ON VERTICES ENFORCING A HAMILTONIAN CYCLE

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Abstract

A nonempty vertex set \( X \subseteq V(G) \) of a hamiltonian graph \( G \) is called an \( H \)-force set of \( G \) if every \( X \)-cycle of \( G \) (i.e. a cycle of \( G \) containing all vertices of \( X \)) is hamiltonian. The \( H \)-force number \( h(G) \) of a graph \( G \) is defined to be the smallest cardinality of an \( H \)-force set of \( G \). In the paper the study of this parameter is introduced and its value or a lower bound for outerplanar graphs, planar graphs, \( k \)-connected graphs and prisms over graphs is determined.

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1. Introduction

One of the most intensively studied areas in graph theory deals with questions concerning cycles. The development of this area has undergone a natural growth and evolution in the questions studied and results obtained. One particular subarea involves questions about cycles containing specific sets of vertices of a graph, see e.g. a recent survey paper [13].

This paper is intended to contribute to this area. Throughout this article we consider finite simple hamiltonian graphs. We shall try to answer the question how small the cardinality of a subset of the vertex set of a given hamiltonian graph can be that the only cycles containing this subset are hamiltonian ones.

We shall use a standard terminology according to [7] except for some terms defined throughout this paper.

For a graph $G$ and a set $X \subseteq V(G)$, an $X$-cycle of $G$ is a cycle containing all vertices of $X$. Let $G$ be a hamiltonian graph. A nonempty vertex set $X \subseteq V(G)$ is called a hamiltonian cycle enforcing set (in short an $H$-force set) of $G$ if every $X$-cycle of $G$ is hamiltonian. For the graph $G$ we define $h(G)$ to be the smallest cardinality of an $H$-force set of $G$ and call it the $H$-force number of $G$.

In this paper we study the $H$-force number for several families of graphs. First we survey known results on this parameter for some families of graphs originally stated in different terms.

The following is obvious

**Proposition 1.** If $X$ is an $H$-force set of a graph $G$ and $X \subseteq Y \subseteq V(G)$, then $Y$ is an $H$-force set of $G$ too.

**Proposition 2.** If $H$ is a hamiltonian spanning subgraph of $G$, then $h(H) \leq h(G)$.

**Proposition 3.** If $C$ is a nonhamiltonian cycle of $G$, then any $H$-force set of $G$ contains a vertex of $V(G) \setminus V(C)$.

The following example demonstrates that the task to determine the $H$-force number of a graph is not easy in general.

**Example 4.** Let $G$ be the dodecahedral graph. Then $h(G) = 15$.

**Proof.** Let $X \subseteq V(G)$ be an $H$-force set of $G$ and let $\bar{X} = V(G) \setminus X$.

It is easy to see that $\bar{X}$ does not contain any of the following configurations (because the subgraph induced on the remaining vertices is hamiltonian, see Figure 1).

(a) two vertices with distance 3 (e.g. 1, 7),
(b) two vertices with distance 5 (e.g. 1, 19),
(c) three vertices inducing a 3-path (e.g. 1, 2, 3).
Suppose that there is an H-force set $X \subseteq V(G)$ with $|X| \leq 14$.

Case 1. If there are two adjacent vertices in $\bar{X}$. Then, without loss of generality, let $4, 5 \in \bar{X}$, then $9, 10, 11, 12, 13, 14, 15, 16, 19, 20 \in X$ by (a), $17, 18 \in X$ by (b), and $1, 3, 6, 8 \in X$ by (c), thus $|X| \geq 16$, a contradiction.

Case 2. If any two vertices of $\bar{X}$ are nonadjacent, then there are two vertices in $\bar{X}$ incident with the same face of $G$; without loss of generality, let $6, 8 \in \bar{X}$. Hence, $4, 5, 7 \in X$ (Case 1), $1, 2, 3, 9, 11, 13, 15, 16, 17, 18, 19 \in X$ by (a), and $10 \in X$ or $20 \in X$ by (a), as well. Finally, $|X| \geq 15$, a contradiction.

It is still necessary to show that there is an H-force set of size 15 in $G$. Let $X_1 = \{1, 2, 3, 4, 5\}$, $X_2 = \{7, 9, 11, 13, 15, 16, 17, 18, 19, 20\}$, $X = X_1 \cup X_2$, $\bar{X} = V(G) \setminus X$ and let $C$ be an $X$-cycle of $G$. The subgraph $G[X]$ of $G$ induced by $X$ consists of two components $G[X_1]$ and $G[X_2]$, the second of them contains five vertices of degree 1. Therefore, $C$ contains at least two edges between $X_1$ and $\bar{X}$ and at least five edges between $X_2$ and $\bar{X}$. Hence, $C$ contains at least seven edges between $X$ and $\bar{X}$, thus at least four vertices of $\bar{X}$. Because $G$ does not contain any cycle of length 19, $C$ must be a hamiltonian cycle of $G$ and $X$ is an H-force set of $G$.

2. 1-hamiltonian Graphs

Through this paper, the number of vertices (the order) of a graph will be denoted by $n$. A graph $G$ is $k$-hamiltonian ($1 \leq k \leq n - 3$) if $G - U$ is hamiltonian for every $U \subseteq V(G)$ with $0 \leq |U| \leq k$. In particular, $G$ is 1-hamiltonian, if it is hamiltonian and for any vertex $u \in V(G)$ the graph $G - u$ is hamiltonian too, i.e. any $n - 1$ vertices lie on a common nonhamiltonian cycle of $G$ and thus there is no H-force set of cardinality $n - 1$ in $G$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}
Proposition 5. The 1-hamiltonian graphs are exactly the graphs with H-force number equal to their order.

Several sufficient conditions for graphs to be 1-hamiltonian have been obtained by various authors. The following two conditions in terms of vertex degrees are of Dirac-type and of Ore-type, respectively.

Theorem 6 (Chartrand, Kapoor, Link [6]). Let $G$ be a graph of order $n \geq 4$. If $\delta(G) \geq \lceil \frac{n}{2} \rceil + 1$, then $G$ is 1-hamiltonian.

Theorem 7 (Chartrand, Kapoor, Link [6]). Let $G$ be a graph of order $n \geq 4$. If for every pair of non-adjacent vertices $x, y \in V(G)$, $\deg_G(x) + \deg_G(y) \geq n + 1$, then $G$ is 1-hamiltonian.

The connectivity and the independence number of a graph $G$ will be denoted by $\kappa(G)$ and $\alpha(G)$, respectively. A simple relationship linking the connectivity, the independence number and hamiltonian properties was discovered by Chvátal and Erdős [9], namely, that a graph $G$ is hamiltonian if $\alpha(G) \leq \kappa(G)$, and, moreover

Theorem 8 (Chvátal, Erdős [9]). If $G$ is a graph with $\kappa(G) \geq 3$ and $\alpha(G) < \kappa(G)$, then $G$ is 1-hamiltonian.

A major theorem of Tutte [21] states that every 4-connected planar graph $G$ is hamiltonian. The following strengthening was obtained by the same proof technique.

Theorem 9 (Nelson [17]). Every 4-connected planar graph $G$ is 1-hamiltonian.

A Halin graph is a union of a tree $T \neq K_2$ without vertices of degree 2 and a cycle $C$ connecting the leaves of $T$ in the cyclic order determined by a plane embedding of $T$. Bondy [2] showed that every Halin graph is hamiltonian and improved this statement to the following (unpublished) result (see [16]).

Theorem 10 (Bondy). Every Halin graph $G$ is 1-hamiltonian.

A graph $G$ is claw-free if it has no induced subgraph isomorphic to $K_{1,3}$ (the claw), and it is locally connected (locally $k$-connected) if, for each vertex $u \in V(G)$, the neighbourhood $N(u)$ of $u$ induces a connected ($k$-connected) subgraph. Oberly and Sumner [18] have shown that every connected, locally connected, claw-free graph of order $\geq 3$ is hamiltonian.

Theorem 11 (Broersma, Veldman [4]). If $G$ is a connected, locally 2-connected, claw-free graph of order $\geq 4$, then $G$ is 1-hamiltonian.

The $k$-th power $G^k$ of a graph $G$ is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if their distance in $G$ is $\leq k$. The famous result of Fleischner [12] states that the square $G^2$ of any 2-connected graph $G$ is hamiltonian.
Theorem 12 (Chartrand et al. [5]). The square $G^2$ of a 2-connected graph $G$ is 1-hamiltonian.

All conditions of Theorems 6–12 are also sufficient for the mentioned graphs to be hamiltonian connected (Erdős, Gallai [11]; Ore [19]; Chvátal, Erdős [9]; Thomassen [20] and Chiba, Nishizeki [8]; Barefoot [1]; Kanetkar, Rao [14]; Chartrand et al. [5]). Recall, that a graph $G$ is hamiltonian connected if any two vertices of $G$ are connected by a hamiltonian path. Nevertheless, there exist graphs that are either 1-hamiltonian or hamiltonian connected. The graph $G_1$ (Figure 2, see Zamfirescu [22]) is 1-hamiltonian, but not hamiltonian connected and the graph $G_c$ (Figure 2) is hamiltonian connected, but not 1-hamiltonian. Both are very probably the smallest graphs of its type.

There are a lot of results concerning $k$-hamiltonian graphs, however, in this paper we start to study the H-force number with the aim to find a decomposition of the class of hamiltonian graphs in which the 1-hamiltonian graphs (including $k$-hamiltonian graphs, $k \geq 2$, as subsets) form an extremal subclass.

3. Graphs with Given H-force Number

Now, we will answer the question for which pairs of integers $k$ and $n$ with $n \geq 3$ and $1 \leq k \leq n$ there exists a hamiltonian graph $G$ of order $n$ such that $h(G) = k$. For the cycle $C_n$ and the wheel $W_n$ of order $n$ it is obvious that $h(C_n) = 1$ and $h(W_n) = n$. But what can we say for $k$ with $2 \leq k \leq n - 1$?

Theorem 13. For all integers $k$ and $n$ where $2 \leq k \leq n - 2$ there exists a (planar) hamiltonian graph $G$ of order $n$ with $h(G) = k$.

Proof. Consider the cycle $C_n = [v_1, v_2, \ldots, v_n]$. Let $G$ be the graph with the vertex set $V = V(C_n)$ and the edge set $E = E(C_n) \cup \{v_2v_n\} \cup \{v_iv_i | 3 \leq i \leq k\} \cup \{v_kv_n\}$. Note that the graph induced by $\{v_1, v_2, \ldots, v_k, v_n\}$ in $G$ is the wheel $W_k$ (or the cycle $C_3$, if $k = 2$). The graph $G$ is hamiltonian and even planar. It
is not difficult to see that \(\{v_1, \ldots, v_{k-1}\} \cup \{u\}\), for any \(u \in \{v_{k+1}, \ldots, v_{n-1}\}\), is the smallest H-force set of \(G\).

**Theorem 14.** For every integer \(n \geq 10\) there exists a hamiltonian graph \(G\) of order \(n\) with \(h(G) = n - 1\).

**Proof.** Consider two complete graphs \(K_3 = (V_1, E_1)\) and \(K_{n-7} = (V_2, E_2)\) with the vertex set \(V_1 = \{y_1, y_2, y_3\}\) and \(V_2 = \{z_1, z_2, \ldots, z_{n-7}\}\), respectively. Let \(G\) be the graph with the vertex set \(V = V_1 \cup V_2 \cup \{x_0, x_1, x_2, x_3\}\) and the edge set \(E = E_1 \cup E_2 \cup \{x_0 u \mid u \in V\} \cup \{x_i y_i, x_i z_i \mid i = 1,2,3\}\). The graph \(G\) is hamiltonian and \(V \setminus \{x_0\}\) is the smallest H-force set of \(G\), because, for any \(u \in V \setminus \{x_0\}\), the graph \(G - u\) is hamiltonian.

The next two theorems provide existence results with respect to the more special class of polyhedral (i.e. 3-connected planar) hamiltonian graphs.

**Theorem 15.** For every integers \(n \geq 9\) and \(k\) where \(5 \leq k \leq n - 4\) there exists a polyhedral hamiltonian graph \(G\) of order \(n\) with \(h(G) = k\).

**Proof.** Let \(C = [x_1, \ldots, x_6]\) be a cycle in the plane with a vertex \(x_0\) in the inner face and with a path \(P = [y_1, \ldots, y_r]\) with \(r \geq 0\) in the outer face. We connect \(x_0\) with every vertex of \(C\), \(x_1\) with every vertex of \(P\) and introduce edges \(x_2 y_1, x_6 y_r\). Moreover, let \(Q = [z_1, \ldots, z_s]\) with \(s \geq 2\) be a path in the unbounded face of the above constructed plane graph. We connect \(z_1\) with \(x_4\) and every vertex of \(Q\) with the vertices \(x_2\) and \(x_6\). The resulting graph \(G = (V, E)\) of order \(n = r + s + 7\) is polyhedral where \([x_1, y_1, \ldots, y_r, x_6, x_5, x_4, z_1, \ldots, z_s, x_2, x_3, x_0]\) is a hamiltonian cycle.

First, we will see that \(G - v\) is hamiltonian for every \(v \in S = \{x_1, x_3, x_5, y_1, \ldots, y_r, z_1, z_s\}\). Hence, every H-force set \(F\) of \(G\) contains \(S\) as a subset. \(G - x_1\) is hamiltonian with \([x_0, x_2, x_3, x_4, z_1, \ldots, z_s, x_6, x_5]\) if \(r = 0\) and with \([x_2, y_1, \ldots, y_r, x_6, x_5, x_0, x_3, x_4, z_1, \ldots, z_s]\), otherwise. \(G - x_3\) is hamiltonian with \([x_1, y_1, \ldots, y_r, x_6, x_5, x_4, z_1, \ldots, z_s, x_2, x_0]\) and, by symmetry \(G - x_5\) is hamiltonian, too. If \(r > 0\) then \(G - y_6\) with \(1 \leq i \leq r\) is hamiltonian with \([x_2, y_1, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_r, x_6, x_5, x_0, x_3, x_4, z_1, \ldots, z_s]\). \(G - z_1\) is hamiltonian with \([x_1, y_1, \ldots, y_r, x_6, z_2, \ldots, z_s, x_2, x_3, x_4, x_5, x_0]\) and, \(G - z_s\) is hamiltonian with \([x_1, y_1, \ldots, y_r, x_6, x_5, x_4, z_1, \ldots, z_{s-1}, x_2, x_3, x_0]\).

Now we prove that \(S\) is an H-force set of \(G\) which implies \(h(G) = |S| = r + 5\). For this purpose it is sufficient to show that \(G - v\) for any \(v \in V \setminus S\) has no \(S\)-cycle. Suppose, for the contrary, that for some \(v \in V \setminus S\) there exists an \(S\)-cycle \(D\) in \(G - v\).

In the case \(v = x_0\) we have \(x_0 x_2 \notin E(G - v)\) for \(i = 1, \ldots, 6\). So, \(x_3, x_5 \in S\) implies that \(D\) contains the path \([x_2, x_3, x_5, x_6]\) and, \(x_2 z_1 \notin E(D)\). By \(z_1 \in S\) we have \(z_1 x_2\) or \(z_1 x_6 \in E(D)\), say \(z_1 x_2 \in E(D)\). Then, \(x_2 z_j \notin E(D)\) for \(j = 1, \ldots, 6\).
2, . . . , s and, because of \( z_s \in S \) the path \([z_1, \ldots, z_s, x_6]\) is contained in \( D \). Thus, \( D = [x_2, \ldots, x_6, z_s, \ldots, z_1] \), a contradiction.

In the case \( v = x_2 \) we have \( x_2x_3, x_2z_j \notin E(G - v) \) for \( j = 1, \ldots, s \). Then, because of \( z_1, z_s \in S \) the path \([x_4, z_1, \ldots, z_s, x_6]\) is contained in \( D \), because otherwise \( D = [z_1, \ldots, z_s, x_6] \), a contradiction. Moreover, \( x_3 \in S \) implies that \( D \) contains the path \([x_0, x_3, x_4]\). Hence, \( x_4x_5 \notin E(D) \) and \( D \) contains also the path \([x_0, x_5, x_6]\) which yields \( D = [x_0, x_3, x_4, z_1, \ldots, z_s, x_6, x_5] \), a contradiction.

In the case \( v = x_6 \) by symmetry we obtain a contradiction, too.

In the case \( v = x_4 \) we have \( x_4x_0, x_4x_3, x_4x_5, x_4z_1 \notin E(G - v) \). Because of \( x_3, x_5 \in S \) the path \([x_2, x_3, x_0, x_5, x_6]\) is contained in \( D \) and, because of \( z_1, z_2 \in S \) exactly one of the paths \([x_2, z_1, \ldots, z_s, x_6]\) and \([x_2, z_s, \ldots, z_1, x_6]\) is contained in \( D \) which gives a contradiction.

Let us consider now the case \( v = z_j \) where \( 1 < j_0 < s \). Because of \( z_s \in S \) there exists a \( j_1 \) with \( j_0 < j_1 \leq s \) such that \( D \) contains one of the two paths \([x_2, z_1, \ldots, z_s, x_6]\), \([x_2, z_s, \ldots, z_1, x_6]\). Without loss of generality, we may assume that \( D \) contains \([x_2, z_1, \ldots, z_s, x_6]\). Moreover, \( z_1 \in S \) implies that there exists a \( j_2 \) with \( 1 \leq j_2 < j_0 \) such that \( D \) contains either (i) one of the two paths \([x_2, z_1, \ldots, z_j, x_6]\), \([x_2, z_{j_2}, \ldots, z_1, x_6]\) or (ii) one of the two paths \([x_4, z_1, \ldots, z_{j_2}, x_2]\), \([x_4, z_1, \ldots, z_{j_2}, x_6]\). In case (i) \( D \) is equal to one of the cycles \([x_2, z_1, \ldots, z_5, x_6, z_{j_2}, \ldots, z_1, x_6]\), \([x_2, z_{j_1}, \ldots, z_5, x_6, z_1, z_{j_2}, \ldots, z_1, x_6]\) which yields a contradiction. In case (ii) by symmetry we may assume that \( D \) contains \([x_4, z_1, \ldots, z_{j_2}, x_2]\). Hence, \( x_2x_3 \notin E(D) \). Then, by \( x_3 \in S \) the path \([x_0, x_3, x_4]\) is contained in \( D \) which implies that \( x_4x_5 \notin E(D) \). Then, because of \( x_3 \in S \) the path \([x_0, x_5, x_6]\) is also contained in \( D \) which yields \( D = [x_2, z_{j_2}, \ldots, z_1, x_4, x_3, x_0, x_5, x_6, z_s, \ldots, z_1] \), a contradiction. Thus, \( S \) is proved to be an \( H \)-force set of \( G \).

If \( n \) is the order and \( k \) the \( H \)-force number of \( G \), then the relations \( n = r + s + 7 \) and \( k = r + 5 \) together with \( r \geq 0 \) and \( s \geq 2 \) imply \( n \geq 9 \) and \( 5 \leq k \leq n - 4 \) which completes the proof.

For the following theorem which considers the remaining three cases \( k = n - 3, n - 2, n - 1 \) we present the construction figures for a proof but (for shortness of this paper) not the complete proof.

**Theorem 16.** For every integers \( n \geq n_0 \) and \( s \in \{1, 2, 3\} \) there exists a polyhedral hamiltonian graph \( G \) of order \( n \) with \( h(G) = n - s \), where \( n_0 = 12, 16, 14 \) for \( s = 3, 2, 1 \), respectively.

**Proof.** In the case \( s = 3 \) let the cycles \( C_1 = [x_1, x_2, x_3], C_2 = [y_1, \ldots, y_6] \) and \( C_3 = [z_1, z_2, z_3] \) be drawn one into each other in the plane such that \( C_1 \) is the outer and \( C_3 \) the inner one and connect the cycles by the edges \( x_1y_1, x_2y_3, x_3y_5, z_1y_2, z_2y_4 \) and \( z_3y_6 \). If \( n \) is greater than \( n_0 = 12 \) then let, in addition, the path \( P = [u_1, \ldots, u_{n-12}] \) be drawn in the unbounded face where \( x_1 \) is connected with
all vertices of $P$ by an edge and $x_2u_1$, $x_3u_{n-12}$ are additional edges. The so constructed polyhedral graph $G$ of order $n$ is hamiltonian and $V(G) \setminus \{y_2, y_4, y_6\}$ is a smallest H-force set of $G$.

In the case $s = 2$ let the cycles $C_1 = [x_1, \ldots, x_4]$, $C_2 = [y_1, y_2, \ldots, y_8]$ and $C_3 = [z_1, \ldots, z_4]$ be drawn one into each other in the plane such that $C_1$ is the outer and $C_3$ the inner one and connect the cycles by the edges $x_1y_1$, $x_2y_3$, $x_3y_5$, $x_4y_7$, $z_1y_2$, $z_2y_4$, $z_3y_6$ and $z_4y_8$. If $n$ is greater than $n_0 = 16$ then let, in addition, the path $P = [u_1, \ldots, u_{n-16}]$ be drawn in the unbounded face where $x_1$ is connected with all vertices of $P$ by an edge and $x_2u_1$, $x_3u_{n-16}$ are additional edges. The so constructed polyhedral graph $G$ of order $n$ is hamiltonian and $V(G) \setminus \{x_2, x_4\}$ is a smallest H-force set of $G$.

In the case $s = 1$ let a cycle $C = [x_1, \ldots, x_9]$ be drawn in the plane and let $z$ be a vertex in the bounded face which is connected with each vertex of $C$ by an edge. Moreover, let $K_{1,3}$ be a claw in the unbounded face with endvertices $y_1$, $y_2$, $y_3$. Let the claw be connected with $C$ by edges $y_1x_2$, $y_1x_3$, $y_2x_5$, $y_2x_6$, $y_3x_8$ and $y_3x_9$. If, now, $n$ is greater than $n_0 = 14$ then let, in addition the path $P = [u_1, \ldots, u_{n-14}]$ be drawn in the unbounded face where $x_1$ is connected with all vertices of $P$ by an edge and $x_2u_1$, $x_9u_{n-14}$ are additional edges. The so constructed polyhedral graph $G$ of order $n$ is hamiltonian and $V(G) \setminus \{z\}$ is a smallest H-force set of $G$.

\section{Bipartite Graphs}

If the number of components of a graph $G$ is denoted by $c(G)$ we have

**Proposition 17.** Let $G$ be a hamiltonian graph of order $n$. If there exists a set $S \subseteq V(G)$ with $c(G - S) = |S|$, then $h(G) \leq n - |S|$.

**Proof.** Let $X = V(G) \setminus S$. Any $X$-cycle of $G$ requires $|S|$ additional vertices, thus it is a hamiltonian one and thereby $X$ is an H-force set of $G$.

There are two noteworthy special cases of the previous statement, the first, if $|S| = 2$

**Corollary 18.** If $G$ is a hamiltonian graph of order $n$ with $\kappa(G) = 2$, then $h(G) \leq n - 2$.

and the second, if every component of $G - S$ is a single vertex.

**Corollary 19.** If $G$ is a hamiltonian graph of order $n$ with $\alpha(G) = \frac{n}{2}$, then $h(G) \leq \frac{n}{2}$.

Applying Corollary 19 to the complete bipartite graph $K_{\frac{n}{2}, \frac{n}{2}}$ and considering that any $k$ vertices with $k < \frac{n}{2}$ are contained in a nonhamiltonian cycle we obtain
Corollary 20. $h(K_{\frac{n}{2}, \frac{n}{2}}) = \frac{n}{2}$.

Dirac [10] proved that any $k$ vertices of a $k$-connected graph lie on a common cycle. We use this result to prove the following theorem.

**Theorem 21.** If $G \neq C_n$ is a Hamiltonian graph, then $h(G) \geq \kappa(G)$.

**Proof.** Let $G$ be a Hamiltonian graph with $\kappa(G) = \kappa$. Since for $\kappa = 2$ the proposition is obvious, let $\kappa \geq 3$.

For any vertex $u \in V(G)$, the graph $G - u$ is $(\kappa - 1)$-connected. For any set $X \subseteq V(G - u)$ with $|X| = \kappa - 1$, by the above mentioned result of Dirac, there is an $X$-cycle in $G - u$ that is obviously non-Hamiltonian in $G$. Therefore, there is no $H$-force set in $G$ consisting of $\kappa - 1$ vertices.

Moreover, graphs resulting from $K_{\frac{n}{2}, \frac{n}{2}}$ by adding any edges in exactly one partite set have $h = \kappa = \frac{n}{2}$, i.e. the lower bound on $H$-force number in the last theorem is tight.

The prism over a graph $G$ is the Cartesian product $G \square K_2$ of $G$ with $K_2$, i.e. the prism over $G$ is obtained by taking two copies of $G$ and joining the two copies of each vertex by a *vertical* edge. We identify $G$ with one of its copies in $G \square K_2$ and denote $\tilde{G}$ the other copy of $G$. This notation is extended, in an obvious way, to vertices, edges and subgraphs of $G \square K_2$. Moreover, if $y = \tilde{x}$, we set $\tilde{y} = x$, in other words, $\tilde{\tilde{x}} = x$.

Theorem 22. Let $G$ be a Hamiltonian graph of order $\frac{n}{2}$. Then

$$h(G \square K_2) = \begin{cases} \frac{n}{2}, & \text{if } G \text{ is bipartite,} \\ n, & \text{if } G \text{ is not bipartite.} \end{cases}$$

**Proof.** Let $G$ be a Hamiltonian graph of order $m = \frac{n}{2}$ and let $C$ be a Hamiltonian cycle of $G$.

**Case 1.** If $G$ is bipartite then the prism $G \square K_2$ over $G$ is bipartite as well and $h(G \square K_2) \leq \frac{n}{2} = m$ by Corollary 19. Moreover, for any set $X \subseteq V(G \square K_2)$ of $m - 1$ vertices, there is a vertical edge $w\tilde{w} \in E(G \square K_2)$ with $w, \tilde{w} \not\in X$, thus $D_1 = [w^+, w^-]_C \cup \tilde{w}^- \cup [\tilde{w}^+, \tilde{w}^-]_C \cup \tilde{w}^+ w^+$ (Figure 3) is a non-Hamiltonian $X$-cycle in $G \square K_2$. Therefore, there is no $H$-force set of cardinality $m - 1$ in $G \square K_2$. 


Case 2. Let $G$ be not bipartite.

Case 2.1. If the order $m$ of $G$ is odd then it is easy to see that, for any vertex $w$ of $G \Box K_2$, there is a cycle $D_2$ of length $n - 1$ in $G \Box K_2$ omitting just the vertex $w$ and containing all vertical edges except of $w\tilde{w}$ (Figure 3). Hence, there is no H-force set of cardinality $n - 1$ in $G \Box K_2$.

Case 2.2. If the order $m$ of $G$ is even then there is an edge $uv \in E(G) \setminus E(C)$ such that $C_1 = [u, v]_G^- \cup uv$ and $C_2 = [u, v]_G^+ \cup uv$ are both odd cycles. Let $G_i$ ($i = 1, 2$) be the graph induced by $V(C_i)$ in $G$. For any vertex of $G \Box K_2$ we look for a cycle omitting just this vertex. Let, without loss of generality, $w \in V(G_1) \setminus \{v\}$. Then $G_1 \Box K_2$ is the prism of a graph of odd order, thus by the previous case, there is a cycle $D'$ in $G_1 \Box K_2$ containing all vertices except of $w$ and all vertical edges except of $w\tilde{w}$. Then $D_3 = (D' - v\tilde{v}) \cup [v, u^+]_G^- \cup u^+\tilde{u}^+ \cup [\tilde{u}^+, v]_G^+$ (Figure 3) is the desired cycle.

5. Planar Graphs

By Theorem 9 of Nelson, the H-force number of every 4-connected planar graph is equal $n$. In section 3, planar graphs of order $n$ and with a given H-force number $k$ were constructed, for any $1 \leq k \leq n$.

A planar graph is outerplanar if it can be embedded in the plane in such a way that all its vertices are incident to the unbounded face. The weak dual $D^*(G)$ of an outerplanar graph $G$ is the graph obtained from the dual of $G$ by removing the vertex corresponding to the unbounded face; it is a tree, if $G$ is 2-connected. In this case let $\ell(G)$ denote the number of leaves of $D^*(G)$.

Theorem 23. If $G \neq C_n$ is an outerplanar hamiltonian graph, then $h(G) = \ell(G) \geq 2$.

Proof. Let $G$ be an outerplanar graph with a hamiltonian cycle $C$ creating the boundary of its outerface. With every leaf of the weak dual $D^*(G)$ there
is associated a face \( \alpha \) of \( G \) incident with a chord \( xy \) of \( C \). All vertices of \( \alpha \) except for \( x \) and \( y \) have degree 2 in \( G \) and every H-force set \( X \) of \( G \) contains at least one of them. Otherwise the cycle \([x, y]_C \cup \{xy\}\) (or \([x, y]_C \cup \{xy\}\)) is a nonhamiltonian cycle of \( G \) omitting all 2-valent vertices of \( \alpha \), a contradiction. Hence, \( |X| = h(G) \geq \ell(G) \).

To prove the converse inequality it is enough to find an H-force set \( X \) consisting of \( \ell(G) \) vertices. If we choose one vertex of degree 2 from each face of \( G \) corresponding to a leaf of the weak dual \( D^*(G) \) we obtain a desired set \( X \). Suppose that there exists a nonhamiltonian \( X \)-cycle \( C' \) in \( G \). Then it has to contain a chord \( xy \in E(G) \). The graph \( G - x - y \) consists of exactly two components each containing a vertex from \( X \), but \( C' \) has an empty intersection with one of them, a contradiction.

For a plane hamiltonian graph \( G \) with a hamiltonian cycle \( C \) let \( G^i_C \) (or \( G^o_C \)) be the graph consisting of the cycle \( C \) and all edges of \( G \) lying inside (outside) of \( C \). Clearly, \( G^i_C \) and \( G^o_C \) are both outerplanar. Taking into consideration the graphs \( G^i_C \) and \( G^o_C \) and the proof of the previous theorem we immediately obtain

**Theorem 24.** If \( G \) is a planar hamiltonian graph with \( \delta(G) \geq 3 \) and \( C \) a hamiltonian cycle of \( G \), then \( h(G) \geq \ell(G^i_C) + \ell(G^o_C) \geq 4 \).

Other results about planar graphs follow in the next section.

### 6. Graphs with Small H-force Number

Let \( C = [v_1, v_2, \ldots, v_n] \) be a hamiltonian cycle of \( G \). We say that a chord \( v_i v_j \) \((i < j - 1)\) separates vertices \( v_k, v_l \) \((k < l - 1)\) on \( C \), if they belong to different components of \( C - v_i - v_j \), and, moreover, crosses the chord \( v_k v_l \) if \( v_k v_l \in E(G) \).

**Theorem 25.** Let \( G \neq C_n \) be a hamiltonian graph and \( C = [v_1, v_2, \ldots, v_n] \) be a hamiltonian cycle of \( G \). Then \( h(G) = 2 \) if and only if

(i) there exist \( x, y \in V(G) \), \( \deg_G(x) = \deg_G(y) = 2 \), such that every chord \( v_i v_j \) \((i < j - 1)\) separates \( x \) and \( y \) on \( C \), and

(ii) for every pair \( v_i v_j \) and \( v_k v_l \) \((i < j - 1, k < l - 1)\) of crossed chords \( v_i v_k, v_j v_l \in E(C) \) holds.

**Proof.** Suppose \( h(G) = 2 \) and let \( F = \{x, y\} \) be an H-force set of \( G \) (i.e. every \( F \)-cycle of \( G \) is hamiltonian). Moreover, we may assume \( v_1 = x \) and \( v_t = y \) where \( 3 \leq t \leq n - 1 \).

**Claim 1.** \( \deg_G(x) = \deg_G(y) = 2 \).
otherwise, if $\deg_G(x) \geq 3$ then, for $x^+ \in N(x) \setminus \{x^-, x^+\}$, one of the cycles $D_1 = [x, x^+]_C^+ \cup xx^*$ and $D_2 = [x, x^+]_C^- \cup xx^*$ contains $y$ but does not contain one of $x^-$ or $x^+$, therefore it is a nonhamiltonian $F$-cycle; a contradiction.

Claim 2. Every chord $uv$ of $C$ separates $x$ and $y$ on $C$,

otherwise one of the cycles $D_3 = [u, w]^+_C \cup uw$ or $D_4 = [u, w]^-_C \cup uw$ is an $F$-cycle omitting $u^-$ or $u^+$; a contradiction.

Claim 3. If $v_i v_j$ and $v_k v_l$ ($1 < i < k < t < j < l$) are two crossed chords of $C$, then $v_i v_k, v_j v_l \in E(C)$ (i.e. $k = i + 1$ and $l = j + 1$),

otherwise, if $v_i v_k \notin E(C)$ then $D_5 = [v_k, v_j]^+_C \cup v_i v_j \cup [v_i, v_l]^+_C \cup v_k v_l$ is an $F$-cycle of $G$ missing vertex $v_{i+1}$; a contradiction.

To prove the converse let $G$ be a graph satisfying properties (i) and (ii). We assume again $v_1 = x$ and $v_t = y$ where $3 \leq t \leq n - 1$.

Claim 4. For every vertex $v_i \in V(G) \setminus \{x, y\}$ there is a vertex $v_i^* \in V(G)$ such that $\{v_i, v_i^*\}$ separates $x$ and $y$ in $G$,

because,

(a) if $\deg_G(v_i) \geq 3$ and $v_i v_j$ is a chord of $C$ crossed by $v_{i+1} v_{j+1}$ then $\{v_i, v_{j+1}\}$ separates $x$ and $y$,

(b) if $\deg_G(v_i) \geq 3$ and $v_i v_j$ is a chord of $C$ crossed by no other chord then $\{v_i, v_j\}$ separates $x$ and $y$, and

(c) for $\deg_G(v_i) = 2$ let $P = [u, w]^+_C$ be the longest subpath of $C$ containing $v_i$ with internal vertices of degree 2 in $G$ only (i.e. $\deg_G(u), \deg_G(w) \geq 3$).

Then $v_i$ separates $x$ and $y$ in $G$ with the same vertex as $w$ does or with one of the vertices $u, w$ (in the case $V(P) \cap \{x, y\} \neq \emptyset$).

Finally, $F = \{x, y\}$ is an $H$-force set of $G$, because otherwise there exists a nonhamiltonian $F$-cycle $C'$ missing a vertex $v_i$. If $\{v_i, v_i^*\}$ separates $x$ and $y$ in $G$,

then the vertices $x$ and $y$ are separated by at most 1 vertex on $C'$; a contradiction.

Thus, any hamiltonian graph with $H$-force number 2 can be considered as the union of two outerplanar hamiltonian graphs with a common hamiltonian cycle which implies

Corollary 26. Every hamiltonian graph $G$ with $h(G) = 2$ is planar.

For a graph $G$ and a set $X \subseteq V(G)$ we denote by $K_X(G)$ the graph with the vertex set $V(G)$ and the edge set $E(G) \cup \{uv \mid u, v \in X\}$, i.e. the smallest spanning supergraph of $G$ in which $X$ induces a clique. Kawarabayashi [15] proved, that for any $k$-connected graph $G$ and any given $\ell$ vertices ($k \leq \ell \leq \frac{3}{2}k$), there is a cycle in $G$ containing exactly $k$ of them.
By Theorem 21, the H-force number of a 3-connected hamiltonian graph is \( \geq 3 \). We prove that there are only four 3-connected graphs with the H-force number 3.

**Theorem 27.** Let \( G \) be a 3-connected hamiltonian graph. Then

(i) \( h(G) \geq 4 \) or

(ii) \( G \) results from \( K_{3,3} \) by adding any edges in exactly one partite set.

**Proof.** Let \( G \) be a 3-connected hamiltonian graph with \( h(G) = 3 \). There exists an H-force set \( F = \{v_1, v_2, v_3\} \subseteq V(G) \) in \( G \) (i.e. every \( F \)-cycle of \( G \) is hamiltonian). Consider an arbitrary vertex \( x \in V(G) \setminus F \). By the above mentioned theorem of Kawarabayashi the graph \( G \) contains a cycle \( C \) through exactly three of the vertices \( v_1, v_2, v_3, x \). Thus, \( C \) is nonhamiltonian and, consequently, it is no \( F \)-cycle which allows to assume that without loss of generality \( v_2, v_3, x \in V(C) \). As \( G \) is 3-connected, there exist three internally disjoint (\( v_1, C \))-paths \( P_1, P_2, P_3 \) with different endvertices \( y_i \in V(P_i) \cap V(C) \), \( i = 1, 2, 3 \). Denote \( Q_i = [y_{i+1}, y_{i+2}]_C \), \( i = 1, 2, 3 \) (indices modulo 3; see Figure 4) and let \( v_j^* \in N(v_j) \setminus \{v_{j-1}, v_{j+1}\} \) for \( j = 2, 3 \).

![Figure 4](image.png)

**Case 1.** If \( v_2, v_3 \) belong to the same path \( Q_i \) (possibly they are its endvertices) then \( Q_i \cup P_{i+1} \cup P_{i+2} \) is an \( F \)-cycle omitting vertex \( y_i \), thus nonhamiltonian, a contradiction.

**Case 2.** Let \( v_2, v_3 \) do not belong to the same path \( Q_i \) and let one of them be identical with a vertex \( y_j \), i.e. assume w.l.o.g. \( v_2 = y_2, v_3 \in Q_2 \) where \( v_3 \notin \{y_1, y_3\} \).

The cycles \( D_1 = P_1 \cup P_2 \cup Q_1 \cup Q_2 \) and \( D_2 = P_2 \cup P_3 \cup Q_2 \cup Q_3 \) (Figure 4) are both \( F \)-cycles, thus hamiltonian. Therefore, \( P_1, P_3, Q_1, Q_3 \) are paths of length 1 (i.e. \( v_1y_1, v_1y_3, y_2y_3, y_1y_2 \in E(G) \)).

**Case 2.1.** If \( v_3^* \in [y_2, y_3]_C \), then \( D_3 = [v_2, v_3^*]_C \cup v_3^*v_3 \cup [v_3, y_1]_C \cup P_1 \cup P_2 \) (Figure 5) is an \( F \)-cycle omitting the vertex \( v_3^* \), a contradiction.

**Case 2.2.** If \( v_3^* \in [v_3, y_1]_C \), then \( D_4 = [v_2, v_3]_C \cup v_3v_3^* \cup [v_3^*, y_1]_C \cup P_1 \cup P_2 \) (Figure 5) is an \( F \)-cycle omitting the vertex \( v_3^* \), a contradiction.
Case 2.3. If $v_3^* = v_2$ then $D_5 = v_2v_3 \cup [v_3, y_1]^C \cup P_1 \cup P_2$ (Figure 5) is an $F$-cycle omitting the vertex $y_3$, a contradiction.

Case 2.4. If $v_3^* \in P_2$, $v_3^* \neq v_2$, then $D_6 = [v_1, v_3^*]P_2 \cup v_3^*v_3 \cup [v_3, y_3]^C \cup P_3$ and $D_7 = [v_2, v_3^*]P_2 \cup v_3^*v_3 \cup [v_3, y_3]^C \cup P_3 \cup P_1 \cup Q_3$ (Figure 6) are both $F$-cycles, thus hamiltonian and therefore $P_2$ and $Q_2$ have length 2, i.e. $K_{3,3}$ is a spanning subgraph of $G$.

Case 3. Let $v_2, v_3$ do not belong to the same path $Q_i$ and let they be different from $y_j$, i.e. assume w.l.o.g. $v_2 \in Q_3$ and $v_3 \in Q_1$.

$D_8 = Q_3 \cup Q_1 \cup P_3 \cup P_1$ (Figure 7) is an $F$-cycle, thus hamiltonian and therefore $P_2$ and $Q_2$ have length 1 (i.e. $v_1y_2, v_3y_1 \in E(G)$).

Case 3.1. If $v_3^* \in [v_3^+, y_3]^C$, then $D_9 = [y_2, v_3]^C \cup v_3v_3^* \cup [v_3^*, y_3]^C \cup P_3 \cup P_1 \cup Q_3$ (Figure 7) is an $F$-cycle omitting the vertex $v_3^*$, a contradiction.

Case 3.2. If $v_3^* \in P_3$ then $D_{10} = [y_2, v_3]^C \cup v_3v_3^* \cup [v_3^*, v_1]P_1 \cup P_1 \cup Q_3$ (Figure 7) is an $F$-cycle omitting the vertex $y_3$, a contradiction.
Case 3.3. If $v_3^* \in [v_2, v_3^-]_C$ then $D_{11} = [v_3, y_3]_C^+ \cup P_3 \cup [y_1, v_3^+]_C^+ \cup v_3^+v_3$ (Figure 8) is an $F$-cycle omitting the vertex $v_3^-$, a contradiction.

Case 3.4. If $v_3^* \in [y_1^+, v_3^+_C]$ then $D_{12} = [y_2, v_3^+_C]^- \cup v_3^+v_3 \cup [v_3, y_3]_C^+ \cup P_3 \cup P_2$ (Figure 8) is an $F$-cycle omitting the vertex $y_1$, a contradiction.

Analogously, we obtain a contradiction in corresponding cases under consideration of $v_2$ and its neighbour $v_2^*$. There are two remaining cases:

Case 3.5. If $v_3^* \in P_3$, $v_3^* \in P_1$ and $v_j^* \neq y_{j+1}$ for $j = 2$ or $j = 3$ then $D_{13} = [v_2, v_3]_C^+ \cup v_3^+v_3 \cup [v_3^*, v_1]_P \cup [v_1, v_2]_P \cup v_2^2v_2$ (Figure 9) is an $F$-cycle omitting the vertex $y_{j+1}$, a contradiction.

Case 3.6. If $v_3^* = y_3$ and $v_3^* = y_1$ then $D_{14} = [v_2, y_2]_C^+ \cup P_2 \cup P_1 \cup y_1v_3 \cup [v_3, y_3]_C^+ \cup y_3v_3$ and $D_{15} = [y_2, v_3]_C^+ \cup v_3y_1 \cup [y_1, v_2]_C^+ \cup v_2y_3 \cup P_3 \cup P_2$ (Figure 9) are both $F$-cycles, thus hamiltonian and therefore paths $P_1$ and $P_3$ have length 1 and paths $Q_1$ and $Q_3$ have length 2, i.e. $K_{3,3}$ is a spanning subgraph of $G$.

In any case, $K_{3,3}$ is a spanning subgraph of $G$. Let $X, Y \subseteq V(K_{3,3}) = V(G)$ be the bipartition of $K_{3,3}$. If $G \subseteq K_X(K_{3,3})$ then $3 \geq h(K_{3,3}) \leq h(G) \leq h(K_X(K_{3,3})) \leq 3$ by Proposition 2 and Corollaries 19 and 20, thus $h(G) = 3$. Otherwise, if $G' = K_{3,3} \cup \{x_1x_2, y_1y_2 \mid x_i \in X, y_i \in Y, i = 1, 2\}$ is a subgraph of $G$, then $h(G) \geq h(G') = 6$, which completes the proof.

In the previous section we proved that the H-force number of a planar hamiltonian graph $G$ with $\delta(G) \geq 3$ is lower-bounded by $\ell(G_C^1) + \ell(G_C^2) \geq 4$.

Theorem 28. Let $G$ be a 3-connected planar hamiltonian graph. Then

1. $h(G) \geq 5$ or
(ii) $G = K_4$ or $G$ results from the graph $Q_3$ of the cube by adding any edges in exactly one partite set.

**Proof.** Let $G$ be a plane 3-connected hamiltonian graph with $h(G) = 4$ and let $C$ be a hamiltonian cycle of $G$. Theorems 23 and 24 imply $\ell(G_C') = \ell(G_C^o) = 2$, i.e. the weak duals $D^s(G_C')$ and $D^s(G_C^o)$ are paths. Let $\alpha, \beta$ and $\gamma, \delta$ be the faces of $G$ corresponding to endvertices of $D^s(G_C')$ and $D^s(G_C^o)$, respectively, and let $F = \{x, y, u, v\}$ be an $H$-force set, where $x \in V(\alpha), \ y \in V(\beta), \ u \in V(\gamma), \ v \in V(\delta)$ and $\deg_{G_C'}(x) = \deg_{G_C'}(y) = \deg_{G_C'}(u) = \deg_{G_C'}(v) = 2$.

**Claim 1.** Every chord $e \in E(G_C')$ (or $e \in E(G_C^o)$) of $C$ separates $x, y$ in $G_C'$ (or $u, v$ in $G_C^o$).

Let $x^*$ be a neighbour of $x$ in $G$, different from $x^+$ and $x^-$, with the smallest distance $d_C(x^*, y)$ from $y$ on $C$ and similarly, $y^* \in N(y) \setminus \{y^+, y^-\}$ with minimum $d_C(y^*, x)$, $u^* \in N(u) \setminus \{u^+, u^-\}$ with minimum $d_C(u^*, v)$ and $v^* \in N(v) \setminus \{v^+, v^-\}$ with minimum $d_C(v^*, u)$.

**Case 1.** Let $xy, uv \in E(G)$ (i.e. $x^* = y, \ y^* = x, \ u^* = v, \ v^* = u$), then $D_1 = [x, u]^+_C \cup uv \cup [y, v]^+_C \cup xy$ and $D_2 = [x, v]^+_C \cup vu \cup [u, y]^+_C \cup vx$ (Figure 10) are $F$-cycles, hence, both are hamiltonian. Therefore, $[x, u]^+_C, \ [v, y]^+_C, \ [x, v]^+_C$ and $[u, y]^+_C$ are paths of length 1 (i.e. $xu, vy, xv, uy \in E(C)$) and finally $G = K_4$.

![Figure 10](image1.png)

**Case 2.** Let, without loss of generality, $xy \in E(G)$ and $uv \notin E(G)$ (i.e. $x^* = y, \ y^* = x, \ u^* \neq v, \ v^* \neq u$).

**Case 2.1.** If $u^* \in [y, v]^+_C$ (and consequently $v^* \in [x, u]^+_C$), then $D_3 = [x, u^*]^+_C \cup u^*u \cup [u, y]^+_C \cup xy$ (Figure 11) is an $F$-cycle omitting vertex $v^*$, a contradiction.

![Figure 11](image2.png)
Case 2.2. If \( u^* \in \{v, x\}_C \) (and consequently \( v^* \in \{u, y\}_C \)), then \( D_4 = [x, u^*_C] \cup uu^* \cup [u^*, y^*_C] \cup yx \) (Figure 11) is a nonhamiltonian \( F \)-cycle, a contradiction.

Case 3. Let \( xy, uv \notin E(G) \) (i.e. \( \{x^*, y^*, u^*, v^*\} \cap \{x, y, u, v\} = \emptyset \)).

Case 3.1. Let each of the paths \( [x, u^*_C], [u, y^*_C], [y, v^*_C], [v, x^*_C] \) contains a vertex from \( \{x^*, y^*, u^*, v^*\} \) (without loss of generality, let \( v^* \in [x, u^*_C], x^* \in [u, y^*_C], u^* \in [y, v^*_C] \) and \( y^* \in [v, x^*_C] \).

Then \( D_3 = [x, v^*_C] \cup v^*v \cup [v, y^*_C] \cup y^*y \cup [y, u^*_C] \cup u^*u \cup [u, x^*_C] \cup x^*x, D_6 = [x, y^*_C] \cup y^*y \cup [y, v^*_C] \cup v^*u \cup [u, x^*_C] \cup x^*x, \) and \( D_7 = [x, u^*_C] \cup uu^* \cup [u^*, y^*_C] \cup y^*y \cup [y, x^*_C] \cup x^*x \) (Figure 12) are \( F \)-cycles, hence all are hamiltonian. Since each of the paths \( [x, v^*_C], [v^*, u^*_C], [u, x^*_C], [x^*, y^*_C], [y, u^*_C], [u^*, v^*_C], [v, y^*_C], [y^*, x^*_C] \) has at least one of the hamiltonian cycles \( D_5, D_6, D_7 \) no inner vertex in common, there is no inner vertex on any of these paths, and, consequently, the cube graph \( Q_3 \) is a spanning subgraph of \( G \).

Case 3.2. Let exactly two of the paths \( [x, u^*_C], [u, y^*_C], [y, v^*_C], [v, x^*_C] \) contain a vertex from \( \{x^*, y^*, u^*, v^*\} \) (without loss of generality and because of claim 1 let \( x^*, v^* \in [u, y^*_C] \) and \( y^*, u^* \in [v, x^*_C] \)).

Case 3.2.1. Let \( u^* \notin [y^*, x^*_C] \) (or analogously \( v^* \notin [x^*, y^*_C] \)). Then \( D_8 = [x, u^*_C] \cup uu^* \cup [u^*, x^*_C] \cup x^*x \) (Figure 13) is a nonhamiltonian \( F \)-cycle, a contradiction.

Case 3.2.2. Let \( u^* \in [y^*, x^*_C] \) and \( v^* \in [x^*, y^*_C] \). Then \( D_9 = [x, u^*_C] \cup uu^* \cup [u^*, y^*_C] \cup y^*y \cup [y, v^*_C] \cup v^*v \cup [v^*, x^*_C] \cup x^*x \) (Figure 13) is an \( F \)-cycle, hence it is hamiltonian. Therefore then the paths \( [u, x^*_C], [v^*, y^*_C], [v, y^*_C], \) and \( [u^*, x^*_C] \) have length 1 (i.e.
ux*, v*y, vy*, u*x ∈ E(G)). Since G is planar and 3-connected, there exists a neighbour of u on [v, u]*C different from u* (otherwise, the set {x*, u*} would be a 2-cut of G, with contradiction), which contradicts the minimality of dC(u*, v).

That means, G = K4 or G contains Q3 as a spanning subgraph. In the second case let X, Y ⊆ V(Q3) = V(G) be the bipartition of Q3. If G ⊆ KX(G) then 4 = h(Q3) ≤ h(G) ≤ h(KX(G)) ≤ 4 by Proposition 2, Corollary 19, and Theorem 22, thus h(G) = 4. Otherwise, if G' = Q3 ∪ {x1x2, y1y2 | xi ∈ X, yi ∈ Y, i = 1, 2} is a subgraph of G, then h(G) ≥ h(G') = 8, which completes the proof.

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