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Dedicated to Mietek Borowiecki on the occasion of his seventieth birthday: best wishes and thanks for all!

A NOTE ON UNIQUELY EMBEDDABLE FORESTS [1]

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Abstract

Let F be a forest of order n. It is well known that if $F \neq S_n$, a star of order n, then there exists an embedding of F into its complement $\overline{F}$. In this note we consider a problem concerning the uniqueness of such an embedding.

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1. Introduction

We shall use standard graph theory notation. We consider only finite, undirected graphs of order $n = |V(G)|$ and size $e(G) = |E(G)|$. All graphs will be assumed to have neither loops nor multiple edges.

We shall need some additional definitions in order to formulate the results. If a graph G has order n and size m, we say that G is an (n, m) *graph*.

Assume now that G_1 and G_2 are two graphs with disjoint vertex sets. The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If a graph G is the union of n (≥ 2) disjoint copies of a graph H, then we write $G = nH$.

For our next operation, the conditions are quite different. Let now G_1 and G_2 be graphs with $V(G_1) = V(G_2)$ and $E(G_1) \cap E(G_2) = \emptyset$. The *edge sum* $G = G_1 \oplus G_2$ has $V(G) = V(G_1) = V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

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An *embedding* of G (in its complement $\overline{G}$) is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$.

In other words, an embedding is an (edge-disjoint) *placement* (or *packing*) of two copies of G into a complete graph K_n.

The following theorem was proved, independently, in [1], [2] and [5].

Theorem 1. *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n - 2$, then G can be embedded in its complement $\overline{G}$.*

The example of the star $S_n = K_{1,n-1}$ shows that Theorem 1 cannot be improved by raising the size of G. However, if a tree is not a star then it is embeddable. This fact was first observed by H.J. Straight (unpublished). The version given below comes from [8].

Theorem 2. *Let T be a non-star tree. Then there exists a cyclic permutation on $V(T)$ being an embedding of T.*

As an immediate consequence of above theorems we get the following corollary.

Corollary 3. *Let F be a non-star forest. Then there exists a cyclic permutation on $V(F)$ being an embedding of F.*

Let us consider now the problem of the uniqueness. First, we have to precise what we mean by *distinct* embeddings.

Let σ be an embedding of the graph $G = (V, E)$. We denote by $\sigma(G)$ the graph with the vertex set V and the edge set $\sigma^*(E)$ where the map σ^* is induced by σ. Since, by definition of an embedding, the sets E and $\sigma^*(E)$ are disjoint we may form the graph $G \oplus \sigma(G)$ (called the *resulting graph* for σ).

Two embeddings σ_1, σ_2 of a graph G are said to be *distinct* if the graphs $G \oplus \sigma_1(G)$ and $G \oplus \sigma_2(G)$ are not isomorphic. A graph G is called *uniquely embeddable* if for all embeddings σ of G, all graphs $G \oplus \sigma(G)$ are isomorphic. Of course, if a graph is edgeless, it is uniquely embeddable. So, we shall assume that all graphs considered in this paper have at least one edge. The next theorem, proved in [9], characterizes all $(n, n-2)$ $(n \geq 3)$ graphs that are uniquely embeddable.

Theorem 4. *Let G be a graph of order n, $n \geq 3$, and size $e(G) = n - 2$. Then either G is not uniquely embeddable or G is isomorphic to one of the seven following graphs: $K_2 \cup K_1$, $2K_2$, $K_3 \cup 2K_1$, $K_3 \cup K_2 \cup K_1$, $K_3 \cup 2K_2$, $2K_3 \cup 2K_1$.*

Remark. The formulation of the above theorem in [9] contains a mistake; actually, the graph $G = C_4 \cup 2K_1$ **is not** uniquely embeddable.

The problem of the uniqueness was considered ([12]) also in the case of cycles.

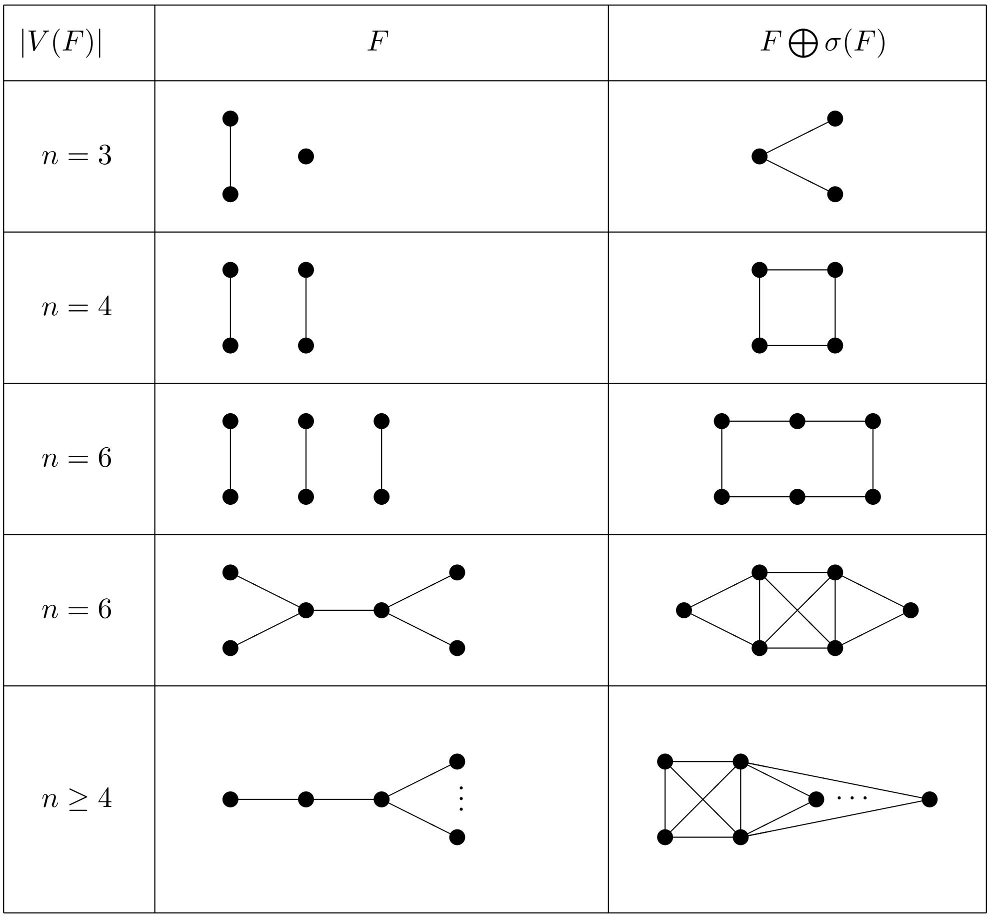

Figure 1. Uniquely embeddable forests.

Theorem 5. *Let C_n be a cycle of order n. The cycles C_3 and C_4 are not embeddable. The cycles C_5 and C_6 are uniquely embeddable. For $n \geq 7$ there exist at least two distinct embeddings of C_n.*

The aim of this note is to consider the problem for acyclic graphs. We need some additional definitions in order to formulate the result. By *double star* $S(p, q)$ we mean a tree obtained from two stars S_{p+1} and S_{q+1} by joining their centers by an edge. (This edge is called *central*). A double star $S(1, q)$ will be denoted also by S'_n ($n \geq 4$, $n = q + 3$). Let us observe that $S'_4 = P_4$, a path of length three.

We have the following characterization of uniquely embeddable forests.

Theorem 6. *Let F be a non-star forest of order n having at least one edge. Then either F is not uniquely embeddable or F is isomorphic to one of the following graphs: $K_2 \cup K_1$, $2K_2$, $3K_2$, $S(2, 2)$ or S'_n, for $n \geq 4$.*

See Figure 1 where the uniquely embeddable forests F as well as the corresponding resulting graphs $F \oplus \sigma(F)$ are illustrated. Note, that in the last case, for $n = 4$, $F = P_4$ and $F \oplus \sigma(F) = K_4$.

The proof of Theorem 6 is given in the next sections. Section 2 contains the case of trees, while Section 3 deals with the general case.

Remark. The main references of the paper and of other packing problems are the following survey papers: [13], [10] or [11].

2. Trees

2.1. Trees of diameter at least five

Let T be a tree. Let us consider first the case where diam(T) ≥ 5. Denote by X the set of leaves of T (*i.e.* vertices of degree one in T). We put $T' = T - X$. Since the diameter of T is at least five, T' is not a star. Therefore, by Theorem 2, T' is embeddable. Moreover, there exists a cyclic packing permutation of T'. Denote it by σ'.

We shall consider now two extensions of this permutation to a packing permutation of T. The first one is defined as follows: we put $\sigma_1(v) = \sigma'(v)$ for $v \in V - X$, and $\sigma_1(v) = v$ for $v \in X$. Since σ' has no fixed points, σ is a packing of T. Let us observe that the graph $T \oplus \sigma_1(F)$ has exactly $|X|$ vertices of degree two; all vertices of $V - X$ have in the graph $T \oplus \sigma_1(T)$ degrees at least four.

The permutation σ_2 is defined in an analogous way. However, we do not remove all end-vertices of T but all but one, say x. More precisely, if x is a leaf of T, we define X' as the set of all leaves of T except for x. The permutation σ_2 is defined as follows: we put $\sigma_2(v) = \sigma''(v)$ for $v \in V - X'$, and $\sigma_2(v) = v$ for $v \in X'$ where σ'' is a cyclic packing permutation of $T'' = T - X'$. As above, since σ'' has no fixed points, σ_2 is a packing of T. This time, the graph $T \oplus \sigma_2(T)$ has exactly $|X| - 1$ vertices of degree two; all vertices of $V - X'$ have in the graph $T \oplus \sigma_2(T)$ degrees at least three.

Hence, the graphs $T \oplus \sigma_1(T)$ and $T \oplus \sigma_2(T)$ are not isomorphic.

2.2. Trees of diameter four

Consider now the case of trees of diameter four. Denote by aa_1cb_1b the consecutive vertices of (one of) the longest paths in T. Denote by X the set of all leaves of T except for a and b.

If X is empty then T is just a path on five vertices. Two different packings of such a path is given in Figure 2.

If X is not empty then define the tree T' by $T' = T - X$. It is easy to see that the tree T' consists of the path joining a and b and (maybe) of some vertices

Figure 2. Two distinct embeddings of P_5.

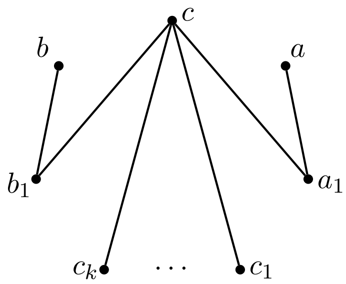

Figure 3. A cyclic packing of T' (Subsection 2.2).

of degree one (in T', but not in T) connected by an edge to the vertex c. Denote them (if exist) by $c_1, \ldots, c_k$.

Since T' contains a path on five vertices, T' is not a star and by Theorem 2 there is a cyclic packing of T'. An example of such a cyclic permutation, defined by $\sigma' = (acbb_1c_k \ldots c_1a_1)$, is given in Figure 3.

Let us observe that both vertices of degree one (in T) are mapped (by σ') on vertices of degree at least two in T. Now, we extend σ' to a packing permutation of T in the following way: we put $\sigma_1(v) = \sigma'(v)$ for $v \in V - X$, and $\sigma_1(v) = v$ for $v \in X$. As above, since σ' has no fixed points, σ is a packing of T. Again, let us observe that the graph $T \oplus \sigma_1(F)$ has exactly $|X|$ vertices of degree two; all vertices of $V - X$ have in the graph $T \oplus \sigma_1(F)$ degrees at least three.

The permutation σ_2 is defined as follows: We put $X'' = X \cup \{b\}$. Then, $T'' = T - X''$ contains a path of length three. Therefore, T'' is packable. If σ'' denotes the cyclic permutation of T'' then by σ_2 we mean a permutation obtained from σ'' by adding fixed points from X''. As above, it is easy to see that σ_2 is a packing of T with the graph $T \oplus \sigma_2(T)$ having exactly $|X| + 1$ vertices of degree two.

Thus, the packings σ_1 and σ_2 are distinct.

2.3. Trees of diameter three

Consider finally the case where T has diameter three. In other words, T is a double star $S(p,q)$. Denote by a and b two vertices of $T = S(p,q)$ of degree at least two and by $a_1,\ldots,a_p$ (respectively by $b_1,\ldots,b_q$) the leaves adjacent to a (respectively to b). Let us start with the case where $p \geq 2$ and $q \geq 2$.

First observe that the central edge ab is a *total edge* in T, *i.e.* the vertices a and b are adjacent to all remaining vertices of T. This implies, in particular, that in a packing the edge ab cannot be mapped on a pair of vertices having a common neighbour. Therefore, if σ is a packing of T, then the central edge ab should be mapped onto non-edge of the form $\{a_i, b_j\}$ for some i, j. Without loss of generality, we may assume that $\sigma(\{a,b\}) = \{a_1, b_1\}$. Next, it is easy to see that if one of central vertices is mapped on a_1, then a leaf adjacent to it should be mapped on b and if it is mapped on b_1, then a leaf adjacent to it should be mapped on a. In consequence, four vertices $\{a, b, a_1, b_1\}$ have to induce K_4 in $T \oplus \sigma(T)$. Two of them are of degree $p+2$ and two are of degree $q+2$ in $T \oplus \sigma(T)$. The remaining vertices of $T \oplus \sigma(T)$ will be of degree two. So, it is impossible to distinguish two packings of a double star by considering only the degree sequences.

The following notion will be useful in the description of the structure of the graph $T \oplus \sigma(T)$. Let G be a graph and let $x \in V(G)$ be a vertex of degree two. Denote by x_1, x_2 the neighbours of x. The pair of vertices $\{x_1, x_2\}$ is called a *base* of x.

We define now two packing permutations for T in the following way:

$\sigma_1(a) = a_1$, $\sigma_1(b) = b_1$, $\sigma_1(a_1) = b$, $\sigma_1(b_1) = a$, $\sigma_1(a_i) = a_i$ for $i \geq 2$, $\sigma_1(b_i) = b_i$ for $i \geq 2$. It is easy to see that in the graph $T \oplus \sigma_1(T)$ there are only two edges being bases of vertices of degree two: aa_1 and bb_1.

The second permutation is defined with a supplementary condition that $q \geq 3$. We put: $\sigma_2(a) = a_1$, $\sigma_2(b) = b_1$, $\sigma_2(a_1) = b$, $\sigma_2(b_1) = a$, $\sigma_2(a_2) = b_2$, $\sigma_2(b_2) = a_2$, $\sigma_2(a_i) = a_i$ for $i \geq 3$ (if exists), $\sigma_2(b_i) = b_i$ for $i \geq 3$. The base of a_2 (which is of degree two in $T \oplus \sigma_2(T)$) is the edge ab_1. The base of b_2 (which is of degree two in $T \oplus \sigma_2(T)$) is the edge a_1b. Finally, the base of b_3 (which is also of degree two in $T \oplus \sigma_2(T)$) is the edge bb_1. Thus, the set of all bases in $T \oplus \sigma_2(T)$ contains at least three edges. We conclude, that the above defined packings are distinct.

We left to the reader the checking of the fact, that the double star $S(2,2)$ is uniquely embeddable.

Examine now the case where T is a double star $S(1,q)$. Let σ be a packing permutation of T. As above, without loss of generality, we may assume that $\sigma(\{a,b\}) = \{a_1, b_1\}$. So, we have two possibilities.

1) $\sigma(a) = a_1$. Then $\sigma(b) = b_1$, $\sigma(a_1) = b$ and without loss of generality, we may assume that $\sigma(b_1) = a$. Now, the set of all remaining leaves adjacent to b

have to be mapped on itself. Thus all vertices of degree two in $T \oplus \sigma(T)$ are based on the same edge bb_1. In the case $q = 1$, $S(1, q)$ is just a path on four vertices, P_4. Then, $T \oplus \sigma(T) = K_4$ contains no vertices of degree two.

2) $\sigma(a) = b_1$. Then $\sigma(b) = a_1$, $\sigma(a_1) = a$ and without loss of generality, we may assume that $\sigma(b_1) = b$. As above, the set of all remaining leaves adjacent to b have to be mapped on itself. Thus all vertices of degree two in $T \oplus \sigma(T)$ (if exist) are based on the same edge ba_1. In the case $q = 1$, $T \oplus \sigma(T) = K_4$.

Evidently, in both cases we get isomorphic graphs. This finishes the part of the proof concerning trees.

3. Forests with Several Components

3.1. Forests with two components

A forest of order n having two components is an $(n, n-2)$ graph, so, by Theorem 4, only two such forests, $K_2 \cup K_1$ and $2K_2$, are uniquely embeddable. An independent proof of this fact can be found in [6].

3.2. Forests with three components

Let $F = T_1 \cup T_2 \cup T_3$ with $|T_1| \le |T_2| \le |T_3|$.

Suppose first, that T_1 is an isolated vertex. If the forest $F' = T_2 \cup T_3$, having two components, is not uniquely embeddable, then it has at least two distinct embeddings which can be easily extended to distinct embeddings of F.

If not, then by the previous case, either $F' = K_1 \cup K_2$ or $F' = 2K_2$. In both cases it is easy to define two distinct packing permutation such that the resulting graph $F \oplus \sigma(F)$ contains either P_3 or $2K_2$, in the first case, or $F \oplus \sigma(F)$ contains either P_5 or C_4, in the second case, respectively.

So, we can assume that T_1 contains at least one edge. Consider first the case where one of trees T_i, $i = 1, 2, 3$, is embeddable. Then, we are done if the remaining part of F is not uniquely embeddable, *i.e.* is not of the form $K_2 \cup K_2$. But $K_2 \cup K_2$ is an embeddable forest. So, actually, the only case where the existence of two distinct packing is not obvious is the case where the embeddable tree (it should be T_3) is uniquely embeddable. In this case, one of possible resulting graph is evidently the graph having two components, one of them being a cycle C_4 obtained as the result of packing $K_2 \cup K_2$ with itself and second of them being the graph $T_3 \oplus \sigma(T_3)$. Another resulting graph, having only one component, is now easy to define: we use two independent vertices of $K_2 \cup K_2$ (one vertex of each K_2) to draw the second copy of T_3 and two remaining vertices of $K_2 \cup K_2$ and two (not yet used) vertices of T_3 to draw the second copy of $K_2 \cup K_2$.

So, we are left with the case where all three trees are stars, S_p, S_q, S_r, say, with $1 \le p \le q \le r$, $p+q+r+3 = n$. Denote by x_0, y_0, z_0 the central vertices of these stars, and by x_1, y_1, z_1 one of their leaves, respectively.

The first packing permutation σ_1 is defined as follows: $\sigma_1(x_0) = y_0$, $\sigma_1(y_0) = z_0$, $\sigma_1(z_0) = x_0$ while the leaves of all stars are fixed points. It is easy to see that the resulting graph $F \oplus \sigma_1(F)$ has three vertices of degree $p+q$, $q+r$ and $r+p$ and $p+q+r$ vertices of degree two.

The second packing permutation σ_2 is defined as follows: $\sigma_2(x_0) = y_1$, $\sigma_2(y_0) = z_1$, $\sigma_2(z_0) = x_1$, $\sigma_2(x_1) = x_0$, $\sigma_2(y_1) = y_0$, $\sigma_2(z_1) = z_0$ while the remaining leaves of all stars are fixed points. It is easy to see that the resulting graph $F \oplus \sigma_2(F)$ has two vertices of degree $p+1$, two vertices of degree $q+1$, two vertices of degree $r+1$, and $p+q+r-3$ vertices of degree two.

Therefore, two resulting graphs $F \oplus \sigma_1(F)$ and $F \oplus \sigma_2(F)$ can be isomorphic only in the case where $p = q = 1$. Then, both of these graphs have two vertices of degree $r+1$ while remaining vertices are of degree two. If $r \ge 2$ then two vertices of maximum degree in $F \oplus \sigma_1(F)$, z_0 and $\sigma(z_0)$ are independent, while two vertices of maximum degree in $F \oplus \sigma_2(F)$, z_0 and $\sigma(z_0)$ are adjacent. Thus, it is easy to see that the graphs $F \oplus \sigma_1(F)$ and $F \oplus \sigma_2(F)$ could be isomorphic only in the case where $p = q = r = 1$, *i.e.* $F = 3K_2$.

3.3. Forests with at least four components

Let $F = T_1 \cup T_2 \cup \cdots \cup T_k$ with $k \ge 4$ and $|T_1| \le |T_2| \le \cdots \le |T_k|$. By putting $F_1 = T_1 \cup T_2$ and $F_2 = T_3 \cup \cdots \cup T_k$ we get two forests, each of them having at least two components. Both of them are embeddable. If at least one of them is not uniquely embeddable, we are done. So, by previous results, $k = 4$ or 5 and F should be an union of isolated vertices and isolated edges and having, by assumptions, at least one edge. For such graphs, two distinct embeddings are easy to define.

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