1}\} \), is the smallest H-force set of \( G \).  

**Theorem 14.** For every integer \( n \geq 10 \) there exists a hamiltonian graph \( G \) of order \( n \) with \( h(G) = n - 1 \).

**Proof.** Consider two complete graphs \( K_3 = (V_1, E_1) \) and \( K_{n-7} = (V_2, E_2) \) with the vertex set \( V_1 = \{y_1, y_2, y_3\} \) and \( V_2 = \{z_1, z_2, \ldots, z_{n-7}\} \), respectively. Let \( G \) be the graph with the vertex set \( V = V_1 \cup V_2 \cup \{x_0, x_1, x_2, x_3\} \) and the edge set \( E = E_1 \cup E_2 \cup \{x_0u \mid u \in V\} \cup \{x_iy_i, x_ix_i \mid i = 1, 2, 3\} \). The graph \( G \) is hamiltonian and \( V \setminus \{x_0\} \) is the smallest H-force set of \( G \), because, for any \( u \in V \setminus \{x_0\} \), the graph \( G - u \) is hamiltonian. 

The next two theorems provide existence results with respect to the more special class of polyhedral (i.e. 3-connected planar) hamiltonian graphs.

**Theorem 15.** For every integers \( n \geq 9 \) and \( k \) where \( 5 \leq k \leq n - 4 \) there exists a polyhedral hamiltonian graph \( G \) of order \( n \) with \( h(G) = k \).

**Proof.** Let \( C = [x_1, \ldots, x_6] \) be a cycle in the plane with a vertex \( x_0 \) in the inner face and with a path \( P = [y_1, \ldots, y_r] \) with \( r \geq 0 \) in the outer face. We connect \( x_0 \) with every vertex of \( C \), \( x_1 \) with every vertex of \( P \) and introduce edges \( x_2y_1, x_6y_r \). Moreover, let \( Q = [z_1, \ldots, z_s] \) with \( s \geq 2 \) be a path in the unbounded face of the above constructed plane graph. We connect \( z_1 \) with \( x_4 \) and every vertex of \( Q \) with the vertices \( x_2 \) and \( x_6 \). The resulting graph \( G = (V, E) \) of order \( n = r + s + 7 \) is polyhedral where \( [x_1, y_1, \ldots, y_r, x_6, x_5, x_4, z_1, \ldots, z_s, x_2, x_3, x_0] \) is a hamiltonian cycle.

First, we will see that \( G - v \) is hamiltonian for every \( v \in S = \{x_1, x_3, x_5, y_1, \ldots, y_r, z_1, z_s\} \). Hence, every H-force set \( F \) of \( G \) contains \( S \) as a subset. \( G - x_1 \) is hamiltonian with \( [x_0, x_2, x_3, x_4, z_1, \ldots, z_s, x_6, x_5] \) if \( r = 0 \) and with \( [x_2, y_1, \ldots, y_r, x_6, x_5, x_0, x_3, x_4, z_1, \ldots, z_s] \), otherwise. \( G - x_3 \) is hamiltonian with \( [x_1, y_1, \ldots, y_r, x_6, x_5, x_4, z_1, \ldots, z_s, x_2, x_0] \) and, by symmetry \( G - x_5 \) is hamiltonian, too. If \( r > 0 \) then \( G - y_6 \) with \( 1 \leq i \leq r \) is hamiltonian with \( [x_2, y_1, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_r, x_6, x_5, x_0, x_3, x_4, z_1, \ldots, z_s] \), \( G - z_1 \) is hamiltonian with \( [x_1, y_1, \ldots, y_r, x_6, z_2, \ldots, z_s, x_2, x_3, x_4, x_5, x_0] \) and, \( G - z_s \) is hamiltonian with \( [x_1, y_1, \ldots, y_r, x_6, x_5, x_4, z_1, \ldots, z_s, x_2, x_3, x_0] \).

Now we prove that \( S \) is an H-force set of \( G \) which implies \( h(G) = |S| = r + 5 \). For this purpose it is sufficient to show that \( G - v \) for any \( v \in V \setminus S \) has no \( S \)-cycle. Suppose, for the contrary, that for some \( v \in V \setminus S \) there exists an \( S \)-cycle \( D \) in \( G - v \).

In the case \( v = x_0 \) we have \( x_0x_i \notin E(G - v) \) for \( i = 1, \ldots, 6 \). So, \( x_3, x_5 \in S \) implies that \( D \) contains the path \( [x_2, x_3, x_4, x_5, x_6] \) and, \( x_4z_1 \notin E(D) \). By \( z_1 \in S \) we have \( z_1x_2 \) or \( z_1x_6 \in E(D) \), say \( z_1x_2 \in E(D) \). Then, \( x_2z_2 \notin E(D) \) for \( j = \ldots \)
2, . . . , s and, because of $z_s \in S$ the path $[z_1, . . . , z_s, x_6]$ is contained in $D$. Thus, $D = [x_2, . . . , x_6, z_s, . . . , z_1]$, a contradiction.

In the case $v = x_2$ we have $x_2x_3, x_2z_j \notin E(G - v)$ for $j = 1, . . . , s$. Then, because of $z_1, z_s \in S$ the path $[x_4, z_1, . . . , z_s, x_6]$ is contained in $D$, because otherwise $D = [z_1, . . . , z_s, x_6]$, a contradiction. Moreover, $x_3 \in S$ implies that $D$ contains the path $[x_0, x_3, x_4]$. Hence, $x_4x_5 \notin E(D)$ and $D$ contains also the path $[x_0, x_5, x_6]$ which yields $D = [x_0, x_3, x_4, z_1, . . . , z_s, x_6, x_5]$, a contradiction.

In the case $v = x_6$ by symmetry we obtain a contradiction, too.

In the case $v = x_4$ we have $x_4x_0, x_4x_3, x_4x_5, x_4z_1 \notin E(G - v)$. Because of $x_3, x_5 \in S$ the path $[x_2, x_3, x_0, x_5, x_6]$ is contained in $D$ and, because of $z_1, z_2 \in S$ exactly one of the paths $[x_2, z_1, . . . , z_s, x_6]$ and $[x_2, z_s, . . . , z_1, x_6]$ is contained in $D$ which gives a contradiction.

Let us consider now the case $v = z_{j_0}$ where $1 < j_0 < s$. Because of $z_s \in S$ there exists a $j_1$ with $j_0 < j_1 < s$ such that $D$ contains one of the two paths $[x_2, z_1, . . . , z_s, x_6]$, $[x_2, z_s, . . . , z_1, x_6]$. Without loss of generality, we may assume that $D$ contains $[x_2, z_1, . . . , z_s, x_6]$. Moreover, $z_1 \in S$ implies that there exists a $j_2$ with $1 \leq j_2 < j_0$ such that $D$ contains either (i) one of the two paths $[x_2, z_1, . . . , z_{j_2}, x_6]$, $[x_2, z_{j_2}, . . . , z_1, x_6]$ or (ii) one of the two paths $[x_4, z_1, . . . , z_{j_2}, x_2]$, $[x_4, z_1, . . . , z_{j_2}, x_6]$. In case (i) $D$ is equal to one of the cycles $[x_2, z_{j_1}, . . . , z_s, x_6, z_{j_2}, . . . , z_1, x_6]$, $[x_2, z_{j_1}, . . . , z_s, x_6, z_1, . . . , z_{j_2}]$, which yields a contradiction. In case (ii) by symmetry we may assume that $D$ contains $[x_4, z_1, . . . , z_{j_2}, x_2]$. Hence, $x_2x_3 \notin E(D)$. Then, by $x_3 \in S$ the path $[x_0, x_3, x_4]$ is contained in $D$ which implies that $x_4x_5 \notin E(D)$. Then, because of $x_5 \in S$ the path $[x_0, x_5, x_6]$ is also contained in $D$ which yields $D = [x_2, z_{j_2}, . . . , z_1, x_4, x_3, x_0, x_5, x_6, z_s, . . . , z_1]$, a contradiction. Thus, $S$ is proved to be an $H$-force set of $G$.

If $n$ is the order and $k$ the $H$-force number of $G$, then the relations $n = r + s + 7$ and $k = r + 5$ together with $r \geq 0$ and $s \geq 2$ imply $n \geq 9$ and $5 \leq k \leq n - 4$ which completes the proof.

For the following theorem which considers the remaining three cases $k = n - 3, n - 2, n - 1$ we present the construction figures for a proof but (for shortness of this paper) not the complete proof.

**Theorem 16.** For every integers $n \geq n_0$ and $s \in \{1, 2, 3\}$ there exists a polyhedral hamiltonian graph $G$ of order $n$ with $h(G) = n - s$, where $n_0 = 12, 16, 14$ for $s = 3, 2, 1$, respectively.

**Proof.** In the case $s = 3$ let the cycles $C_1 = [x_1, x_2, x_3]$, $C_2 = [y_1, . . . , y_6]$ and $C_3 = [z_1, z_2, z_3]$ be drawn one into each other in the plane such that $C_1$ is the outer and $C_3$ the inner one and connect the cycles by the edges $x_1y_1, x_2y_3, x_3y_5, z_1y_2, z_2y_4$ and $z_3y_6$. If $n$ is greater than $n_0 = 12$ then let, in addition, the path $P = [u_1, . . . , u_{n-12}]$ be drawn in the unbounded face where $x_1$ is connected with
all vertices of $P$ by an edge and $x_2u_1$, $x_3u_{n-12}$ are additional edges. The so constructed polyhedral graph $G$ of order $n$ is hamiltonian and $V(G) \setminus \{y_2, y_4, y_6\}$ is a smallest $H$-force set of $G$.

In the case $s = 2$ let the cycles $C_1 = [x_1, \ldots, x_4]$, $C_2 = [y_1, y_2, \ldots, y_8]$ and $C_3 = [z_1, \ldots, z_4]$ be drawn one into each other in the plane such that $C_1$ is the outer and $C_3$ the inner one and connect the cycles by the edges $x_1y_1$, $x_2y_3$, $x_3y_5$, $x_4y_7$, $z_1y_2$, $z_2y_4$, $z_3y_6$ and $z_4y_8$. If $n$ is greater than $n_0 = 16$ then let, in addition, the path $P = [u_1, \ldots, u_{n-16}]$ be drawn in the unbounded face where $x_1$ is connected with all vertices of $P$ by an edge and $x_2u_1$, $x_3u_{n-16}$ are additional edges. The so constructed polyhedral graph $G$ of order $n$ is hamiltonian and $V(G) \setminus \{x_2, x_4\}$ is a smallest $H$-force set of $G$.

In the case $s = 1$ let a cycle $C = [x_1, \ldots, x_9]$ be drawn in the plane and let $z$ be a vertex in the bounded face which is connected with each vertex of $C$ by an edge. Moreover, let $K_{1,3}$ be a claw in the unbounded face with endvertices $y_1$, $y_2$, $y_3$. Let the claw be connected with $C$ by edges $y_1x_2$, $y_1x_3$, $y_2x_5$, $y_2x_6$, $y_3x_8$ and $y_3x_9$. If, now, $n$ is greater than $n_0 = 14$ then let, in addition the path $P = [u_1, \ldots, u_{n-14}]$ be drawn in the unbounded face where $x_1$ is connected with all vertices of $P$ by an edge and $x_2u_1$, $x_9u_{n-14}$ are additional edges. The so constructed polyhedral graph $G$ of order $n$ is hamiltonian and $V(G) \setminus \{z\}$ is a smallest $H$-force set of $G$.

4. Bipartite Graphs

If the number of components of a graph $G$ is denoted by $c(G)$ we have

**Proposition 17.** Let $G$ be a hamiltonian graph of order $n$. If there exists a set $S \subseteq V(G)$ with $c(G - S) = |S|$, then $h(G) \leq n - |S|$.

**Proof.** Let $X = V(G) \setminus S$. Any $X$-cycle of $G$ requires $|S|$ additional vertices, thus it is a hamiltonian one and thereby $X$ is an $H$-force set of $G$. \hfill \blacksquare

There are two noteworthy special cases of the previous statement, the first, if $|S| = 2$

**Corollary 18.** If $G$ is a hamiltonian graph of order $n$ with $\kappa(G) = 2$, then $h(G) \leq n - 2$.

and the second, if every component of $G - S$ is a single vertex.

**Corollary 19.** If $G$ is a hamiltonian graph of order $n$ with $\alpha(G) = \frac{n}{2}$, then $h(G) \leq \frac{n}{2}$.

Applying Corollary 19 to the complete bipartite graph $K_{\frac{n}{2}, \frac{n}{2}}$ and considering that any $k$ vertices with $k < \frac{n}{2}$ are contained in a nonhamiltonian cycle we obtain
Corollary 20. \( h(K_{\frac{n}{2}}, \frac{n}{2}) = \frac{n}{2} \). 

Dirac [10] proved that any \( k \) vertices of a \( k \)-connected graph lie on a common cycle. We use this result to prove the following.

**Theorem 21.** If \( G \neq C_n \) is a hamiltonian graph, then \( h(G) \geq \kappa(G) \).

**Proof.** Let \( G \) be a hamiltonian graph with \( \kappa(G) = \kappa \). Since for \( \kappa = 2 \) the proposition is obvious, let \( \kappa \geq 3 \).

For any vertex \( u \in V(G) \), the graph \( G - u \) is \( (\kappa - 1) \)-connected. For any set \( X \subseteq V(G - u) \) with \( |X| = \kappa - 1 \), by the above mentioned result of Dirac, there is an \( X \)-cycle in \( G - u \) that is obviously nonhamiltonian in \( G \). Therefore, there is no H-force set in \( G \) consisting of \( \kappa - 1 \) vertices.

Moreover, graphs resulting from \( K_{\frac{n}{2}}, \frac{n}{2} \) by adding any edges in exactly one partite set have \( h = \kappa = \frac{n}{2} \), i.e. the lower bound on H-force number in the last theorem is tight.

The **prism** over a graph \( G \) is the Cartesian product \( G \boxtimes K_2 \) of \( G \) with \( K_2 \), i.e. the prism over \( G \) is obtained by taking two copies of \( G \) and joining the two copies of each vertex by a vertical edge. We identify \( G \) with one of its copies in \( G \boxtimes K_2 \) and denote \( \tilde{G} \) the other copy of \( G \). This notation is extended, in an obvious way, to vertices, edges and subgraphs of \( G \boxtimes K_2 \). Moreover, if \( y = \tilde{x} \), we set \( \tilde{y} = x \), in other words, \( \tilde{\tilde{x}} = x \).

For a path \( P \) and two vertices \( x, y \in V(P) \) let \( [x, y]_P \) be the subpath of \( P \) from \( x \) to \( y \) and for a cycle \( C \) and two vertices \( x, y \in V(C) \) let \( [x, y]_C \) (\( [x, y]_{\tilde{C}} \)) be the path from \( x \) to \( y \) on \( C \) following the anticlockwise (clockwise) orientation of \( C \). For a vertex \( x \in V(C) \), \( x^+ \) (\( x^- \)) denotes its successor (predecessor) on \( C \) according to the anticlockwise orientation.

**Theorem 22.** Let \( G \) be a hamiltonian graph of order \( \frac{n}{2} \). Then

\[
h(G \boxtimes K_2) = \begin{cases} 
\frac{n}{2}, & \text{if } G \text{ is bipartite,} \\
n, & \text{if } G \text{ is not bipartite.}
\end{cases}
\]

**Proof.** Let \( G \) be a hamiltonian graph of order \( m = \frac{n}{2} \) and let \( C \) be a hamiltonian cycle of \( G \).

**Case 1.** If \( G \) is bipartite then the prism \( G \boxtimes K_2 \) over \( G \) is bipartite as well and \( h(G \boxtimes K_2) \leq \frac{n}{2} = m \) by Corollary 19. Moreover, for any set \( X \subseteq V(G \boxtimes K_2) \) of \( m - 1 \) vertices, there is a vertical edge \( w\tilde{w} \in E(G \boxtimes K_2) \) with \( w, \tilde{w} \notin X \), thus \( D_1 = [w^+, w^-]_C \cup w^-\tilde{w}^- \cup [\tilde{w}^+, \tilde{w}^+]_{\tilde{C}} \cup \tilde{w}^+w^+ \) (Figure 3) is a nonhamiltonian \( X \)-cycle in \( G \boxtimes K_2 \). Therefore, there is no H-force set of cardinality \( m - 1 \) in \( G \boxtimes K_2 \).
Case 2. Let \( G \) be not bipartite.

Case 2.1. If the order \( m \) of \( G \) is odd then it is easy to see that, for any vertex \( w \) of \( G \square K_2 \), there is a cycle \( D_2 \) of length \( n - 1 \) in \( G \square K_2 \) omitting just the vertex \( w \) and containing all vertical edges except of \( w \bar{w} \) (Figure 3). Hence, there is no H-force set of cardinality \( n - 1 \) in \( G \square K_2 \).

Case 2.2. If the order \( m \) of \( G \) is even then there is an edge \( uv \in E(G) \setminus E(C) \) such that \( C_1 = [u, v]^- \cup uv \) and \( C_2 = [u, v]^+ \cup uv \) are both odd cycles. Let \( G_i \) \((i = 1, 2)\) be the graph induced by \( V(C_i) \) in \( G \). For any vertex of \( G \square K_2 \) we look for a cycle omitting just this vertex. Let, without loss of generality, \( w \in V(G_1) \setminus \{v\} \). Then \( G_1 \square K_2 \) is the prism of a graph of odd order, thus by the previous case, there is a cycle \( D' \) in \( G_1 \square K_2 \) containing all vertices except of \( w \) and all vertical edges except of \( w \bar{w} \). Then \( D_3 = (D' - v\bar{v}) \cup [v, u^+]_{\overline{C}} \cup u^+ \bar{u}^+ \cup [\bar{u}^+, \bar{v}]^+_{\overline{C}} \) (Figure 3) is the desired cycle.

5. Planar Graphs

By Theorem 9 of Nelson, the H-force number of every 4-connected planar graph is equal \( n \). In section 3, planar graphs of order \( n \) and with a given H-force number \( k \) were constructed, for any \( 1 \leq k \leq n \).

A planar graph is outerplanar if it can be embedded in the plane in such a way that all its vertices are incident to the unbounded face. The weak dual \( D^*(G) \) of an outerplanar graph \( G \) is the graph obtained from the dual of \( G \) by removing the vertex corresponding to the unbounded face; it is a tree, if \( G \) is 2-connected. In this case let \( \ell(G) \) denote the number of leaves of \( D^*(G) \).

**Theorem 23.** If \( G \neq C_n \) is an outerplanar hamiltonian graph, then \( h(G) = \ell(G) \geq 2 \).

**Proof.** Let \( G \) be an outerplanar graph with a hamiltonian cycle \( C \) creating the boundary of its outerface. With every leaf of the weak dual \( D^*(G) \) there...
is associated a face $\alpha$ of $G$ incident with a chord $xy$ of $C$. All vertices of $\alpha$ except for $x$ and $y$ have degree 2 in $G$ and every H-force set $X$ of $G$ contains at least one of them. Otherwise the cycle $[x, y]_C \cup \{xy\}$ (or $[x, y]_C \cup \{xy\}$) is a nonhamiltonian cycle of $G$ omitting all 2-valent vertices of $\alpha$, a contradiction. Hence, $|X| = h(G) \geq \ell(G)$.

To prove the converse inequality it is enough to find an H-force set $X$ consisting of $\ell(G)$ vertices. If we choose one vertex of degree 2 from each face of $G$ corresponding to a leaf of the weak dual $D^*(G)$ we obtain a desired set $X$. Suppose that there exists a nonhamiltonian $X$-cycle $C'$ in $G$. Then it has to contain a chord $xy \in E(G)$. The graph $G - x - y$ consists of exactly two components each containing a vertex from $X$, but $C'$ has an empty intersection with one of them, a contradiction.

For a plane hamiltonian graph $G$ with a hamiltonian cycle $C$ let $G_C^i$ (or $G_C^o$) be the graph consisting of the cycle $C$ and all edges of $G$ lying inside (outside) of $C$. Clearly, $G_C^i$ and $G_C^o$ are both outerplanar. Taking into consideration the graphs $G_C^i$ and $G_C^o$ and the proof of the previous theorem we immediately obtain

**Theorem 24.** If $G$ is a planar hamiltonian graph with $\delta(G) \geq 3$ and $C$ a hamiltonian cycle of $G$, then $h(G) \geq \ell(G_C^i) + \ell(G_C^o) \geq 4$.

Other results about planar graphs follow in the next section.

### 6. Graphs with Small H-force Number

Let $C = [v_1, v_2, \ldots, v_n]$ be a hamiltonian cycle of $G$. We say that a chord $v_iv_j$ $(i < j - 1)$ separates vertices $v_k$, $v_l$ $(k < l - 1)$ on $C$, if they belong to different components of $C - v_i - v_j$ and, moreover, crosses the chord $v_kv_l$, if $v_kv_l \in E(G)$.

**Theorem 25.** Let $G \neq C_n$ be a hamiltonian graph and $C = [v_1, v_2, \ldots, v_n]$ be a hamiltonian cycle of $G$. Then $h(G) = 2$ if and only if

(i) there exist $x, y \in V(G)$, $\deg_G(x) = \deg_G(y) = 2$, such that every chord $v_iv_j$ $(i < j - 1)$ separates $x$ and $y$ on $C$, and

(ii) for every pair $v_iv_j$ and $v_kv_l$ $(i < j - 1, k < l - 1)$ of crossed chords $v_iv_k, v_jv_l \in E(C)$ holds.

**Proof.** Suppose $h(G) = 2$ and let $F = \{x, y\}$ be an H-force set of $G$ (i.e. every $F$-cycle of $G$ is hamiltonian). Moreover, we may assume $v_1 = x$ and $v_t = y$ where $3 \leq t \leq n - 1$.

**Claim 1.** $\deg_G(x) = \deg_G(y) = 2$,
otherwise, if \( \deg_G(x) \geq 3 \) then, for \( x^* \in N(x) \setminus \{x^-, x^+\} \), one of the cycles \( D_1 = [x, x^+T]_C \cup xx^* \) and \( D_2 = [x, x^-T]_C \cup xx^* \) contains \( y \) but does not contain one of \( x^- \) or \( x^+ \), therefore it is a nonhamiltonian \( F \)-cycle; a contradiction.

**Claim 2.** Every chord \( uw \) of \( C \) separates \( x \) and \( y \) on \( C \),

otherwise one of the cycles \( D_3 = [u, w]^+T \cup uw \) or \( D_4 = [u, w]^-T \cup uw \) is an \( F \)-cycle omitting \( u^- \) or \( u^+ \); a contradiction.

**Claim 3.** If \( v_iv_j \) and \( v_kv_l \) (\( 1 < i < k < t < j < l \)) are two crossed chords of \( C \), then \( v_iv_k, v_jv_l \in E(C) \) (i.e. \( k = i + 1 \) and \( l = j + 1 \)),

otherwise, if \( v_iv_k \not\in E(C) \) then \( D_5 = [v_k, v_j]^+T \cup v_iv_j \cup [v_i, v_l]^-T \cup v_kv_l \) is an \( F \)-cycle of \( G \) missing vertex \( v_{i+1} \); a contradiction.

To prove the converse let \( G \) be a graph satisfying properties (i) and (ii). We assume again \( v_1 = x \) and \( v_t = y \) where \( 3 \leq t \leq n - 1 \).

**Claim 4.** For every vertex \( v_i \in V(G) \setminus \{x, y\} \) there is a vertex \( v_i^* \in V(G) \) such that \( \{v_i, v_i^*\} \) separates \( x \) and \( y \) in \( G \),

because,

(a) if \( \deg_C(v_i) \geq 3 \) and \( v_iv_j \) is a chord of \( C \) crossed by \( v_{i+1}v_{j+1} \) then \( \{v_i, v_{j+1}\} \) separates \( x \) and \( y \),

(b) if \( \deg_C(v_i) \geq 3 \) and \( v_iv_j \) is a chord of \( C \) crossed by no other chord then \( \{v_i, v_j\} \) separates \( x \) and \( y \),

(c) for \( \deg_C(v_i) = 2 \) let \( P = [u, w]^+T \) be the longest subpath of \( C \) containing \( v_i \) with internal vertices of degree 2 (in \( G \)) only (i.e. \( \deg_G(u), \deg_G(w) \geq 3 \)). Then \( v_i \) separates \( x \) and \( y \) in \( G \) with the same vertex as \( w \) does or with one of the vertices \( u, w \) (in the case \( V(P) \cap \{x, y\} \neq \emptyset \)).

Finally, \( F = \{x, y\} \) is an \( H \)-force set of \( G \), because otherwise there exists a nonhamiltonian \( F \)-cycle \( C' \) missing a vertex \( v_i \). If \( \{v_i, v_i^*\} \) separates \( x \) and \( y \) in \( G \), then the vertices \( x \) and \( y \) are separated by at most 1 vertex on \( C' \); a contradiction.

Thus, any hamiltonian graph with \( H \)-force number 2 can be considered as the union of two outerplanar hamiltonian graphs with a common hamiltonian cycle which implies

**Corollary 26.** Every hamiltonian graph \( G \) with \( h(G) = 2 \) is planar.

For a graph \( G \) and a set \( X \subseteq V(G) \) we denote by \( K_X(G) \) the graph with the vertex set \( V(G) \) and the edge set \( E(G) \cup \{uv \mid u, v \in X\} \), i.e. the smallest spanning supergraph of \( G \) in which \( X \) induces a clique. Kawarabayashi [15] proved, that for any \( k \)-connected graph \( G \) and any given \( \ell \) vertices (\( k \leq \ell \leq \frac{3}{2}k \)), there is a cycle in \( G \) containing exactly \( k \) of them.
By Theorem 21, the H-force number of a 3-connected hamiltonian graph is $\geq 3$. We prove that there are only four 3-connected graphs with the H-force number 3.

**Theorem 27.** Let $G$ be a 3-connected hamiltonian graph. Then

(i) $h(G) \geq 4$ or

(ii) $G$ results from $K_{3,3}$ by adding any edges in exactly one partite set.

**Proof.** Let $G$ be a 3-connected hamiltonian graph with $h(G) = 3$. There exists an H-force set $F = \{v_1, v_2, v_3\} \subseteq V(G)$ in $G$ (i.e. every $F$-cycle of $G$ is hamiltonian). Consider an arbitrary vertex $x \in V(G) \setminus F$. By the above mentioned theorem of Kawarabayashi the graph $G$ contains a cycle $C$ through exactly three of the vertices $v_1, v_2, v_3, x$. Thus, $C$ is nonhamiltonian and, consequently, it is no $F$-cycle which allows to assume that without loss of generality $v_2, v_3, x \in V(C)$. As $G$ is 3-connected, there exist three internally disjoint $(v_1, C)$-paths $P_1, P_2, P_3$ with different endvertices $y_i \in V(P_i) \cap V(C)$, $i = 1, 2, 3$. Denote $Q_i = [y_{i+1}, y_{i+2}]_C^+$, $i = 1, 2, 3$ (indices modulo 3; see Figure 4) and let $v_j^* \in N(v_j) \setminus \{v_j^-, v_j^+\}$ for $j = 2, 3$.

![Figure 4](image)

**Case 1.** If $v_2, v_3$ belong to the same path $Q_i$ (possibly they are its endvertices) then $Q_i \cup P_{i+1} \cup P_{i+2}$ is an $F$-cycle omitting vertex $y_i$, thus nonhamiltonian, a contradiction.

**Case 2.** Let $v_2, v_3$ do not belong to the same path $Q_i$ and let one of them be identical with a vertex $y_j$, i.e. assume w.l.o.g. $v_2 = y_2, v_3 \in Q_2$ where $v_3 \notin \{y_1, y_3\}$.

The cycles $D_1 = P_1 \cup P_2 \cup Q_1 \cup Q_2$ and $D_2 = P_2 \cup P_3 \cup Q_2 \cup Q_3$ (Figure 4) are both $F$-cycles, thus hamiltonian. Therefore, $P_1, P_3, Q_1, Q_3$ are paths of length 1 (i.e. $v_1y_1, v_1y_3, y_2y_3, y_1y_2 \in E(G)$).

**Case 2.1.** If $v_3^* \in [y_3, v_3^+]_C^+$ then $D_3 = [v_2, v_3^+]_C^+ \cup v_3^*v_3 \cup [v_3, y_1]_C^+ \cup P_1 \cup P_2$ (Figure 5) is an $F$-cycle omitting the vertex $v_3^+$, a contradiction.

**Case 2.2.** If $v_3^* \in [v_3^+, y_1]_C^+$ then $D_4 = [v_2, v_3]_C^+ \cup v_3^*v_3 \cup [v_3^+, y_1]_C^+ \cup P_1 \cup P_2$ (Figure 5) is an $F$-cycle omitting the vertex $v_3^+$, a contradiction.
Case 2.3. If \( v^*_3 = v_2 \) then \( D_5 = v_2v_3 \cup [v_3, y_1]^+_C \cup P_1 \cup P_2 \) (Figure 5) is an \( F \)-cycle omitting the vertex \( y_3 \), a contradiction.

Case 2.4. If \( v^*_3 \in P_2, v^*_3 \neq v_2 \), then \( D_6 = [v_1, v^*_3]_P \cup v^*_3v_3 \cup [v_3, y_3]^+_C \cup P_3 \) and \( D_7 = [v_2, v^*_3]_P \cup v^*_3v_3 \cup [v_3, y_3]^+_C \cup P_3 \cup P_1 \cup Q_3 \) (Figure 6) are both \( F \)-cycles, thus hamiltonian and therefore \( P_2 \) and \( Q_2 \) have length 2, i.e. \( K_{3,3} \) is a spanning subgraph of \( G \).

Case 3. Let \( v_2, v_3 \) do not belong to the same path \( Q_i \) and let they be different from \( y_j \), i.e. assume w.l.o.g. \( v_2 \in Q_3 \) and \( v_3 \in Q_1 \).

\( D_8 = Q_3 \cup Q_1 \cup P_3 \cup P_1 \) (Figure 7) is an \( F \)-cycle, thus hamiltonian and therefore \( P_2 \) and \( Q_2 \) have length 1 (i.e. \( v_1y_2, v_3y_1 \in E(G) \)).

Case 3.1. If \( v^*_3 \in [v_2^+, y_3]^+_C \) then \( D_9 = [y_2, v_3]^+_C \cup v_3v^*_3 \cup [v^*_3, y_3]^+_C \cup P_3 \cup P_1 \cup Q_3 \) (Figure 7) is an \( F \)-cycle omitting the vertex \( v^*_3 \), a contradiction.

Case 3.2. If \( v^*_3 \in P_3 \) then \( D_{10} = [y_2, v_3]^+_C \cup v_3v^*_3 \cup [v^*_3, v_1]^+_C \cup P_1 \cup P_1 \cup Q_3 \) (Figure 7) is an \( F \)-cycle omitting the vertex \( y_3 \), a contradiction.
Case 3.3. If $v_3^* \in [v_2, v_3^-)_C$ then $D_{11} = [v_3, y_3]^+_C \cup P_3 \cup [y_1, v_3^*]^+_C \cup v_3^*v_3$ (Figure 8) is an $F$-cycle omitting the vertex $v_3^*$, a contradiction.

Case 3.4. If $v_1^* \in [y_1^+, v_2^*]_C$ then $D_{12} = [y_2, v_3^*]_C \cup v_3^*v_3 \cup [v_3, y_3]^+_C \cup P_3 \cup P_2$ (Figure 8) is an $F$-cycle omitting the vertex $y_1$, a contradiction.

Analogously, we obtain a contradiction in corresponding cases under consideration of $v_2$ and its neighbour $v_2^*$. There are two remaining cases:

Case 3.5. If $v_3^* \in P_3$, $v_3^* \in P_1$ and $v_j^* \neq y_{j+1}$ for $j = 2$ or $j = 3$ then $D_{13} = [v_2, v_3^*]_C \cup v_3^*v_3 \cup [v_3^*, v_1]_P \cup [v_1, v_2^*]_P \cup v_2^*v_2$ (Figure 9) is an $F$-cycle omitting the vertex $y_{j+1}$, a contradiction.

Case 3.6. If $v_3^* = y_3$ and $v_2^* = y_1$ then $D_{14} = [v_2, y_2]_C \cup P_2 \cup P_1 \cup y_1v_3 \cup [v_3, y_3]^+_C \cup y_3v_2$ and $D_{15} = [y_2, v_3]_C \cup v_3y_1 \cup [y_1, v_2]_C \cup v_2y_3 \cup P_3 \cup P_2$ (Figure 9) are both $F$-cycles, thus hamiltonian and therefore paths $P_1$ and $P_3$ have length 1 and paths $Q_1$ and $Q_3$ have length 2, i.e. $K_{3,3}$ is a spanning subgraph of $G$.

In any case, $K_{3,3}$ is a spanning subgraph of $G$. Let $X, Y \subseteq V(K_{3,3}) = V(G)$ be the bipartition of $K_{3,3}$. If $G \subseteq K_X(K_{3,3})$ then $3 = h(K_{3,3}) \leq h(G) \leq h(K_X(K_{3,3})) \leq 3$ by Proposition 2 and Corollaries 19 and 20, thus $h(G) = 3$. Otherwise, if $G' = K_{3,3} \cup \{x_1x_2, y_1y_2 \mid x_i \in X, y_i \in Y, i = 1, 2\}$ is a subgraph of $G$, then $h(G) \geq h(G') = 6$, which completes the proof.

In the previous section we proved that the H-force number of a planar hamiltonian graph $G$ with $\delta(G) \geq 3$ is lower-bounded by $\ell(G')_C + \ell(G''_C) \geq 4$.

**Theorem 28.** Let $G$ be a 3-connected planar hamiltonian graph. Then

(i) $h(G) \geq 5$ or
(ii) $G = K_4$ or $G$ results from the graph $Q_3$ of the cube by adding any edges in exactly one partite set.

**Proof.** Let $G$ be a plane 3-connected hamiltonian graph with $h(G) = 4$ and let $C$ be a hamiltonian cycle of $G$. Theorems 23 and 24 imply $\ell(G^*_C) = \ell(G^o_C) = 2$, i.e. the weak duals $D^*(G^i_C)$ and $D^*(G^o_C)$ are paths. Let $\alpha, \beta$ and $\gamma, \delta$ be the faces of $G$ corresponding to endvertices of $D^*(G^i_C)$ and $D^*(G^o_C)$, respectively, and let $F = \{x, y, u, v\}$ be an H-force set, where $x, y \in V(\alpha)$, $u \in V(\beta)$, $v \in V(\gamma)$, $v \in V(\delta)$ and $\deg_{G^*_C}(x) = \deg_{G^*_C}(y) = \deg_{G^*_C}(u) = \deg_{G^*_C}(v) = 2$.

**Claim 1.** Every chord $e \in E(G^*_C)$ (or $e \in E(G^o_C)$) of $C$ separates $x, y$ in $G^* (or u, v in C_C)$.

Let $x^*$ be a neighbour of $x$ in $G$, different from $x^+$ and $x^-$, with the smallest distance $d_C(x^*, y)$ from $y$ on $C$ and similarly, $y^* \in N(y) \setminus \{y^+, y^-, x^+, x^-, x^+, x^+\}$ with minimum $d_C(y^*, x)$, $u^* \in N(u) \setminus \{u^+, u^-, x^+, x^-, u^+, u^-\}$ with minimum $d_C(u^*, v)$ and $v^* \in N(v) \setminus \{v^+, v^-, x^+, x^-, v^+, v^-\}$ with minimum $d_C(v^*, u)$.

**Case 1.** Let $xy$, $uv \in E(G)$ (i.e. $x^* = y$, $y^* = x$, $u^* = v$, $v^* = u$), then $D_1 = [x, u]_C^+ \cup uv \cup [v, y]_C^+ \cup vy$ and $D_2 = [x, v]_C^- \cup vu \cup [u, y]_C^+ \cup vx$ (Figure 10) are $F$-cycles, hence, both are hamiltonian. Therefore, $[x, u]_C^+$, $[v, y]_C^+$, $[x, v]_C^-$ and $[u, y]_C^+$ are paths of length 1 (i.e. $xu, vy, xv, uy \in E(C)$) and finally $G = K_4$.

![Figure 10](image1.png)

**Case 2.** Let, without loss of generality, $xy \in E(G)$ and $uv \notin E(G)$ (i.e. $x^* = y$, $y^* = x$, $u^* \neq v$, $v^* \neq u$).

**Case 2.1.** If $u^* \in [y, v]_C^+$ (and consequently $v^* \in [x, u]_C^+$), then $D_3 = [x, u]_C^- \cup u^*u \cup [u, y]_C^+ \cup vy$ (Figure 11) is an $F$-cycle omitting vertex $v^*$, a contradiction.

![Figure 11](image2.png)
Case 2.2. If \( u^* \in [v, x]_C^+ \) (and consequently \( v^* \in [u, y]_C^+ \)), then \( D_4 = [x, u]_C^+ \cup uu^* \cup [u^*, y]_C^- \cup yx \) (Figure 11) is a nonhamiltonian \( F \)-cycle, a contradiction.

Case 3. Let \( xy, uv \notin E(G) \) (i.e. \( \{x^*, y^*, u^*, v^*\} \cap \{x, y, u, v\} = \emptyset \)).

Case 3.1. Let each of the paths \([x, u]_C^+, [u, y]_C^+, [y, v]_C^+, [v, x]_C^+\) contains a vertex from \( \{x^*, y^*, u^*, v^*\} \) (without loss of generality, let \( v^* \in [x, u]_C^+, x^* \in [u, y]_C^+, u^* \in [y, v]_C^+, y^* \in [v, x]_C^+\).

Then \( D_3 = [x, v]_C^+ \cup v^*x \cup [v, y]_C^+ \cup y^*y \cup [y, u]_C^+ \cup u^*u \cup [u, x]_C^+ \cup x^*x \), \( D_6 = [x, y]_C^+ \cup y^*y \cup [y, v]_C^+ \cup vv^* \cup [v, x]_C^+ \cup x^*x \), and \( D_7 = [x, u]_C^+ \cup uu^* \cup [u^*, y^*]_C^- \cup y^*y \cup [y, x^*]_C^- \cup x^*x \) (Figure 12) are \( F \)-cycles, hence all are hamiltonian. Since each of the paths \([x, v]_C^+, [v, u]_C^+, [u, x]_C^+, [x, y]_C^+, [y, u]_C^+, [u, v]_C^+, [v, y]_C^+, [y, x]_C^+\) has at least one of the hamiltonian cycles \( D_5, D_6, D_7 \) no inner vertex in common, there is no inner vertex on any of these paths, and, consequently, the cube graph \( Q_3 \) is a spanning subgraph of \( G \).

![Figure 12](image-url)

Case 3.2. Let exactly two of the paths \([x, u]_C^+, [u, y]_C^+, [y, v]_C^+, [v, x]_C^+\) contain a vertex from \( \{x^*, y^*, u^*, v^*\} \) (without loss of generality and because of claim 1 let \( x^*, v^* \in [u, y]_C^+ \) and \( y^*, u^* \in [v, x]_C^+ \)).

Case 3.2.1. Let \( u^* \notin [y^*, x]_C^+ \) (or analogously \( v^* \notin [x^*, y]_C^+ \)). Then \( D_8 = [x, u]_C^+ \cup uu^* \cup [u^*, x]_C^- \cup x^*x \) (Figure 13) is a nonhamiltonian \( F \)-cycle, a contradiction.

![Figure 13](image-url)

Case 3.2.2. Let \( u^* \in [y^*, x]_C^+ \) and \( v^* \in [x^*, y]_C^+ \). \( D_9 = [x, u]_C^+ \cup uu^* \cup [u^*, y^*]_C^- \cup y^*y \cup [y, v]_C^+ \cup vv^* \cup [v^*, x]_C^- \cup x^*x \) (Figure 13) is an \( F \)-cycle, hence it is hamiltonian. Therefore then the paths \([u, x^*]_C^+, [v^*, y]_C^+, [v, y^*]_C^+, [u^*, x]_C^+\) have length 1 (i.e.
ux*, v*y, vy*, u*x ∈ E(G)). Since G is planar and 3-connected, there exists a neighbour of u on [v, u*x] different from u* (otherwise, the set {x*, u*} would be a 2-cut of G, with contradiction), which contradicts the minimality of d_C(u*, v).

That means, G = K_4 or G contains Q_3 as a spanning subgraph. In the second case let X, Y ⊆ V(Q_3) = V(G) be the bipartition of Q_3. If G ⊆ K_X(G) then 4 = h(Q_3) ≤ h(G) ≤ h(K_X(G)) ≤ 4 by Proposition 2, Corollary 19, and Theorem 22, thus h(G) = 4. Otherwise, if G′ = Q_3 ∪ \{x_1x_2, y_1y_2 \mid x_i ∈ X, y_i ∈ Y, i = 1, 2\} is a subgraph of G, then h(G) ≥ h(G′) = 8, which completes the proof.

References

doi:10.1007/BF02579268
doi:10.1002/jgt.3190110314
doi:10.1016/0095-8956(74)90075-6
doi:10.1016/S0021-9800(70)80069-2
doi:10.1002/jgt.3190100404
doi:10.1016/0012-365X(72)90079-9
doi:10.1002/mana.19600220107


[18] D.J. Oberly and D.P. Sumner, *Every connected, locally connected nontrivial graph with no induced claw is hamiltonian*, J. Graph Theory **3** (1979) 351–356. doi:10.1002/jgt.3190030405


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