

INDEPENDENT DETOUR TRANSVERSALS IN 3-DEFICIENT DIGRAPHS¹

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Abstract

In 1982 Laborde, Payan and Xuong [*Independent sets and longest directed paths in digraphs*, in: *Graphs and other combinatorial topics* (Prague, 1982) 173–177 (Teubner-Texte Math., 59 1983)] conjectured that every digraph has an independent detour transversal (IDT), i.e. an independent set which intersects every longest path. Havet [*Stable set meeting every longest path*, *Discrete Math.* 289 (2004) 169–173] showed that the conjecture holds for digraphs with independence number two. A digraph is *p-deficient* if its order is exactly p more than the order of its longest paths. It follows easily from Havet's result that for $p = 1, 2$ every p -deficient digraph has an independent detour transversal. This paper explores the existence of independent detour transversals in 3-deficient digraphs.

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1. INTRODUCTION

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. The number of vertices of D is called its *order* and denoted by $n(D)$. For any nonempty subset S of $V(D)$, $\langle S \rangle$ denotes the subdigraph of D induced by S . The cardinality of S is denoted by $|S|$.

If v is a vertex in a digraph D , we denote the sets of *out-neighbours* and *in-neighbours* of v by $N^+(v)$ and $N^-(v)$ and the cardinality of these sets by $d^+(v)$ and $d^-(v)$, respectively. The *neighbourhood* of v , denoted by $N(v)$ is defined by $N(v) = N^+(v) \cup N^-(v)$ and the *degree* of v in D is defined as $d(v) = d^+(v) + d^-(v)$. The *closed neighbourhood* of v , denoted by $N[v]$ is defined by $N[v] = N(v) \cup \{v\}$. If $N[v] = V(D)$, we call v a *universal vertex* of D . We denote $V(D) - N[v]$ by $N^o(v)$.

A *directed path* (cycle) in a digraph will simply be called a path (cycle). A path (cycle) of order m is called an m -path (m -cycle). A *detour* in D is a longest path in D and the *detour order* of D , denoted by $\lambda(D)$, is the order of a detour in D . Following Galeana-Sánchez and Gómez [8], we denote the sets of initial and terminal vertices of detours in D by $L^+(D)$ and $L^-(D)$, respectively. If v is a vertex in D , then the order of a longest path starting at v is denoted by $s_D(v)$ and the order of a detour ending at v is denoted by $e_D(v)$. The *detour deficiency* of D is defined as $p(D) = n(D) - \lambda(D)$. A digraph with detour deficiency p is called p -deficient. The *circumference* $c(D)$ of D is the order of a longest cycle in D .

A digraph D is called *traceable* if it contains a *hamiltonian path* (a path containing all vertices of D). A digraph D is *hamiltonian* if it contains a *hamiltonian cycle* (a cycle containing all vertices of D).

A digraph D is *strong* (or *strongly connected*) if for every two distinct vertices x and y of D there is a path from x to y . A maximal strong subdigraph of D is called a *strong component* of D . The strong components of D have an *acyclic ordering*, i.e. they may be labeled D_1, D_2, \dots, D_t such that if there is an arc from D_i to D_j , then $i \leq j$ (cf. [3], p. 17). If X and Y are distinct strong components in D such that some vertex in X dominates a vertex in Y , then vertices in Y are said to lie *below* X and vertices in X are said to lie *above* Y .

An *oriented graph* is a digraph that is obtained from a simple graph by assigning a direction to each edge, i.e. it is a digraph that has no 2-cycles. An orientation of a complete graph is called a *tournament*.

An *independent set* (or *stable set*) in D is a set of pairwise nonadjacent vertices. The *independence number* (or *stability number*) of D , denoted by $\alpha(D)$, is the maximum integer k such that D has an independent set of cardinality k . Suppose S is a set of vertices in a digraph D . Then S is a *dominating set* of D if for every $x \in V(D) - S$ there is a $y \in S$ such that $yx \in A(D)$. The set S

is an *absorbant set* of D if for every $x \in V(D) - S$ there is a $y \in S$ such that $xy \in A(D)$.

An *independent detour transversal* in D , denoted by IDT, is an independent set that intersects every longest path in D . For undefined concepts we refer the reader to [3].

In 1982 Laborde, Payan and Xuong [11] conjectured that every digraph has an IDT. The conjecture (which we refer to as the LPX Conjecture) clearly holds for every digraph having an independent dominating set or an independent absorbant set. Richardson [12] proved that every digraph without odd cycles has an independent dominating set, and consequently the LPX Conjecture holds for digraphs without odd cycles. In [9] Galeana-Sánchez and Rincón-Mejía presented a number of sufficient conditions for a digraph to have an IDT.

We note that the LPX Conjecture is a particular case of the Directed Path Partition Conjecture (DPPC) (see [1]) which states: For every digraph D and every pair of positive integers λ_1 and λ_2 , such that $\lambda(D) = \lambda_1 + \lambda_2$, there exists a partition (V_1, V_2) of the vertex set $V(D)$ such that $\lambda(\langle V_i \rangle) \leq \lambda_i$, for $i = 1, 2$. Bang-Jensen, Nielsen and Yeo [4] showed that the DPPC, and consequently the LPX Conjecture (which is the case of the DPPC where $\lambda_1 = 1$) holds for special classes of digraphs which are generalizations of tournaments. In [8] Galeana-Sánchez and Gómez also showed that line digraphs as well as certain generalizations of tournaments possess an independent set that intersects every non-augmentable path and hence every longest path.

In 2004 Havet [10] proved the following result.

Theorem 1 [10]. *Every digraph D with independence number at most 2 has an IDT.*

The next result is a direct consequence of Theorem 1.

Corollary 2. *If D is a digraph and $p(D) \leq 2$, then D has an IDT.*

Proof. If $\alpha(D) \leq 2$, then according to Theorem 1, D has an IDT.

Now suppose $\alpha(D) \geq 3$. Let S be an independent set with 3 vertices. It then follows that $|V(D - S)| = n - 3 < \lambda(D)$ and hence S is an IDT of D . ■

It is therefore natural to consider the existence of IDTs in 3-deficient digraphs. Obviously, every p -deficient digraph with independence number greater than p has an IDT. Thus another consequence of Havet's Theorem is the following.

Corollary 3. *If D is a 3-deficient digraph with $\alpha(D) \neq 3$, then D has an IDT.*

In view of Corollary 3, the restriction of the LPX Conjecture to 3-deficient digraphs should be considerably easier to settle than the general conjecture.

In Section 2 we present some elementary but useful results concerning IDTs in digraphs, and in Section 3 we present some sufficient conditions for the existence of IDTs in 3-deficient digraphs. Our main result is that if D is a strong 3-deficient digraph with circumference at least $n(D) - 5$, then D has an IDT. However, proving the LPX Conjecture for 3-deficient digraphs seems more difficult than expected— even for digraphs of small order. Therefore, in Section 4, we restrict our attention to 3-deficient *oriented* graphs. We show that those of order at most 8 have IDTs. We also show that every *strong* 3-deficient oriented graph of order at most 10 has an IDT.

2. GENERAL RESULTS

We begin this section by stating some results on detours that will be used frequently. First, we have an easy observation.

Observation 4. *Suppose D is a digraph and H and F are two subdigraphs of D such that $V(H) \cap V(F) = \emptyset$. Let $v \in V(H)$ and $w \in V(F)$. If $vw \in A(D)$, then $e_H(v) + s_F(w) \leq \lambda(D)$.*

The following two lemmas are direct consequences of this observation.

Lemma 5 (The Lollipop Lemma [5]). *Let C be a cycle in a digraph D and let $P = v_1v_2 \dots v_p$ be a path in $D - V(C)$. If v_1 has an in-neighbour on C or v_p has an out-neighbour on C , then $\lambda(D) \geq n(C) + n(P)$.*

Lemma 6. *Suppose D is a digraph and $w \in V(D)$. If $D - \{w\}$ has a detour $P = v_1 \dots v_i v_{i+1} \dots v_\lambda$ of D , then*

- (i) $w \notin N^-(v_1)$,
- (ii) $w \notin N^+(v_\lambda)$,
- (iii) $w \notin N^+(v_i) \cap N^-(v_{i+1})$.

Corollary 7. *Suppose D is a digraph with a detour P . If $x \notin V(P)$, then $N_P^o(x) \neq \emptyset$.*

It is a well known fact that every semi-complete digraph (i.e. every digraph with independence number 1) is traceable. Chen and Manalastas [7] extended this result as follows.

Theorem 8 [7]. *Every strong digraph with independence number at most 2 is traceable.*

The following extension of Theorem 8 is given in [2].

Theorem 9 [2]. *If D is a connected digraph with at most two strong components and $\alpha(D) \leq 2$, then D is traceable.*

Next we present some elementary but useful sufficient conditions for the existence of IDTs in digraphs. The first result follows from Corollary 7.

Lemma 10. *Suppose I is an independent set in a digraph D . If $N(w) = V(D) - I$ for some $w \in I$, then I is an IDT of D . In particular, if D has a universal vertex x , then $\{x\}$ is an IDT of D .*

Lemma 11. *Let D be a p -deficient digraph and suppose I is an independent set in D with p vertices. If there is an $x \in V(D) - I$ such that $\emptyset \neq N^-(x) \subseteq I$ or $\emptyset \neq N^+(x) \subseteq I$, then I is an IDT of D .*

Proof. Suppose that I is not an IDT of D . Then the subdigraph $D - I$ has a hamiltonian path P . If v is not the initial vertex of P , then v has an in-neighbour on P , so $N^-(v) \not\subseteq I$. If v is the initial vertex of P , then v has no in-neighbour in I , so either $N^-(v) = \emptyset$ or $N^-(v) \not\subseteq I$. A similar argument holds with respect to out-neighbours of vertices on P . ■

For digraphs that are *not strong* we need the following results.

Lemma 12. *For any digraph D the following hold.*

- (i) *Suppose x and y are two vertices in different strong components of D that are both in $L^+(D)$ (or both in $L^-(D)$). Then x and y are nonadjacent.*
- (ii) *Suppose D has a hamiltonian strong component X and P is a detour of D that starts in X . Then $V(X) \subseteq V(P)$.*

Proof. (i) Suppose $x, y \in L^+(D)$ and $xy \in A(D)$. Let P be a detour of D with y as initial vertex. If x and y are in different strong components of D , then x lies above y , so P does not contain x and hence xP is a path of order greater than $\lambda(D)$.

(ii) Let P' be the subpath of P contained in X and let z be the terminal vertex of P' . Since X has a hamiltonian cycle, z is the terminal vertex of a hamiltonian path Q of X . If $V(X) \not\subseteq V(P')$, then a path with more vertices than P is obtained by replacing P' with Q . ■

In [1] it is shown that a digraph D for which $\langle L^+(D) \rangle$ has an independent dominating set ($\langle L^-(D) \rangle$ has an independent absorbant set), has an IDT. Combining this result with that of Richardson [12] mentioned in Section 1 we have the following.

Lemma 13. *Suppose D is a digraph. If $\langle L^+(D) \rangle$ or $\langle L^-(D) \rangle$ contains no odd cycles, then D has an IDT.*

We now prove the following more general result.

Lemma 14. *A digraph D has an IDT if D satisfies one of the following.*

- (i) *If X is a strong component of D that contains a vertex of $L^+(D)$, then X is hamiltonian or $\langle L^+(D) \cap V(X) \rangle$ has an independent dominating set.*
- (ii) *If X is a strong component of D that contains a vertex of $L^-(D)$, then X is hamiltonian or $\langle L^-(D) \cap V(X) \rangle$ has an independent absorbant set.*

Proof. If D satisfies (i), we construct a subset S of $L^+(D)$ as follows.

Consider every strong component X of D that contains vertices of $L^+(D)$. If X is hamiltonian, put exactly one vertex of $V(X) \cap L^+(D)$ into S . If X is nonhamiltonian, then put all the vertices in an independent dominating set of $\langle L^+(D) \cap V(X) \rangle$ into S . By Observation 12(i), S is an independent set.

Let λ be the detour order of D . If S is not an IDT of D , let P be a λ -path in $D - S$ with initial vertex a and let X be the strong component containing a . If X is hamiltonian then Lemma 12(ii) implies that $V(X) \subseteq V(P)$. However, $V(X) \not\subseteq V(D - S)$, since S contains a vertex of X . Thus X is nonhamiltonian and hence S contains an independent dominating set of $\langle L^+(D) \cap V(X) \rangle$. But then a is dominated by a vertex s in S , so sP is a path of order $\lambda + 1$. This contradiction proves that S is an IDT of D .

If D satisfies (ii), we use a symmetric argument to construct an IDT of D that is contained in $L^-(D)$. ■

Corollary 15. *If D is a digraph that has no IDT, then D contains a nonhamiltonian strong component X such that $\langle L^+(D) \cap V(X) \rangle$ has an odd cycle and a nonhamiltonian strong component Y such that $\langle L^-(D) \cap V(Y) \rangle$ has an odd cycle (X and Y may be the same component).*

3. IDTs IN 3-DEFICIENT DIGRAPHS

We now focus our attention on 3-deficient digraphs and present some sufficient conditions for such digraphs to have IDTs.

Lemma 16. *Let D be a 3-deficient digraph and suppose x and y are two nonadjacent vertices of D . Then an independent set I of D is an IDT of D if one of the following holds.*

- (i) $d^+(x) \geq 1$, $d^+(y) \geq 1$ and $N^+(x) \cup N^+(y) \subseteq I$,
- (ii) $d^-(x) \geq 1$, $d^-(y) \geq 1$ and $N^-(x) \cup N^-(y) \subseteq I$,
- (iii) $d^-(x) \geq 1$, $d^+(y) \geq 1$ and $N^-(x) \cup N^+(y) \subseteq I$.

Proof. We prove (i). The proofs of (ii) and (iii) are similar. If $|I| \geq 4$, then I is an IDT of D since $\lambda(D - I) \leq n(D - I) \leq n - 4 < \lambda(D)$. If $|I| = 3$, then

Lemma 11 implies that I is an IDT of D . Now suppose $|I| = 1$, say $I = \{w\}$. If w is a universal vertex of D , then I is an IDT of D by Lemma 10. If w is not a universal vertex of D , let $z \in N^o(w)$. Then $\{w, z\}$ is an independent set and $N^+(x) \cup N^+(y) \subseteq \{w, z\}$. Thus we need only consider the case where $|I| = 2$.

Suppose I is not an IDT of D . Then $D - I$ contains a λ -path P . Since $n(D - I) = \lambda(D) + 1$, at least one of x and y , say x , is in P . But x has no out-neighbours in $D - I$, so x is the terminal vertex of P . However, by our assumption, x has an out-neighbour, say z , in I and hence Pz is a $(\lambda + 1)$ -path in D . ■

Lemma 17. *Suppose D is a strong 3-deficient digraph of order n . Then D has an IDT if $|N^o(w)| \leq 2$ for some vertex $w \in V(D)$.*

Proof. The case $|N^o(w)| = 0$ follows immediately from Lemma 10.

Suppose $|N^o(w)| = 1$. Then there exists a vertex $x \in V(D)$ such that $I = \{x, w\}$ is an independent set. Then, according to Lemma 10, I is an IDT of D .

Suppose $|N^o(w)| = 2$. Then there exist vertices $x, y \in V(D)$ such that $x, y \notin N(w)$. If x and y are nonadjacent, then according to Lemma 10, $I = \{x, y, w\}$ is an IDT of D . Now suppose $xy \in A(D)$ and that D does not have an IDT. Let $U = \langle N^-(w) \rangle$ and $Z = \langle N(w) - N^-(w) \rangle$. We first observe the following. Suppose $P = v_1 v_2 \dots v_\lambda$ is a λ -path in $D - \{x, w\}$ or in $D - \{y, w\}$. If $v_i \in U$, then $v_{i+1} \notin Z$, otherwise $v_1 \dots v_i w v_{i+1} \dots v_\lambda$ is a $(\lambda + 1)$ -path in D . Also $v_1 \notin Z$, otherwise wP is a $(\lambda + 1)$ -path in D and $v_\lambda \notin U$, otherwise Pw is a $(\lambda + 1)$ -path.

Suppose $V(Z) = \emptyset$. Then $D - \{y, w\}$ has a λ -path which ends at x . But then Px is a $(\lambda + 1)$ -path in D . Thus we may assume $V(Z) \neq \emptyset$. Then $D - \{x, w\}$ has a λ -path $P = P_u y P_z$, where P_u and P_z are paths in U and Z , respectively and where P_z can be \emptyset . Also $D - \{y, w\}$ has a λ -path $Q = Q_u x Q_z$, where Q_u and Q_z are paths in U and Z , respectively and $Q_z \neq \emptyset$. If $|V(P_u)| = |V(Q_u)|$, then $Q_u x y P_z$ is a $(\lambda + 1)$ -path in D . On the other hand, if $|V(P_u)| > |V(Q_u)|$, then $P_u w Q_z$ is a $(\lambda + 1)$ -path in D . Similarly if $|V(P_u)| < |V(Q_u)|$, then $Q_u w P_z$ is a $(\lambda + 1)$ -path in D . ■

We now show that the LPX Conjecture holds for strong 3-deficient digraphs with circumference at least $n - 5$.

Theorem 18. *Let D be a strong 3-deficient digraph. If $c(D) \geq n - 5$, then D has an IDT.*

Proof. Assume that D does not have an IDT. By Lemma 5 and since $\lambda(D) = n - 3$ and $\alpha(D) = 3$ it follows that $c(D) \leq n - 5$. Now suppose $c(D) = n - 5$. Let $C = v_1 v_2 \dots v_{n-5} v_1$ be a circumference cycle in D and let $X = D - V(C)$.

Suppose X contains a cycle C_X of order greater than 2. Then since D is strong there is an arc or a path from a vertex on C to a vertex on C_X . But then

by Lemma 5, $\lambda(D) \geq n(C) + n(C_X) \geq n - 5 + 3 = n - 2$. Hence X contains no cycle of order greater than 2.

First we show that $\lambda(X) \leq 2$. Assume $\lambda(X) \geq 3$ and let $P = x_1x_2 \dots x_\lambda$ be a detour of X . Since D is strong there is an $x_i - x_{i-1}$ path in D for $i \in \{2, \dots, \lambda\}$. But since $n(C_X) \leq 2$ for any cycle C_X in X and $s_X(x_i) \geq 3$, $i = 1, \dots, \lambda - 2$ and $e_X(x_i) \geq 3$, $i = 3, \dots, \lambda$, it follows from Lemma 5 that $x_ix_{i-1} \in A(D)$ for $i = 2, \dots, \lambda$. Then clearly, since there are no cycles of order greater than 2 in X and $\lambda(X) \geq 3$, x_1 and x_λ are nonadjacent. We now choose a vertex $y \in V(X)$ such that $y = x_3$ if $\lambda(X) = 5$, and $y \in V(X) - V(P)$ if $\lambda(X) < 5$ and let $Y = \{x_1, x_\lambda, y\}$. Then Y is an independent set and since no vertex on C is a neighbour of x_1 or x_λ and $\alpha(D) = 3$, it follows that $V(C) \subseteq N(y)$. But, since C is a longest cycle in D , no successor of an in-neighbour of y is an out-neighbour of y ; hence $V(C)$ is either contained in $N^-(y)$ or in $N^+(y)$. By symmetry, we may assume that $V(C) \subseteq N^-(y)$. But then it follows from Lemma 5 and the fact that $s_X(x_3) \geq 3$ that $y \neq x_3$ (i.e. $\lambda(X) < 5$ and $y \in V(X) - V(P)$). Then $s_X(x_i) \geq 2$ for every $x_i \in V(P)$. Also, since D is strong there is a $y - x_i$ path in D . But again from Lemma 5 we have a contradiction.

Thus $\lambda(X) \leq 2$. But since $\alpha(D) = 3$, $\lambda(X) \geq 2$ and therefore $\lambda(X) = 2$. It is also easy to see that since $\alpha(D) = 3$ and $\lambda(X) = 2$, there is at most one component of X of order 1.

First suppose every component of X has order at least two. Then we may assume w.l.o.g. that $V(X) = \{u_1, u_2, u_3, w_1, w_2\}$ and $\{u_1w_1, u_3w_1, u_2w_2\} \subseteq A(X)$. Let $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2\}$. Then $U \subseteq L^+(X)$ and $W \subseteq L^-(X)$, and both U and W are independent sets. Moreover, since $\lambda(X) = 2$, u_1 has no in-neighbours in X and hence, since D is strong, there exists a $v_i \in V(C)$ such that $v_i \in N^-(u_1)$. But then $v_{i+1} \in L^+(D)$. Since U is an independent set of cardinality 3, v_{i+1} has a neighbour in U . If v_{i+1} has an in-neighbour in U , then either we get an $(n - 4)$ -cycle (if $u_1v_{i+1} \in A(D)$), or if say $u_2v_{i+1} \in A(D)$, then $u_2v_{i+1}v_{i+2} \dots v_iu_1w_1$ is an $(n - 2)$ -path. Hence, $v_{i+1} \in N^-(U)$ and again $v_{i+2} \in L^+(D)$. Continuing this argument it follows that $V(C) \subseteq N^-(U)$. But then no vertex of $V(C)$ is in $N^+(W)$ otherwise we either get a cycle of order greater than $c(D)$ or a path of order greater than $\lambda(D)$. Hence $N^+(w_1) = \emptyset$ and therefore D is not strong.

Hence exactly one component of X has order 1. Let x be the vertex of this component. Then $X - x$ contains two vertex disjoint paths u_1w_1 and u_2w_2 . Let $U = \{u_1, u_2\}$ and $W = \{w_1, w_2\}$. Since $\lambda(X) = 2$ it follows that $U \cup \{x\}$ and $W \cup \{x\}$ are independent sets. We now prove the following two claims.

Claim 1. *Each of u_1 and u_2 has an in-neighbour on C and each of w_1 and w_2 has an out-neighbour on C .*

Proof. Suppose $N^-(u_i) \cap V(C) = \emptyset$ for some $u_i \in U$. Then $N^-(u_i) \subset (W \cup \{x\})$

and by Lemma 11, D has an IDT. Similarly, if $N^+(w_i) \cap V(C) = \emptyset$ for some $w_i \in W$, then $N^+(w_i) \subset (U \cup \{x\})$ and again by Lemma 11 we get a contradiction. This proves Claim 1. \square

Claim 2. *There exist distinct vertices v_k and v_ℓ on C such that $v_k \in N^-(U)$ and $v_\ell \in N^+(W)$.*

Proof. Now suppose $|N_{V(C)}^-(U) \cup N_{V(C)}^+(W)| = 1$ and suppose $v_k \in V(C)$ such that $\{v_k\} = N_{V(C)}^-(U) \cup N_{V(C)}^+(W)$. By Lemma 10 there exists a vertex $y \in N^o(v_k)$. Now let $I = \{v_k, y\}$ and let P be an $(n-3)$ -path in $D - I$. Then P contains at least one vertex in U . Suppose $u_i \in V(P)$ for some $i \in \{1, 2\}$. Then u_i is not an initial vertex of P otherwise $v_k P$ is $(\lambda+1)$ -path in D . Since v_k is the only in-neighbour of u_i on $V(C)$, and w_i the only possible in-neighbour of u_i in X , $w_i u_i$ is a subpath of P . But then replacing this subpath of P with $w_i v_k u_i$ we get a $(\lambda+1)$ -path in D . This proves Claim 2. \square

Now, by relabeling the vertices of C if necessary, we may choose k and ℓ such that $1 \leq k < \ell \leq n-5$ and $v_k \in N^-(U)$, $v_\ell \in N^+(W)$ and $v_i \notin N^-(U)$ and $v_i \notin N^+(W)$, for $i = k+1, \dots, \ell-1$. We may assume w.l.o.g. that $v_k \in N^-(u_1)$. Then the path $v_{k+1} \dots v_{n-5} v_1 \dots v_k u_1 w_1$ has order $n-3$. Hence, if u_2, w_2 or x is an in-neighbour of v_{k+1} , then D has an $(n-2)$ -path or an $(n-1)$ -path, and if u_1 or w_1 is an in-neighbour of v_{k+1} , then D has an $(n-4)$ -cycle or an $(n-3)$ -cycle. These contradictions show that v_{k+1} has no in-neighbours in X . This implies that v_{k+1} has an out-neighbour in the independent set $U \cup \{x\}$. But, by our choice of ℓ and k , neither u_1 nor u_2 is an out-neighbour of v_{k+1} , so $x \in N^+(v_{k+1})$. A similar argument shows that $x \in N^-(v_{\ell-1})$. But no successor of an in-neighbour of x on C is an out-neighbour of x (otherwise D has a cycle of order $n-4$), so $\ell \geq k+4$ and there exist $r, s \in \{k+1, \dots, \ell-1\}$ with $s \geq r+2$ such that $v_r \in N^-(x)$, $v_s \in N^+(x)$ and $v_i \in N^o(x)$ for $i = r+1, \dots, s-1$. But v_{r+1} has a neighbour in the independent set $U \cup \{x\}$, so v_{r+1} is an out-neighbour of u_i , with $i = 1$ or 2 . Similarly, v_{s-1} is an in-neighbour of w_j , with $j = 1$ or 2 . Now $\widehat{C} = x v_s v_{s+1} \dots v_{r-1} v_r x$ is a cycle of order $(n-5) - (s-r-1) + 1 = n - (s-r+3)$ and $u_i v_{r+1} \dots v_{s-1} w_j$ is a path of order $(s-1) - (r+1) + 1 + 2 = s-r+1$. But, by Claim 1 and the choice of k and ℓ , u_i has an in-neighbour on \widehat{C} and hence, by Lemma 5, $\lambda(D) \geq n - (s-r+3) + (s-r+1) = n-2$, contradicting the fact that $\lambda(D) = n-3$. \blacksquare

As a corollary of Theorem 18, we have the following sufficient condition.

Corollary 19. *Suppose D is a strong 3-deficient digraph. If $x \in V(D)$ such that $d(x) \leq 3$ and x does not lie in an independent set of cardinality 3, then D has an IDT.*

Proof. By our assumption on x , the subdigraph $\langle N^o(x) \rangle$ has order at least $n - 4$ and contains a spanning tournament. If $\langle N^o(x) \rangle$ is strong, then it is hamiltonian and hence $c(D) \geq n - 4$. But then according to Theorem 18, D has an IDT.

Now suppose $\langle N^o(x) \rangle$ is not strong. Then, since D is strong, some vertex a in the first component of $\langle N^o(x) \rangle$ has an in-neighbour u in $N(x)$ and some vertex z in its last component has an out-neighbour w in $N(x)$. Since $\langle N^o(x) \rangle$ contains a spanning tournament, it is traceable and each of its non-trivial strong components is hamiltonian, so it has a hamiltonian path P with a as initial vertex and z as terminal vertex. Hence $u = w$; otherwise uPw is an $(n - 2)$ -path. But then $c(D) \geq n - 3$, so again D has an IDT. ■

4. IDTs IN 3-DEFICIENT ORIENTED GRAPHS OF SMALL ORDER

We require the following two lemmas in order to prove that a strong 3-deficient oriented graph of order at most 10 has an IDT.

Lemma 20. *Let D be a strong 3-deficient oriented graph which does not have an IDT. Let $U = \{u_1, u_2, u_3\}$ be an independent set in D and $P = v_1v_2 \dots v_\lambda$ be an $(n - 3)$ -path in $D - U$. If $v_t \in N^-(v_1)$ for some $t \in \{3, \dots, \lambda - 3\}$, then $v_{t+1} \notin N^+(v_\lambda)$.*

Proof. Suppose $v_t \in N^-(v_1)$ and $v_{t+1} \in N^+(v_\lambda)$. Let C^1 be the cycle $v_1v_2 \dots v_tv_1$ and C^2 be the cycle $v_{t+1}v_{t+2} \dots v_\lambda v_{t+1}$. If a vertex u in U has an in-neighbour in one of these two cycles, then u cannot have an out-neighbour in the other one; otherwise D would have a $(\lambda + 1)$ -path, by Lemma 5. Hence $N(u)$ is contained in exactly one of the sets $V(C^1)$ and $V(C^2)$. But, since $\alpha(D) = 3$, every vertex on P is adjacent to a vertex in U . Thus we may assume w.l.o.g. that $N(u_1) \cup N(u_2) = V(C^1)$ and $N(u_3) = V(C^2)$. Then, by Lemma 6(ii), $v_\lambda \in N^+(u_3)$. But then it follows from Lemma 6 that u_3 has no in-neighbour, which contradicts that D is strong. ■

Lemma 21. *Suppose D is a strong 3-deficient oriented graph of order n . Then D has an IDT if one of the following holds:*

- (i) $\Delta(D) \geq n - 3$.
- (ii) D has an independent set $\{x, w, z\}$ such that $d(w) = n - 4$ and $d^-(w) \leq 3$ or $d^+(w) \leq 3$.

Proof. (i) This follows from Lemma 17.

(ii) Suppose $\{x, w, z\}$ is an independent set in D such that $d(w) = n - 4$ and $d^-(w) \leq 3$ or $d^+(w) \leq 3$. Assume that D does not have an IDT. Then $D - \{x, w, z\}$ has a hamiltonian path $P = v_1v_2 \dots v_\lambda$. Since $d(w) = n - 4$,

there is exactly one vertex v_t on P that is not in $N(w)$ and, by Lemma 6(iii), $\{v_1, \dots, v_{t-1}\} = N^-(w)$ and $\{v_{t+1}, \dots, v_\lambda\} = N^+(w)$.

Now $v_i \in N^-(v_1)$ for some $i \in \{3, \dots, \lambda - 3\}$. If $i \geq t$, then the path $v_2v_3 \dots v_iv_1vv_{i+1} \dots v_\lambda$ has order $\lambda + 1$. Thus $i \leq t - 1$, i.e. v_1 has an in-neighbour in $\{v_3, \dots, v_{t-1}\}$. Similarly, v_λ has an out-neighbour in $\{v_{t+1}, \dots, v_{\lambda-2}\}$. Hence both $d^-(w)$ and $d^+(w)$ are at least 3 and one of them equals 3.

Now suppose $d^-(w) = 3$. Then $t = 4$ and $v_3 \in N^-(v_1)$. Then each of the vertices v_1, v_2, v_3 is an initial vertex of a λ -path in $D - \{v_4, x, z\}$. Thus $\{v_1, v_2, v_3, w\}$ has no in-neighbours in $\{v_4, x, z\}$.

Also, if $i \geq 4$, then $v_i \notin N^-(\{v_1, v_2, v_3, w\})$, otherwise the path $v_4 \dots v_iQwv_{i+1} \dots v_\lambda$, where Q is a 3-path in the cycle $v_1v_2v_3v_1$, has order $\lambda + 1$. Consequently $\{v_1, v_2, v_3, w\}$ has no in-neighbours in $D - \{v_1, v_2, v_3, w\}$. Hence D is not strong, a contradiction.

Similarly, $d^+(w) = 3$ leads to a contradiction.

Thus D has an IDT. ■

Theorem 22. *Suppose D is a strong 3-deficient oriented graph of order $n \leq 10$. Then D has an IDT.*

Proof. Assume D does not have an IDT. Let $U = \{u_1, u_2, u_3\} \subset V(D)$ be an independent set. Then $D - U$ contains an $(n - 3)$ -path and $P = v_1v_2 \dots v_{n-3}$. Since D is strong it follows from Lemma 6 that $N^-(v_1) \cap V(P) \neq \emptyset$ and $N^+(v_\lambda) \cap V(P) \neq \emptyset$. It follows easily from Theorem 18 that we need only consider $n \geq 9$.

For $n = 9$ and $\lambda(D) = 6$, it follows from Theorem 18 and Lemma 20 that $N^-(v_1) \cap V(P) = \emptyset$ and $N^+(v_6) \cap V(P) = \emptyset$.

We now consider $n = 10$ and $\lambda(D) = 7$, and note, from Lemma 21, that $\Delta(D) \leq 6$. It follows from symmetry, Theorem 18 and Lemma 20 that we need only consider the cases where $v_3 \in N^-(v_1)$ and $v_5 \in N^+(v_7)$; and where $v_4 \in N^-(v_1) \cap N^+(v_7)$.

Case 1. $v_3 \in N^-(v_1)$ and $v_5 \in N^+(v_7)$. We first prove that $N_U^+(v_1) \cap N_U^-(v_7) = \emptyset$. Suppose $u_1 \in N_U^+(v_1) \cap N_U^-(v_7)$. Then u_1 and v_4 are nonadjacent, since if $u_1 \in N^-(v_4)$, then $v_2v_3v_1u_1v_4v_5v_6v_7$ is an 8-path and if $u_1 \in N^+(v_4)$, then $v_1v_2v_3v_4u_1v_7v_5v_6$ is an 8-path. W.l.o.g. we assume u_2 is adjacent with v_4 . Suppose $v_4 \in N^+(u_2)$. By Lemma 6, $v_3, v_7 \notin N^-(u_2)$. Also $v_1, v_2 \notin N^-(u_2)$; otherwise $v_2v_3v_1u_2v_4v_5v_6v_7$ or $v_3v_1v_2u_2v_4v_5v_6v_7$ would be an 8-path in D . Also, $v_5, v_6 \notin N^-(u_2)$; otherwise D would contain $v_2v_3v_1u_1v_7v_5(v_6)u_2v_4$, which has order greater than 7. Hence $N^-(u_2) = \emptyset$ which contradicts the fact that D is strong. The case $v_4 \in N^-(u_2)$ is similar.

Assume that $u_1 \in N^+(v_1)$ and $u_3 \in N^-(v_7)$. We first show that v_4 is nonadjacent with both u_1 and u_3 . Suppose $v_4 \in N^-(u_1)$ (by symmetry we need only consider this case). Let $H = \langle \{v_1, \dots, v_4, u_1\} \rangle$ and $F = \langle \{v_5, v_6, v_7, u_3\} \rangle$. Since $e_H(u_1) = 5$ and $s_F(v_i) \geq 3$ for $v_i \in V(F)$, it follows from Observation 4

and Lemma 6 that $N^+(u_1) = \{v_3\}$. Since $s_F(u_3) = 4$ and $e_H(v_i) < 4$ only if $i = 3$, v_3 is the only possible in-neighbour of u_3 in H . Hence by Lemma 16(iii), $v_5 \in N^-(u_3)$. Now v_2 and u_1 are nonadjacent and v_6 and u_3 are nonadjacent. Also since $s_H(v_2) \geq 4$, $e_H(v_2) = 5$ and $s_F(u_3) = e_F(u_3) = 4$, v_2 and u_3 are nonadjacent. Also $s_H(u_1) \geq 4$, $e_H(u_1) = 5$ and $s_F(v_6) = e_F(v_6) = 4$, and thus u_1 and v_6 are nonadjacent. This implies that v_2 and v_6 are both adjacent with u_2 and that they are either both in- or out- neighbours of u_2 . Either case leads to a contradiction.

Hence v_4 is nonadjacent with u_1 and u_3 and therefore $v_4 \in N(u_2)$. First suppose $v_4 \in N^-(u_2)$. Let $H = \langle \{v_1, \dots, v_4, u_1, u_2\} \rangle$ and $F = \langle \{v_5, v_6, v_7, u_3\} \rangle$. We have $N^+(u_2) \subseteq \{v_2, v_3\}$ and therefore $e_H(u_1) \geq 5$. Hence $N^+(u_1) = \{v_3\}$ and by Lemma 16(i), $v_2 \in N^+(u_2)$. Now $s_H(u_1) \geq 5$, $e_H(u_1) = 6$, $s_H(u_2) \geq 5$ and $e_H(u_2) \geq 5$, and hence by Observation 4, u_1 and u_2 are nonadjacent with both v_5 and v_6 . Hence $\{u_1, u_2, v_5\}$ is independent and thus according to Lemma 11, v_7 needs an out-neighbour in $D - \{u_1, u_2, v_5\}$. But according to Theorem 18 and Lemma 20 this is impossible. The case $v_4 \in N^+(u_2)$ is similar.

Case 2. $v_4 \in N^-(v_1) \cap N^+(v_7)$. Suppose v_4 is nonadjacent with two of u_1, u_2 and u_3 , say u_1 and u_2 . Then $\{u_1, u_2, v_4\}$ is independent and thus according to Lemma 11, v_1 needs an in-neighbour and v_7 needs an out-neighbour in $D - \{u_1, u_2, v_4\}$. But then, according to Theorem 18 and Lemmas 6 and 20, the only possibility is that $v_3 \in N^-(v_1)$ and $v_5 \in N^+(v_7)$, so we have Case 1.

We may thus assume $u_1, u_2 \in N(v_4)$. Then $d(v_4) = 6$. Hence, according to Lemma 21, $\{v_2, v_4, v_6\}$ cannot be independent and consequently v_2 and v_6 are adjacent. From Theorem 18 it follows that $v_2 \in N^-(v_6)$. But then $v_2v_6v_7v_4v_1v_2$ is a 5-cycle which contradicts Theorem 18. ■

If we now relax the condition that a 3-deficient oriented graph need be strong then we have the following result.

Theorem 23. *If D is a 3-deficient oriented graph of order at most 8, then D has an IDT.*

Proof. If D is disconnected, D obviously has an IDT, since any component X of D either contains no detour of D or X is at most 2-deficient. If D is strong, then D has an IDT by Theorem 22. Thus we assume that D is connected but D has more than one strong component.

Suppose D has no IDT. Then, according to Corollary 15, D has nonhamiltonian strong components X and Y such that $\langle L^+(D) \cap V(X) \rangle$ contains an odd cycle A and $\langle L^-(D) \cap V(Y) \rangle$ contains an odd cycle Z . Since D is connected and $\lambda(D) \leq 5$, both A and Z have fewer than 5 vertices, so they are both 3-cycles. Hence, by Lemma 5, $n = 7$ or 8 and A and Z are not vertex disjoint. Thus A and Z are contained in the same strong component X of D . Let $A = a_1a_2a_3a_1$ and

$Z = z_1 z_2 z_3 z_1$. Then, by Lemma 2.3 (i) and (ii), no vertex above X is adjacent to any vertex in A and no vertex in Z is adjacent to any vertex below X and therefore $|X| \geq 4$. W.l.o.g. we may also assume that D has a vertex, say v , that lies above X in D and that v has an out-neighbour in $X - V(A)$ (since a symmetric argument holds when v lies below X and v has an in-neighbour in $X - V(Z)$). We note the following obvious, but useful observations, the first of which is due to Lemma 5.

Observation 1. Since X is strong, there is a path from every vertex in $X - V(A)$ to a vertex in A . Hence, by Lemma 5, every path in $D - V(A)$ that ends in X has order at most $\lambda(D) - 3$. Similarly every path in $D - V(Z)$ that starts in X has order at most $\lambda(D) - 3$ and consequently no detour in D contains two consecutive vertices in $D - V(X)$.

Observation 1 immediately implies that $\lambda(D) = 5$ and also the following.

Observation 2. $N_X^+(v)$ is independent and no 5-path in $D - N_X^+(v)$ contains v . Hence $N_X^+(v)$ is an independent set of order at most 2 in X .

Case 1. $n(X) = 4$. Let $V(X) - V(A) = \{x\}$. Since X is strong but not hamiltonian, we may assume w.l.o.g. that $a_1 \in N^o(x)$, $a_2 \in N^+(x)$ and $a_3 \in N^-(x)$. Now x is the only vertex in X that can be an out-neighbour of any vertex above X , and hence a_1 is the only vertex in X that can be the terminal vertex of a 5-path of D , contradicting that $Z \subseteq \langle V(X) \cap L^-(D) \rangle$.

Case 2. $n(X) = 5$. Let $V(X) - V(A) = \{x, y\}$ and let $x \in N_X^+(v)$. Since $\lambda(D) = 5$, the digraph $\langle V(X) \cup \{v\} \rangle$ is nontraceable and hence Theorem 9 implies that $\langle V(X) \cup \{v\} \rangle$ has an independent set I with $|I| = 3$. By Observation 2, $x \notin I$ and we may therefore w.l.o.g. assume that $I = \{v, y, a_1\}$. Now let P be a 5-path in $D - I$. Let Q be the subpath of P that intersects X . Then by Observation 1, $P = uQw$ where u and w are vertices that lie above and below X respectively. But then u and v are nonadjacent and $ux \in A(D)$. By Observation 1, $N_X^+(u) \cup N_X^+(v)$ is contained in an independent set and again by Observation 1 and Lemma 16 (i), we get a contradiction.

Case 3. $n(X) = 6$. Since $\lambda(X) \leq 5$, X is nontraceable. Thus, it follows from Theorem 8 that $\alpha(X) = 3$. Let U be an independent set of cardinality 3 in X . Since $\alpha(D) = 3$ we may assume v is adjacent to some vertex $u \in U$ and therefore it follows from Observation 2 that u is not the only out-neighbour of v in X and therefore $d_X^+(v) = 2$. Now let $V(X) = \{a_1, a_2, a_3, u, w, x\}$ and let $N_X^+(v) = \{u, w\}$. We also note by Lemma 5 that the graph induced by $V(A)$ together with any other vertex in $X - V(A)$ cannot contain a 4-cycle.

Since u and w do not lie in an independent set of cardinality 3 in X , we may assume w.l.o.g. that x and w are adjacent and that $U = \{u, x, a_3\}$, and therefore a_3 and w are adjacent. Hence, since D is strong, $N_X(u) = \{a_1, a_2\}$ and since

$\langle V(A) \cup \{u\} \rangle$ is not a 4-cycle, $a_2u, ua_1 \in A(D)$. Then $wa_3 \notin A(D)$ otherwise $vwa_3a_1a_2u$ is a 6-path in D . But if $a_3w \in A(D)$, then $vua_1a_2a_3w$ is a 6-path in D .

Case 4. $n(X) = 7$. By a similar argument as that in the previous case, we may assume $N^+(v) = \{u, w\}$ where u and w are nonadjacent. Now let $P = v_1v_2v_3v_4v_5$ be a 5-path in $D - \{u, w\}$. Then v is not on P and by Lemma 5, $\langle V(P) \rangle$ does not contain a cycle of order greater than 3. Hence, since X is strong, $v_3v_1, v_5v_3 \in A(D)$. Also, by Observation 2, every vertex on P is adjacent with at least one vertex in $\{u, w\}$. We may therefore w.l.o.g. assume that $v_1u \in A(D)$. Now since $e_{\langle P \rangle}(v_2) = 5$, $v_2 \notin N^-(u) \cup N^-(w)$ and since $s_{\langle P \rangle}(v_2) \geq 4$, $v_2 \notin N^+(u) \cup N^+(w)$. ■

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