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Dedicated to Mietek Borowiecki on the occasion of his 70^{th} birthday.

ON VERTICES ENFORCING A HAMILTONIAN CYCLE

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Abstract

A nonempty vertex set $X \subseteq V(G)$ of a hamiltonian graph G is called an *H-force set* of G if every X-cycle of G (i.e. a cycle of G containing all vertices of X) is hamiltonian. The *H-force number* $h(G)$ of a graph G is defined to be the smallest cardinality of an H-force set of G. In the paper the study of this parameter is introduced and its value or a lower bound for outerplanar graphs, planar graphs, k-connected graphs and prisms over graphs is determined.

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1. Introduction

One of the most intensively studied areas in graph theory deals with questions concerning cycles. The development of this area has undergone a natural growth and evolution in the qestions studied and results obtained. One particular subarea involves questions about cycles containing specific sets of vertices of a graph, see e.g. a recent survey paper [13].

This paper is intended to contribute to this area. Throughout this article we consider finite simple hamiltonian graphs. We shall try to answer the question how small the cardinality of a subset of the vertex set of a given hamiltonian graph can be that the only cycles containing this subset are hamiltonian ones.

We shall use a standard terminology according to [7] except for some terms defined throughout this paper.

For a graph G and a set $X \subseteq V(G)$, an *X-cycle* of G is a cycle containing all vertices of X. Let G be a hamiltonian graph. A nonempty vertex set $X \subseteq V(G)$ is called a *hamiltonian cycle enforcing set* (in short an *H-force set*) of G if every X-cycle of G is hamiltonian. For the graph G we define $h(G)$ to be the smallest cardinality of an H-force set of G and call it the *H-force number* of G.

In this paper we study the H-force number for several families of graphs. First we survey known results on this parameter for some families of graphs originally stated in different terms.

The following is obvious

Proposition 1. *If X is an H-force set of a graph G and $X \subseteq Y \subseteq V(G)$, then Y is an H-force set of G too.*

Proposition 2. *If H is a hamiltonian spanning subgraph of G, then $h(H) \leq h(G)$.*

Proposition 3. *If C is a nonhamiltonian cycle of G, then any H-force set of G contains a vertex of $V(G) \setminus V(C)$.*

The following example demonstrates that the task to determine the H-force number of a graph is not easy in general.

Example 4. Let G be the dodecahedral graph. Then $h(G) = 15$.

Proof. Let $X \subseteq V(G)$ be an H-force set of G and let $\bar{X} = V(G) \setminus X$.

It is easy to see that $\bar{X}$ does not contain any of the following configurations (because the subgraph induced on the remaining vertices is hamiltonian, see Figure 1).

(a) two vertices with distance 3 (e.g. 1, 7),
(b) two vertices with distance 5 (e.g. 1, 19),
(c) three vertices inducing a 3-path (e.g. 1, 2, 3).

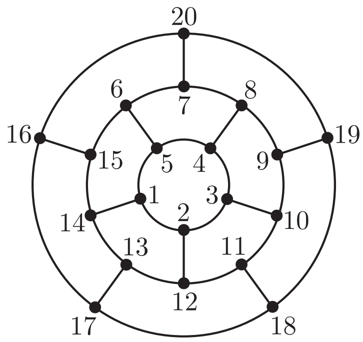


Figure 1

Suppose that there is an H-force set $X \subseteq V(G)$ with $|X| \leq 14$.

Case 1. If there are two adjacent vertices in $\bar{X}$. Then, without loss of generality, let $4, 5 \in \bar{X}$, then $9, 10, 11, 12, 13, 14, 15, 16, 19, 20 \in X$ by (a), $17, 18 \in X$ by (b), and $1, 3, 6, 8 \in X$ by (c), thus $|X| \geq 16$, a contradiction.

Case 2. If any two vertices of $\bar{X}$ are nonadjacent, then there are two vertices in $\bar{X}$ incident with the same face of G; without loss of generality, let $6, 8 \in \bar{X}$. Hence, $4, 5, 7 \in X$ (Case 1), $1, 2, 3, 9, 11, 13, 15, 16, 17, 18, 19 \in X$ by (a), and $10 \in X$ or $20 \in X$ by (a), as well. Finally, $|X| \geq 15$, a contradiction.

It is still necessary to show that there is an H-force set of size 15 in G. Let $X_1 = \{1, 2, 3, 4, 5\}$, $X_2 = \{7, 9, 11, 13, 15, 16, 17, 18, 19, 20\}$, $X = X_1 \cup X_2$, $\bar{X} = V(G) \setminus X$ and let C be an X-cycle of G. The subgraph $G[X]$ of G induced by X consists of two components $G[X_1]$ and $G[X_2]$, the second of them contains five vertices of degree 1. Therefore, C contains at least two edges between X_1 and $\bar{X}$ and at least five edges between X_2 and $\bar{X}$. Hence, C contains at least seven edges between X and $\bar{X}$, thus at least four vertices of $\bar{X}$. Because G does not contain any cycle of length 19, C must be a hamiltonian cycle of G and X is an H-force set of G. ■

2. 1-hamiltonian Graphs

Through this paper, the number of vertices (the order) of a graph will be denoted by n. A graph G is *k-hamiltonian* ($1 \leq k \leq n-3$) if $G - U$ is hamiltonian for every $U \subseteq V(G)$ with $0 \leq |U| \leq k$. In particular, G is 1-hamiltonian, if it is hamiltonian and for any vertex $u \in V(G)$ the graph $G - u$ is hamiltonian too, i.e. any $n-1$ vertices lie on a common nonhamiltonian cycle of G and thus there is no H-force set of cardinality $n-1$ in G.

Proposition 5. *The* 1*-hamiltonian graphs are exactly the graphs with H-force number equal to their order.*

Several sufficient conditions for graphs to be 1-hamiltonian have been obtained by various authors. The following two conditions in terms of vertex degrees are of Dirac-type and of Ore-type, respectively.

Theorem 6 (Chartrand, Kapoor, Link [6])**.** *Let G be a graph of order $n \geq 4$. If $\delta(G) \geq \lfloor \frac{n}{2} \rfloor + 1$, then G is* 1*-hamiltonian.*

Theorem 7 (Chartrand, Kapoor, Link [6])**.** *Let G be a graph of order $n \geq 4$. If for every pair of non-adjacent vertices $x, y \in V(G)$, $\deg_G(x) + \deg_G(y) \geq n + 1$, then G is* 1*-hamiltonian.*

The connectivity and the independence number of a graph G will be denoted by $\kappa(G)$ and $\alpha(G)$, respectively. A simple relationship linking the connectivity, the independence number and hamiltonian properties was discovered by Chvátal and Erdős [9], namely, that a graph G is hamiltonian if $\alpha(G) \leq \kappa(G)$, and, moreover

Theorem 8 (Chvátal, Erdős [9])**.** *If G is a graph with $\kappa(G) \geq 3$ and $\alpha(G) < \kappa(G)$, then G is* 1*-hamiltonian.*

A major theorem of Tutte [21] states that every 4-connected planar graph G is hamiltonian. The following strengthening was obtained by the same proof technique.

Theorem 9 (Nelson [17])**.** *Every* 4*-connected planar graph G is* 1*-hamiltonian.*

A *Halin graph* is a union of a tree $T \neq K_2$ without vertices of degree 2 and a cycle C connecting the leaves of T in the cyclic order determined by a plane embedding of T. Bondy [2] showed that every Halin graph is hamiltonian and improved this statement to the following (unpublished) result (see [16]).

Theorem 10 (Bondy)**.** *Every Halin graph G is* 1*-hamiltonian.*

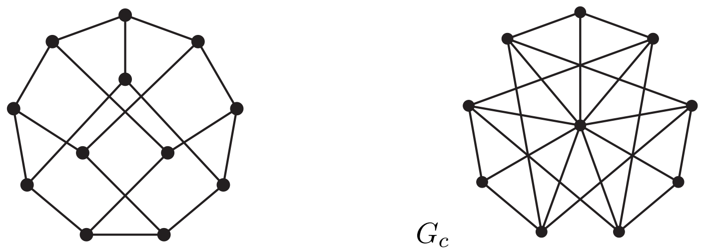
A graph G is *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$ (the claw), and it is *locally connected* (*locally k-connected*) if, for each vertex $u \in V(G)$, the neighbourhood $N(u)$ of u induces a connected (k-connected) subgraph. Oberly and Sumner [18] have shown that every connected, locally connected, claw-free graph of order ≥ 3 is hamiltonian.

Theorem 11 (Broersma, Veldman [4])**.** *If G is a connected, locally* 2*-connected, claw-free graph of order ≥ 4, then G is* 1*-hamiltonian.*

The *k-th power* G^k of a graph G is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if their distance in G is $\leq k$. The famous result of Fleischner [12] states that the square G^2 of any 2-connected graph G is hamiltonian.

Theorem 12 (Chartrand *et al.* [5]). *The square G^2 of a 2-connected graph G is 1-hamiltonian.*

All conditions of Theorems 6–12 are also sufficient for the mentioned graphs to be hamiltonian connected (Erdős, Gallai [11]; Ore [19]; Chvátal, Erdős [9]; Thomassen [20] and Chiba, Nishizeki [8]; Barefoot [1]; Kanetkar, Rao [14]; Chartrand *et al.* [5]). Recall, that a graph G is *hamiltonian connected* if any two vertices of G are connected by a hamiltonian path. Nevertheless, there exist graphs that are either 1-hamiltonian or hamiltonian connected. The graph G_1 (Figure 2, see Zamfirescu [22]) is 1-hamiltonian, but not hamiltonian connected and the graph G_c (Figure 2) is hamiltonian connected, but not 1-hamiltonian. Both are very probably the smallest graphs of its type.



G_1 G_c

Figure 2

There are a lot of results concerning k-hamiltonian graphs, however, in this paper we start to study the H-force number with the aim to find a decomposition of the class of hamiltonian graphs in which the 1-hamiltonian graphs (including k-hamiltonian graphs, $k \geq 2$, as subsets) form an extremal subclass.

3. Graphs with Given H-force Number

Now, we will answer the question for which pairs of integers k and n with $n \geq 3$ and $1 \leq k \leq n$ there exists a hamiltonian graph G of order n such that $h(G) = k$. For the cycle C_n and the wheel W_n of order n it is obvious that $h(C_n) = 1$ and $h(W_n) = n$. But what can we say for k with $2 \leq k \leq n-1$?

Theorem 13. *For all integers k and n where $2 \leq k \leq n-2$ there exists a (planar) hamiltonian graph G of order n with $h(G) = k$.*

Proof. Consider the cycle $C_n = [v_1, v_2, \ldots, v_n]$. Let G be the graph with the vertex set $V = V(C_n)$ and the edge set $E = E(C_n) \cup \{v_2 v_n\} \cup \{v_1 v_i \,|\, 3 \leq i \leq k\} \cup \{v_k v_n\}$. Note that the graph induced by $\{v_1, v_2, \ldots, v_k, v_n\}$ in G is the wheel W_k (or the cycle C_3, if $k = 2$). The graph G is hamiltonian and even planar. It is not difficult to see that $\{v_1, \ldots, v_{k-1}\} \cup \{u\}$, for any $u \in \{v_{k+1}, \ldots, v_{n-1}\}$, is the smallest H-force set of G. ■

Theorem 14. *For every integer $n \geq 10$ there exists a hamiltonian graph G of order n with $h(G) = n - 1$.*

Proof. Consider two complete graphs $K_3 = (V_1, E_1)$ and $K_{n-7} = (V_2, E_2)$ with the vertex set $V_1 = \{y_1, y_2, y_3\}$ and $V_2 = \{z_1, z_2, \ldots, z_{n-7}\}$, respectively. Let G be the graph with the vertex set $V = V_1 \cup V_2 \cup \{x_0, x_1, x_2, x_3\}$ and the edge set $E = E_1 \cup E_2 \cup \{x_0u \mid u \in V\} \cup \{x_iy_i, x_iz_i \mid i = 1, 2, 3\}$. The graph G is hamiltonian and $V \setminus \{x_0\}$ is the smallest H-force set of G, because, for any $u \in V \setminus \{x_0\}$, the graph $G - u$ is hamiltonian. ■

The next two theorems provide existence results with respect to the more special class of polyhedral (i.e. 3-connected planar) hamiltonian graphs.

Theorem 15. *For every integers $n \geq 9$ and k where $5 \leq k \leq n - 4$ there exists a polyhedral hamiltonian graph G of order n with $h(G) = k$.*

Proof. Let $C = [x_1, \ldots, x_6]$ be a cycle in the plane with a vertex x_0 in the inner face and with a path $P = [y_1, \ldots, y_r]$ with $r \geq 0$ in the outer face. We connect x_0 with every vertex of C, x_1 with every vertex of P and introduce edges x_2y_1, x_6y_r. Moreover, let $Q = [z_1, \ldots, z_s]$ with $s \geq 2$ be a path in the unbounded face of the above constructed plane graph. We connect z_1 with x_4 and every vertex of Q with the vertices x_2 and x_6. The resulting graph $G = (V, E)$ of order $n = r+s+7$ is polyhedral where $[x_1, y_1, \ldots, y_r, x_6, x_5, x_4, z_1, \ldots, z_s, x_2, x_3, x_0]$ is a hamiltonian cycle.

First, we will see that $G - v$ is hamiltonian for every $v \in S = \{x_1, x_3, x_5, y_1, \ldots, y_r, z_1, z_s\}$. Hence, every H-force set F of G contains S as a subset. $G - x_1$ is hamiltonian with $[x_0, x_2, x_3, x_4, z_1, \ldots, z_s, x_6, x_5]$ if $r = 0$ and with $[x_2, y_1, \ldots, y_r, x_6, x_5, x_0, x_3, x_4, z_1, \ldots, z_s]$, otherwise. $G - x_3$ is hamiltonian with $[x_1, y_1, \ldots, y_r, x_6, x_5, x_4, z_1, \ldots, z_s, x_2, x_0]$ and, by symmetry $G - x_5$ is hamiltonian, too. If $r > 0$ then $G - y_i$ with $1 \leq i \leq r$ is hamiltonian with $[x_2, y_1, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_r, x_6, x_5, x_0, x_3, x_4, z_1, \ldots, z_s]$. $G - z_1$ is hamiltonian with $[x_1, y_1, \ldots, y_r, x_6, z_2, \ldots, z_s, x_2, x_3, x_4, x_5, x_0]$ and, $G - z_s$ is hamiltonian with $[x_1, y_1, \ldots, y_r, x_6, x_5, x_4, z_1, \ldots, z_{s-1}, x_2, x_3, x_0]$.

Now we prove that S is an H-force set of G which implies $h(G) = |S| = r+5$. For this purpose it is sufficient to show that $G - v$ for any $v \in V \setminus S$ has no S-cycle. Suppose, for the contrary, that for some $v \in V \setminus S$ there exists an S-cycle D in $G - v$.

In the case $v = x_0$ we have $x_0x_i \notin E(G - v)$ for $i = 1, \ldots, 6$. So, $x_3, x_5 \in S$ implies that D contains the path $[x_2, x_3, x_4, x_5, x_6]$ and, $x_4z_1 \notin E(D)$. By $z_1 \in S$ we have z_1x_2 or $z_1x_6 \in E(D)$, say $z_1x_2 \in E(D)$. Then, $x_2z_j \notin E(D)$ for $j =$

$2, \ldots, s$ and, because of $z_s \in S$ the path $[z_1, \ldots, z_s, x_6]$ is contained in D. Thus, $D = [x_2, \ldots, x_6, z_s, \ldots, z_1]$, a contradiction.

In the case $v = x_2$ we have $x_2x_3, x_2z_j \notin E(G - v)$ for $j = 1, \ldots, s$. Then, because of $z_1, z_s \in S$ the path $[x_4, z_1, \ldots, z_s, x_6]$ is contained in D, because otherwise $D = [z_1, \ldots, z_s, x_6]$, a contradiction. Moreover, $x_3 \in S$ implies that D contains the path $[x_0, x_3, x_4]$. Hence, $x_4x_5 \notin E(D)$ and D contains also the path $[x_0, x_5, x_6]$ which yields $D = [x_0, x_3, x_4, z_1, \ldots, z_s, x_6, x_5]$, a contradiction.

In the case $v = x_6$ by symmetry we obtain a contradiction, too.

In the case $v = x_4$ we have $x_4x_0, x_4x_3, x_4x_5, x_4z_1 \notin E(G - v)$. Because of $x_3, x_5 \in S$ the path $[x_2, x_3, x_0, x_5, x_6]$ is contained in D and, because of $z_1, z_2 \in S$ exactly one of the paths $[x_2, z_1, \ldots, z_s, x_6]$ and $[x_2, z_s, \ldots, z_1, x_6]$ is contained in D which gives a contradiction.

Let us consider now the case $v = z_{j_0}$ where $1 < j_0 < s$. Because of $z_s \in S$ there exists a j_1 with $j_0 < j_1 \leq s$ such that D contains one of the two paths $[x_2, z_{j_1}, \ldots, z_s, x_6]$, $[x_2, z_s, \ldots, z_{j_1}, x_6]$. Without loss of generality, we may assume that D contains $[x_2, z_{j_1}, \ldots, z_s, x_6]$. Moreover, $z_1 \in S$ implies that there exists a j_2 with $1 \leq j_2 < j_0$ such that D contains either (i) one of the two paths $[x_2, z_1, \ldots, z_{j_2}, x_6]$, $[x_2, z_{j_2}, \ldots, z_1, x_6]$ or (ii) one of the two paths $[x_4, z_1, \ldots, z_{j_2}, x_2]$, $[x_4, z_1, \ldots, z_{j_2}, x_6]$. In case (i) D is equal to one of the cycles $[x_2, z_{j_1}, \ldots, z_s, x_6, z_{j_2}, \ldots, z_1]$, $[x_2, z_{j_1}, \ldots, z_s, x_6, z_1, \ldots, z_{j_2}]$ which yields a contradiction. In case (ii) by symmetry we may assume that D contains $[x_4, z_1, \ldots, z_{j_2}, x_2]$. Hence, $x_2x_3 \notin E(D)$. Then, by $x_3 \in S$ the path $[x_0, x_3, x_4]$ is contained in D which implies that $x_4x_5 \notin E(D)$. Then, because of $x_5 \in S$ the path $[x_0, x_5, x_6]$ is also contained in D which yields $D = [x_2, z_{j_2}, \ldots, z_1, x_4, x_3, x_0, x_5, x_6, z_s, \ldots, z_{j_1}]$, a contradiction. Thus, S is proved to be an H-force set of G.

If n is the order and k the H-force number of G, then the relations $n = r+s+7$ and $k = r + 5$ together with $r \geq 0$ and $s \geq 2$ imply $n \geq 9$ and $5 \leq k \leq n - 4$ which completes the proof. ■

For the following theorem which considers the remaining three cases $k = n - 3, n - 2, n - 1$ we present the construction figures for a proof but (for shortness of this paper) not the complete proof.

Theorem 16. *For every integers $n \geq n_0$ and $s \in \{1, 2, 3\}$ there exists a polyhedral hamiltonian graph G of order n with $h(G) = n - s$, where $n_0 = 12, 16, 14$ for $s = 3, 2, 1$, respectively.*

Proof. In the case $s = 3$ let the cycles $C_1 = [x_1, x_2, x_3]$, $C_2 = [y_1, \ldots, y_6]$ and $C_3 = [z_1, z_2, z_3]$ be drawn one into each other in the plane such that C_1 is the outer and C_3 the inner one and connect the cycles by the edges x_1y_1, x_2y_3, x_3y_5, z_1y_2, z_2y_4 and z_3y_6. If n is greater than $n_0 = 12$ then let, in addition, the path $P = [u_1, \ldots, u_{n-12}]$ be drawn in the unbounded face where x_1 is connected with all vertices of P by an edge and x_2u_1, x_3u_{n-12} are additional edges. The so constructed polyhedral graph G of order n is hamiltonian and $V(G) \setminus \{y_2, y_4, y_6\}$ is a smallest H-force set of G.

In the case $s = 2$ let the cycles $C_1 = [x_1, \dots, x_4]$, $C_2 = [y_1, y_2, \dots, y_8]$ and $C_3 = [z_1, \dots, z_4]$ be drawn one into each other in the plane such that C_1 is the outer and C_3 the inner one and connect the cycles by the edges x_1y_1, x_2y_3, x_3y_5, x_4y_7 z_1y_2, z_2y_4, z_3y_6 and z_4y_8. If n is greater than $n_0 = 16$ then let, in addition, the path $P = [u_1, \dots, u_{n-16}]$ be drawn in the unbounded face where x_1 is connected with all vertices of P by an edge and x_2u_1, x_3u_{n-16} are additional edges. The so constructed polyhedral graph G of order n is hamiltonian and $V(G) \setminus \{x_2, x_4\}$ is a smallest H-force set of G.

In the case $s = 1$ let a cycle $C = [x_1, \dots, x_9]$ be drawn in the plane and let z be a vertex in the bounded face which is connected with each vertex of C by an edge. Moreover, let $K_{1,3}$ be a claw in the unbounded face with endvertices y_1, y_2, y_3. Let the claw be connected with C by edges y_1x_2, y_1x_3, y_2x_5, y_2x_6, y_3x_8 and y_3x_9. If, now, n is greater than $n_0 = 14$ then let, in addition the path $P = [u_1, \dots, u_{n-14}]$ be drawn in the unbounded face where x_1 is connected with all vertices of P by an edge and x_2u_1, x_9u_{n-14} are additional edges. The so constructed polyhedral graph G of order n is hamiltonian and $V(G) \setminus \{z\}$ is a smallest H-force set of G. ■

4. Bipartite Graphs

If the number of components of a graph G is denoted by $c(G)$ we have

Proposition 17. *Let G be a hamiltonian graph of order n. If there exists a set $S \subseteq V(G)$ with $c(G - S) = |S|$, then $h(G) \leq n - |S|$.*

Proof. Let $X = V(G) \setminus S$. Any X-cycle of G requires $|S|$ additional vertices, thus it is a hamiltonian one and thereby X is an H-force set of G. ■

There are two noteworthy special cases of the previous statement, the first, if $|S| = 2$

Corollary 18. *If G is a hamiltonian graph of order n with $\kappa(G) = 2$, then $h(G) \leq n - 2$.*

and the second, if every component of $G - S$ is a single vertex.

Corollary 19. *If G is a hamiltonian graph of order n with $\alpha(G) = \frac{n}{2}$, then $h(G) \leq \frac{n}{2}$.*

Applying Corollary 19 to the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$ and considering that any k vertices with $k < \frac{n}{2}$ are contained in a nonhamiltonian cycle we obtain

Corollary 20. $h(K_{\frac{n}{2},\frac{n}{2}}) = \frac{n}{2}$.

Dirac [10] proved that any k vertices of a k-connected graph lie on a common cycle. We use this result to prove the following

Theorem 21. *If $G \neq C_n$ is a hamiltonian graph, then $h(G) \geq \kappa(G)$.*

Proof. Let G be a hamiltonian graph with $\kappa(G) = \kappa$. Since for $\kappa = 2$ the proposition is obvious, let $\kappa \geq 3$.

For any vertex $u \in V(G)$, the graph $G - u$ is $(\kappa - 1)$-connected. For any set $X \subseteq V(G - u)$ with $|X| = \kappa - 1$, by the above mentioned result of Dirac, there is an X-cycle in $G - u$ that is obviously nonhamiltonian in G. Therefore, there is no H-force set in G consisting of $\kappa - 1$ vertices. ∎

Moreover, graphs resulting from $K_{\frac{n}{2},\frac{n}{2}}$ by adding any edges in exactly one partite set have $h = \kappa = \frac{n}{2}$, i.e. the lower bound on H-force number in the last theorem is tight.

The *prism* over a graph G is the Cartesian product $G\Box K_2$ of G with K_2, i.e. the prism over G is obtained by taking two copies of G and joining the two copies of each vertex by a *vertical* edge. We identify G with one of its copies in $G\Box K_2$ and denote $\tilde{G}$ the other copy of G. This notation is extended, in an obvious way, to vertices, edges and subgraphs of $G\Box K_2$. Moreover, if $y = \tilde{x}$, we set $\tilde{y} = x$, in other words, $\tilde{\tilde{x}} = x$.

For a path P and two vertices $x, y \in V(P)$ let $[x, y]_P$ be the subpath of P from x to y and for a cycle C and two vertices $x, y \in V(C)$ let $[x, y]_C^+$ ($[x, y]_C^-$) be the path from x to y on C following the anticlockwise (clockwise) orientation of C. For a vertex $x \in V(C)$, x^+ (x^-) denotes its successor (predecessor) on C according to the anticlockwise orientation.

Theorem 22. *Let G be a hamiltonian graph of order $\frac{n}{2}$. Then*

$$h(G\Box K_2) = \begin{cases} \frac{n}{2}, & \text{if } G \text{ is bipartite,} \\ n, & \text{if } G \text{ is not bipartite.} \end{cases}$$

Proof. Let G be a hamiltonian graph of order $m = \frac{n}{2}$ and let C be a hamiltonian cycle of G.

Case 1. If G is bipartite then the prism $G\Box K_2$ over G is bipartite as well and $h(G\Box K_2) \leq \frac{n}{2} = m$ by Corollary 19. Moreover, for any set $X \subseteq V(G\Box K_2)$ of $m - 1$ vertices, there is a vertical edge $w\tilde{w} \in E(G\Box K_2)$ with $w, \tilde{w} \notin X$, thus $D_1 = [w^+, w^-]_C^+ \cup w^-\tilde{w}^- \cup [\tilde{w}^-, \tilde{w}^+]_{\tilde{C}}^- \cup \tilde{w}^+ w^+$ (Figure 3) is a nonhamiltonian X-cycle in $G\Box K_2$. Therefore, there is no H-force set of cardinality $m - 1$ in $G\Box K_2$.

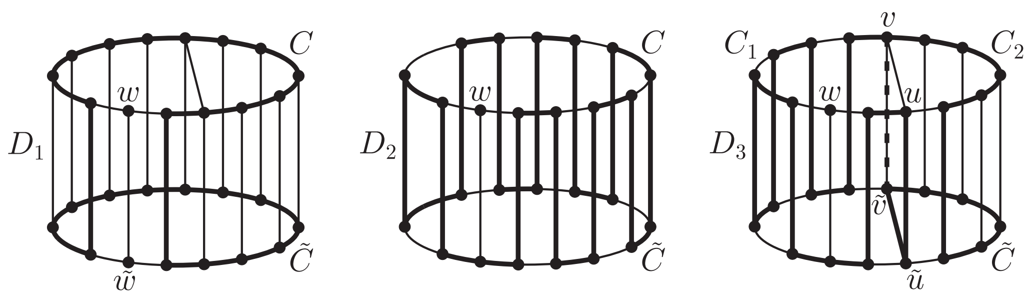


Figure 3

Case 2. Let G be not bipartite.

Case 2.1. If the order m of G is odd then it is easy to see that, for any vertex w of $G\Box K_2$, there is a cycle D_2 of length $n-1$ in $G\Box K_2$ omitting just the vertex w and containing all vertical edges except of $w\tilde{w}$ (Figure 3). Hence, there is no H-force set of cardinality $n-1$ in $G\Box K_2$.

Case 2.2. If the order m of G is even then there is an edge $uv \in E(G) \setminus E(C)$ such that $C_1 = [u,v]_C^- \cup uv$ and $C_2 = [u,v]_C^+ \cup uv$ are both odd cycles. Let G_i $(i = 1,2)$ be the graph induced by $V(C_i)$ in G. For any vertex of $G\Box K_2$ we look for a cycle omitting just this vertex. Let, without loss of generality, $w \in V(G_1) \setminus \{v\}$. Then $G_1\Box K_2$ is the prism of a graph of odd order, thus by the previous case, there is a cycle D' in $G_1\Box K_2$ containing all vertices except of w and all vertical edges except of $w\tilde{w}$. Then $D_3 = (D' - v\tilde{v}) \cup [v,u^+]_C^- \cup u^+\tilde{u}^+ \cup [\tilde{u}^+,\tilde{v}]_{\tilde{C}}^+$ (Figure 3) is the desired cycle. ■

5. Planar Graphs

By Theorem 9 of Nelson, the H-force number of every 4-connected planar graph is equal n. In section 3, planar graphs of order n and with a given H-force number k were constructed, for any $1 \le k \le n$.

A planar graph is *outerplanar* if it can be embedded in the plane in such a way that all its vertices are incident to the unbounded face. The *weak dual* $D^\star(G)$ of an outerplanar graph G is the graph obtained from the dual of G by removing the vertex corresponding to the unbounded face; it is a tree, if G is 2-connected. In this case let $\ell(G)$ denote the number of leaves of $D^\star(G)$.

Theorem 23. *If $G \neq C_n$ is an outerplanar hamiltonian graph, then $h(G) = \ell(G) \ge 2$.*

Proof. Let G be an outerplanar graph with a hamiltonian cycle C creating the boundary of its outerface. With every leaf of the weak dual $D^\star(G)$ there is associated a face α of G incident with a chord xy of C. All vertices of α except for x and y have degree 2 in G and every H-force set X of G contains at least one of them. Otherwise the cycle $[x,y]_C^+ \cup \{xy\}$ (or $[x,y]_C^- \cup \{xy\}$) is a nonhamiltonian cycle of G omitting all 2-valent vertices of α, a contradiction. Hence, $|X| = h(G) \geq \ell(G)$.

To prove the converse inequality it is enough to find an H-force set X consisting of $\ell(G)$ vertices. If we choose one vertex of degree 2 from each face of G corresponding to a leaf of the weak dual $D^\star(G)$ we obtain a desired set X. Suppose that there exists a nonhamiltonian X-cycle C' in G. Then it has to contain a chord $xy \in E(G)$. The graph $G - x - y$ consists of exactly two components each containing a vertex from X, but C' has an empty intersection with one of them, a contradiction. ■

For a plane hamiltonian graph G with a hamiltonian cycle C let G_C^i (or G_C^o) be the graph consisting of the cycle C and all edges of G lying inside (outside) of C. Clearly, G_C^i and G_C^o are both outerplanar. Taking into consideration the graphs G_C^i and G_C^o and the proof of the previous theorem we immediately obtain

Theorem 24. *If G is a planar hamiltonian graph with $\delta(G) \geq 3$ and C a hamiltonian cycle of G, then $h(G) \geq \ell(G_C^i) + \ell(G_C^o) \geq 4$.*

Other results about planar graphs follow in the next section.

6. Graphs with Small H-force Number

Let $C = [v_1, v_2, \ldots, v_n]$ be a hamiltonian cycle of G. We say that a chord v_iv_j $(i < j-1)$ *separates* vertices v_k, v_l $(k < l-1)$ on C, if they belong to different components of $C - v_i - v_j$, and, moreover, *crosses* the chord v_kv_l, if $v_kv_l \in E(G)$.

Theorem 25. *Let $G \neq C_n$ be a hamiltonian graph and $C = [v_1, v_2, \ldots, v_n]$ be a hamiltonian cycle of G. Then $h(G) = 2$ if and only if*

- (i) *there exist $x, y \in V(G)$, $\deg_G(x) = \deg_G(y) = 2$, such that every chord v_iv_j $(i < j-1)$ separates x and y on C, and*
- (ii) *for every pair v_iv_j and v_kv_l $(i < j-1,\ k < l-1)$ of crossed chords $v_iv_k, v_jv_l \in E(C)$ holds.*

Proof. Suppose $h(G) = 2$ and let $F = \{x, y\}$ be an H-force set of G (i.e. every F-cycle of G is hamiltonian). Moreover, we may assume $v_1 = x$ and $v_t = y$ where $3 \leq t \leq n-1$.

Claim 1. $\deg_G(x) = \deg_G(y) = 2$, otherwise, if $\deg_G(x) \geq 3$ then, for $x^* \in N(x) \setminus \{x^-, x^+\}$, one of the cycles $D_1 = [x, x^*]_C^+ \cup xx^*$ and $D_2 = [x, x^*]_C^- \cup xx^*$ contains y but does not contain one of x^- or x^+, therefore it is a nonhamiltonian F-cycle; a contradiction.

Claim 2. *Every chord uw of C separates x and y on C,*

otherwise one of the cycles $D_3 = [u, w]_C^+ \cup uw$ or $D_4 = [u, w]_C^- \cup uw$ is an F-cycle omitting u^- or u^+; a contradiction.

Claim 3. *If v_iv_j and v_kv_l $(1 < i < k < t < j < l)$ are two crossed chords of C, then $v_iv_k, v_jv_l \in E(C)$ (i.e. $k = i + 1$ and $l = j + 1$),*

otherwise, if $v_iv_k \notin E(C)$ then $D_5 = [v_k, v_j]_C^+ \cup v_iv_j \cup [v_i, v_l]_C^- \cup v_kv_l$ is an F-cycle of G missing vertex v_{i+1}; a contradiction.

To prove the converse let G be a graph satisfying properties (i) and (ii). We assume again $v_1 = x$ and $v_t = y$ where $3 \leq t \leq n - 1$.

Claim 4. *For every vertex $v_i \in V(G) \setminus \{x, y\}$ there is a vertex $v_i^* \in V(G)$ such that $\{v_i, v_i^*\}$ separates x and y in G,*

because,

(a) if $\deg_G(v_i) \geq 3$ and v_iv_j is a chord of C crossed by $v_{i+1}v_{j+1}$ then $\{v_i, v_{j+1}\}$ separates x and y,

(b) if $\deg_G(v_i) \geq 3$ and v_iv_j is a chord of C crossed by no other chord then $\{v_i, v_j\}$ separates x and y, and

(c) for $\deg_G(v_i) = 2$ let $P = [u, w]_C^+$ be the longest subpath of C containing v_i with internal vertices of degree 2 (in G) only (i.e. $\deg_G(u), \deg_G(w) \geq 3$). Then v_i separates x and y in G with the same vertex as w does or with one of the vertices u, w (in the case $V(P) \cap \{x, y\} \neq \emptyset$).

Finally, $F = \{x, y\}$ is an H-force set of G, because otherwise there exists a nonhamiltonian F-cycle C' missing a vertex v_i. If $\{v_i, v_i^*\}$ separates x and y in G, then the vertices x and y are separated by at most 1 vertex on C'; a contradiction. ■

Thus, any hamiltonian graph with H-force number 2 can be considered as the union of two outerplanar hamiltonian graphs with a common hamiltonian cycle which implies

Corollary 26. *Every hamiltonian graph G with $h(G) = 2$ is planar.*

For a graph G and a set $X \subseteq V(G)$ we denote by $K_X(G)$ the graph with the vertex set $V(G)$ and the edge set $E(G) \cup \{uv \mid u, v \in X\}$, i.e. the smallest spanning supergraph of G in which X induces a clique. Kawarabayashi [15] proved, that for any k-connected graph G and any given ℓ vertices ($k \leq \ell \leq \frac{3}{2}k$), there is a cycle in G containing exactly k of them.

By Theorem 21, the H-force number of a 3-connected hamiltonian graph is ≥ 3. We prove that there are only four 3-connected graphs with the H-force number 3.

Theorem 27. *Let G be a 3-connected hamiltonian graph. Then*
(i) $h(G) \geq 4$ *or*
(ii) *G results from $K_{3,3}$ by adding any edges in exactly one partite set.*

Proof. Let G be a 3-connected hamiltonian graph with $h(G) = 3$. There exists an H-force set $F = \{v_1, v_2, v_3\} \subseteq V(G)$ in G (i.e. every F-cycle of G is hamiltonian). Consider an arbitrary vertex $x \in V(G) \setminus F$. By the above mentioned theorem of Kawarabayashi the graph G contains a cycle C through exactly three of the vertices v_1, v_2, v_3, x. Thus, C is nonhamiltonian and, consequently, it is no F-cycle which allows to assume that without loss of generality $v_2, v_3, x \in V(C)$. As G is 3-connected, there exist three internally disjoint (v_1, C)-paths P_1, P_2, P_3 with different endvertices $y_i \in V(P_i) \cap V(C)$, $i = 1, 2, 3$. Denote $Q_i = [y_{i+1}, y_{i+2}]_C^+$, $i = 1, 2, 3$ (indices modulo 3; see Figure 4) and let $v_j^* \in N(v_j) \setminus \{v_j^-, v_j^+\}$ for $j = 2, 3$.

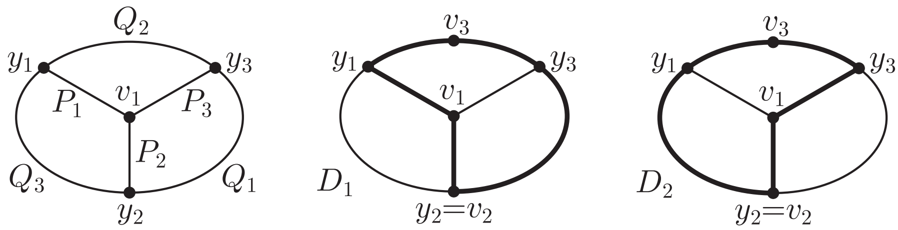


Figure 4

Case 1. If v_2, v_3 belong to the same path Q_i (possibly they are its endvertices) then $Q_i \cup P_{i+1} \cup P_{i+2}$ is an F-cycle omitting vertex y_i, thus nonhamiltonian, a contradiction.

Case 2. Let v_2, v_3 do not belong to the same path Q_i and let one of them be identical with a vertex y_j, i.e. assume w.l.o.g. $v_2 = y_2$, $v_3 \in Q_2$ where $v_3 \notin \{y_1, y_3\}$.

The cycles $D_1 = P_1 \cup P_2 \cup Q_1 \cup Q_2$ and $D_2 = P_2 \cup P_3 \cup Q_2 \cup Q_3$ (Figure 4) are both F-cycles, thus hamiltonian. Therefore, P_1, P_3, Q_1, Q_3 are paths of length 1 (i.e. $v_1y_1, v_1y_3, y_2y_3, y_1y_2 \in E(G)$).

Case 2.1. If $v_3^* \in [y_3, v_3^-]_C^+$ then $D_3 = [v_2, v_3^*]_C^+ \cup v_3^*v_3 \cup [v_3, y_1]_C^+ \cup P_1 \cup P_2$ (Figure 5) is an F-cycle omitting the vertex v_3^-, a contradiction.

Case 2.2. If $v_3^* \in [v_3^+, y_1]_C^+$ then $D_4 = [v_2, v_3]_C^+ \cup v_3v_3^* \cup [v_3^*, y_1]_C^+ \cup P_1 \cup P_2$ (Figure 5) is an F-cycle omitting the vertex v_3^+, a contradiction.

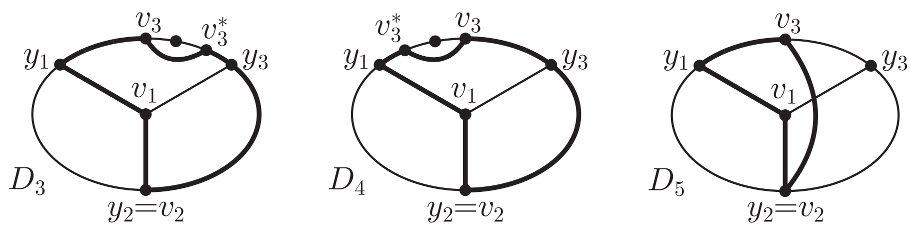


Figure 5

Case 2.3. If $v_3^* = v_2$ then $D_5 = v_2v_3 \cup [v_3, y_1]_C^+ \cup P_1 \cup P_2$ (Figure 5) is an F-cycle omitting the vertex y_3, a contradiction.

Figure 6

Case 2.4. If $v_3^* \in P_2$, $v_3^* \neq v_2$, then $D_6 = [v_1, v_3^*]_{P_2} \cup v_3^*v_3 \cup [v_3, y_3]_C^+ \cup P_3$ and $D_7 = [v_2, v_3^*]_{P_2} \cup v_3^*v_3 \cup [v_3, y_3]_C^- \cup P_3 \cup P_1 \cup Q_3$ (Figure 6) are both F-cycles, thus hamiltonian and therefore P_2 and Q_2 have length 2, i.e. $K_{3,3}$ is a spanning subgraph of G.

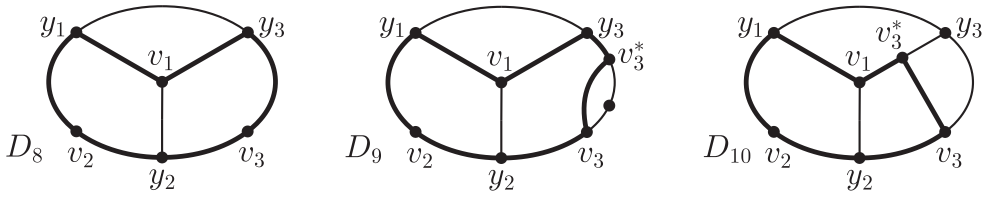


Figure 7

Case 3. Let v_2, v_3 do not belong to the same path Q_i and let they be different from y_j, i.e. assume w.l.o.g. $v_2 \in Q_3$ and $v_3 \in Q_1$.

$D_8 = Q_3 \cup Q_1 \cup P_3 \cup P_1$ (Figure 7) is an F-cycle, thus hamiltonian and therefore P_2 and Q_2 have length 1 (i.e. $v_1y_2, v_3y_1 \in E(G)$).

Case 3.1. If $v_3^* \in [v_3^+, y_3]_C^+$ then $D_9 = [y_2, v_3]_C^+ \cup v_3v_3^* \cup [v_3^*, y_3]_C^+ \cup P_3 \cup P_1 \cup Q_3$ (Figure 7) is an F-cycle omitting the vertex v_3^+, a contradiction.

Case 3.2. If $v_3^* \in P_3$ then $D_{10} = [y_2, v_3]_C^+ \cup v_3v_3^* \cup [v_3^*, v_1]_{P_3} \cup P_1 \cup Q_3$ (Figure 7) is an F-cycle omitting the vertex y_3, a contradiction.

Figure 8

Case 3.3. If $v_3^* \in [v_2, v_3^-]_C^+$ then $D_{11} = [v_3, y_3]_C^+ \cup P_3 \cup P_1 \cup [y_1, v_3^*]_C^+ \cup v_3^* v_3$ (Figure 8) is an F-cycle omitting the vertex v_3^-, a contradiction.

Case 3.4. If $v_3^* \in [y_1^+, v_2]_C^+$ then $D_{12} = [y_2, v_3^*]_C^- \cup v_3^* v_3 \cup [v_3, y_3]_C^+ \cup P_3 \cup P_2$ (Figure 8) is an F-cycle omitting the vertex y_1, a contradiction.

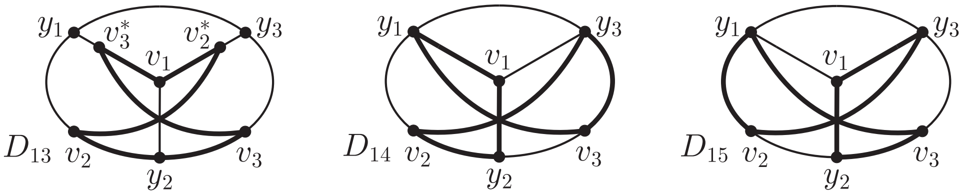


Figure 9

Analogously, we obtain a contradiction in corresponding cases under consideration of v_2 and its neighbour v_2^*. There are two remaining cases:

Case 3.5. If $v_2^* \in P_3$, $v_3^* \in P_1$ and $v_j^* \neq y_{j+1}$ for $j = 2$ or $j = 3$ then $D_{13} = [v_2, v_3]_C^+ \cup v_3^* v_3 \cup [v_3^*, v_1]_{P_1} \cup [v_1, v_2^*]_{P_3} \cup v_2^* v_2$ (Figure 9) is an F-cycle omitting the vertex y_{j+1}, a contradiction.

Case 3.6. If $v_2^* = y_3$ and $v_3^* = y_1$ then $D_{14} = [v_2, y_2]_C^+ \cup P_2 \cup P_1 \cup y_1 v_3 \cup [v_3, y_3]_C^+ \cup y_3 v_2$ and $D_{15} = [y_2, v_3]_C^+ \cup v_3 y_1 \cup [y_1, v_2]_C^+ \cup v_2 y_3 \cup P_3 \cup P_2$ (Figure 9) are both F-cycles, thus hamiltonian and therefore paths P_1 and P_3 have length 1 and paths Q_1 and Q_3 have length 2, i.e. $K_{3,3}$ is a spanning subgraph of G.

In any case, $K_{3,3}$ is a spanning subgraph of G. Let $X, Y \subseteq V(K_{3,3}) = V(G)$ be the bipartition of $K_{3,3}$. If $G \subseteq K_X(K_{3,3})$ then $3 = h(K_{3,3}) \leq h(G) \leq h(K_X(K_{3,3})) \leq 3$ by Proposition 2 and Corollaries 19 and 20, thus $h(G) = 3$. Otherwise, if $G' = K_{3,3} \cup \{x_1 x_2, y_1 y_2 \,|\, x_i \in X, y_i \in Y, i = 1, 2\}$ is a subgraph of G, then $h(G) \geq h(G') = 6$, which completes the proof. ■

In the previous section we proved that the H-force number of a planar hamiltonian graph G with $\delta(G) \geq 3$ is lower-bounded by $\ell(G_C^i) + \ell(G_C^o) \geq 4$.

Theorem 28. *Let G be a 3-connected planar hamiltonian graph. Then*

(i) $h(G) \geq 5$ *or*

(ii) *$G = K_4$ or G results from the graph Q_3 of the cube by adding any edges in exactly one partite set.*

Proof. Let G be a plane 3-connected hamiltonian graph with $h(G) = 4$ and let C be a hamiltonian cycle of G. Theorems 23 and 24 imply $\ell(G^i_C) = \ell(G^o_C) = 2$, i.e. the weak duals $D^\star(G^i_C)$ and $D^\star(G^o_C)$ are paths. Let α, β and γ, δ be the faces of G corresponding to endvertices of $D^\star(G^i_C)$ and $D^\star(G^o_C)$, respectively, and let $F = \{x, y, u, v\}$ be an H-force set, where $x \in V(\alpha)$, $y \in V(\beta)$, $u \in V(\gamma)$, $v \in V(\delta)$ and $\deg_{G^i_C}(x) = \deg_{G^i_C}(y) = \deg_{G^o_C}(u) = \deg_{G^o_C}(v) = 2$.

Claim 1. *Every chord $e \in E(G^i_C)$ (or $e \in E(G^o_C)$) of C separates x, y in G^i_C (or u, v in G^o_C).*

Let x^* be a neighbour of x in G, different from x^+ and x^-, with the smallest distance $d_C(x^*, y)$ from y on C and similarly, $y^* \in N(y)\backslash\{y^+, y^-\}$ with minimum $d_C(y^*, x)$, $u^* \in N(u)\backslash\{u^+, u^-\}$ with minimum $d_C(u^*, v)$ and $v^* \in N(v)\backslash\{v^+, v^-\}$ with minimum $d_C(v^*, u)$.

Case 1. Let $xy, uv \in E(G)$ (i.e. $x^* = y$, $y^* = x$, $u^* = v$, $v^* = u$), then $D_1 = [x, u]^+_C \cup uv \cup [v, y]^-_C \cup yx$ and $D_2 = [x, v]^-_C \cup vu \cup [u, y]^+_C \cup yx$ (Figure 10) are F-cycles, hence, both are hamiltonian. Therefore, $[x, u]^+_C$, $[v, y]^-_C$, $[x, v]^-_C$ and $[u, y]^+_C$ are paths of length 1 (i.e. $xu, vy, xv, uy \in E(C)$) and finally $G = K_4$.

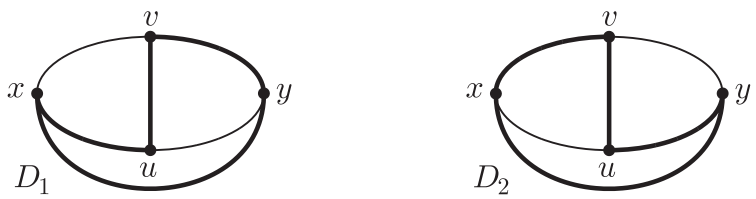


Figure 10

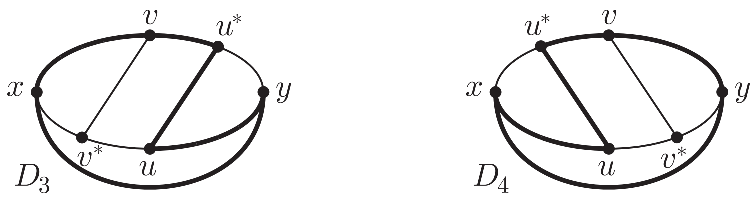


Figure 11

Case 2. Let, without loss of generality, $xy \in E(G)$ and $uv \notin E(G)$ (i.e. $x^* = y$, $y^* = x$, $u^* \neq v$, $v^* \neq u$).

Case 2.1. If $u^* \in [y, v]^+_C$ (and consequently $v^* \in [x, u]^+_C$), then $D_3 = [x, u^*]^-_C \cup u^*u \cup [u, y]^+_C \cup yx$ (Figure 11) is an F-cycle omitting vertex v^*, a contradiction.

Case 2.2. If $u^* \in [v,x]_C^+$ (and consequently $v^* \in [u,y]_C^+$), then $D_4 = [x,u]_C^+ \cup uu^* \cup [u^*,y]_C^- \cup yx$ (Figure 11) is a nonhamiltonian F-cycle, a contradiction.

Case 3. Let $xy, uv \notin E(G)$ (i.e. $\{x^*,y^*,u^*,v^*\} \cap \{x,y,u,v\} = \emptyset$).

Case 3.1. Let each of the paths $[x,u]_C^+$, $[u,y]_C^+$, $[y,v]_C^+$, $[v,x]_C^+$ contains a vertex from $\{x^*,y^*,u^*,v^*\}$ (without loss of generality, let $v^* \in [x,u]_C^+$, $x^* \in [u,y]_C^+$, $u^* \in [y,v]_C^+$ and $y^* \in [v,x]_C^+$).

Then $D_5 = [x,v^*]_C^+ \cup v^*v \cup [v,y^*]_C^+ \cup y^*y \cup [y,u^*]_C^+ \cup u^*u \cup [u,x^*]_C^+ \cup x^*x$, $D_6 = [x,y^*]_C^- \cup y^*y \cup [y,v]_C^+ \cup vv^* \cup [v^*,x^*]_C^+ \cup x^*x$, and $D_7 = [x,u]_C^+ \cup uu^* \cup [u^*,y^*]_C^+ \cup y^*y \cup [y,x^*]_C^- \cup x^*x$ (Figure 12) are F-cycles, hence all are hamiltonian. Since each of the paths $[x,v^*]_C^+$, $[v^*,u]_C^+$, $[u,x^*]_C^+$, $[x^*,y]_C^+$, $[y,u^*]_C^+$, $[u^*,v]_C^+$, $[v,y^*]_C^+$, $[y^*,x]_C^+$ has with at least one of the hamiltonian cycles D_5, D_6, D_7 no inner vertex in common, there is no inner vertex on any of these paths, and, consequently, the cube graph Q_3 is a spanning subgraph of G.

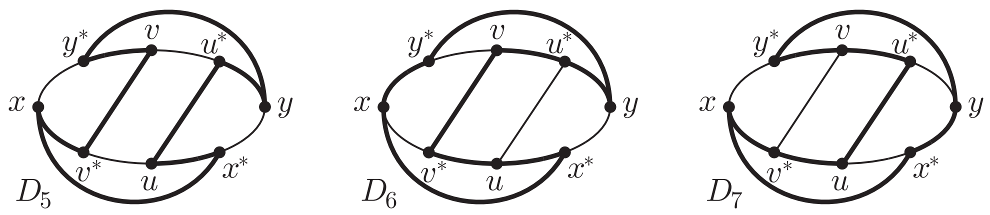


Figure 12

Case 3.2. Let exactly two of the paths $[x,u]_C^+$, $[u,y]_C^+$, $[y,v]_C^+$, $[v,x]_C^+$ contain a vertex from $\{x^*,y^*,u^*,v^*\}$ (without loss of generality and because of claim 1 let $x^*, v^* \in [u,y]_C^+$ and $y^*, u^* \in [v,x]_C^+$).

Case 3.2.1. Let $u^* \notin [y^*,x]_C^+$ (or analogously $v^* \notin [x^*,y]_C^+$). Then $D_8 = [x,u]_C^+ \cup uu^* \cup [u^*,x^*]_C^- \cup x^*x$ (Figure 13) is a nonhamiltonian F-cycle, a contradiction.

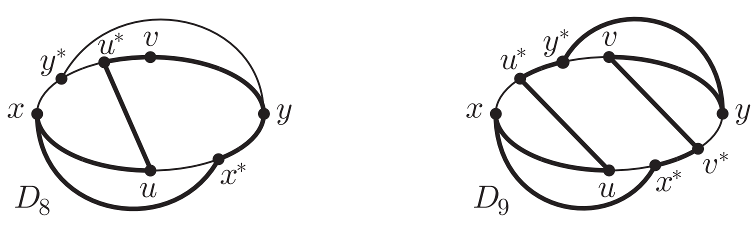


Figure 13

Case 3.2.2. Let $u^* \in [y^*,x]_C^+$ and $v^* \in [x^*,y]_C^+$. $D_9 = [x,u]_C^+ \cup uu^* \cup [u^*,y^*]_C^- \cup y^*y \cup [y,v]_C^+ \cup vv^* \cup [v^*,x^*]_C^- \cup x^*x$ (Figure 13) is an F-cycle, hence it is hamiltonian. Therefore then the paths $[u,x^*]_C^+$, $[v^*,y]_C^+$, $[v,y^*]_C^+$, and $[u^*,x]_C^+$ have length 1 (i.e.

$ux^*, v^*y, vy^*, u^*x \in E(G)$). Since G is planar and 3-connected, there exists a neighbour of u on $[v, u^*]_C^+$ different from u^* (otherwise, the set $\{x^*, u^*\}$ would be a 2-cut of G, with contradiction), which contradicts the minimality of $d_C(u^*, v)$.

That means, $G = K_4$ or G contains Q_3 as a spanning subgraph. In the second case let $X, Y \subseteq V(Q_3) = V(G)$ be the bipartition of Q_3. If $G \subseteq K_X(G)$ then $4 = h(Q_3) \leq h(G) \leq h(K_X(G)) \leq 4$ by Proposition 2, Corollary 19, and Theorem 22, thus $h(G) = 4$. Otherwise, if $G' = Q_3 \cup \{x_1x_2, y_1y_2 \,|\, x_i \in X, y_i \in Y, i = 1, 2\}$ is a subgraph of G, then $h(G) \geq h(G') = 8$, which completes the proof. ■

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