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Dedicated to Mieczysław Borowiecki on the occasion of his 70th birthday

FRACTIONAL $(\mathcal{P}, \mathcal{Q})$-TOTAL LIST COLORINGS OF GRAPHS

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Abstract

Let $r, s \in \mathbb{N}$, $r \geq s$, and $\mathcal{P}$ and $\mathcal{Q}$ be two additive and hereditary graph properties. A $(\mathcal{P}, \mathcal{Q})$-total (r, s)-coloring of a graph $G = (V, E)$ is a coloring of the vertices and edges of G by s-element subsets of $\mathbb{Z}_r$ such that for each color i, $0 \leq i \leq r-1$, the vertices colored by subsets containing i induce a subgraph of G with property $\mathcal{P}$, the edges colored by subsets containing i induce a subgraph of G with property $\mathcal{Q}$, and color sets of incident vertices and edges are disjoint. The fractional $(\mathcal{P}, \mathcal{Q})$-total chromatic number $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ of G is defined as the infimum of all ratios r/s such that G has a $(\mathcal{P}, \mathcal{Q})$-total (r, s)-coloring.

A $(\mathcal{P},\mathcal{Q})$-total independent set $T = V_T \cup E_T \subseteq V \cup E$ is the union of a set V_T of vertices and a set E_T of edges of G such that for the graphs induced by the sets V_T and E_T it holds that $G[V_T] \in \mathcal{P}$, $G[E_T] \in \mathcal{Q}$, and $G[V_T]$ and $G[E_T]$ are disjoint. Let $\mathcal{T}_{\mathcal{P},\mathcal{Q}}$ be the set of all $(\mathcal{P},\mathcal{Q})$-total independent sets of G.

Let $L(x)$ be a set of admissible colors for every element $x \in V \cup E$. The graph G is called $(\mathcal{P},\mathcal{Q})$-total (a,b)-list colorable if for each list assignment L with $|L(x)| = a$ for all $x \in V \cup E$ it is possible to choose a subset $C(x) \subseteq L(x)$ with $|C(x)| = b$ for all $x \in V \cup E$ such that the set T_i which is defined by $T_i = \{x \in V \cup E : \ i \in C(x)\}$ belongs to $\mathcal{T}_{\mathcal{P},\mathcal{Q}}$ for every color i. The $(\mathcal{P},\mathcal{Q})$-choice ratio $\mathrm{chr}_{\mathcal{P},\mathcal{Q}}(G)$ of G is defined as the infimum of all ratios a/b such that G is $(\mathcal{P},\mathcal{Q})$-total (a,b)-list colorable.

We give a direct proof of $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \mathrm{chr}_{\mathcal{P},\mathcal{Q}}(G)$ for all simple graphs G and we present for some properties $\mathcal{P}$ and $\mathcal{Q}$ new bounds for the $(\mathcal{P},\mathcal{Q})$-total chromatic number and for the $(\mathcal{P},\mathcal{Q})$-choice ratio of a graph G.

Keywords: graph property, total coloring, $(\mathcal{P},\mathcal{Q})$-total coloring, fractional coloring, fractional $(\mathcal{P},\mathcal{Q})$-total chromatic number, circular coloring, circular $(\mathcal{P},\mathcal{Q})$-total chromatic number, list coloring, $(\mathcal{P},\mathcal{Q})$-total (a,b)-list colorings.

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1. Introduction

We denote the class of all finite simple graphs by $\mathcal{I}$ (see [6]). A *graph property* $\mathcal{P}$ is any non-empty isomorphism-closed subclass of $\mathcal{I}$. A property $\mathcal{P}$ of graphs is called *hereditary* if it is closed under taking subgraphs, i. e., $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$. A property $\mathcal{P}$ is called *additive* if it is closed under disjoint union of graphs, i. e., $G \in \mathcal{P}$ and $H \in \mathcal{P}$ implies $G \cup H \in \mathcal{P}$.

Some well-known hereditary and additive graph properties are (see [8]):

$$\begin{aligned}
\mathcal{O} &= \{G \in \mathcal{I} : E(G) = \emptyset\},\\
\mathcal{O}_k &= \{G \in \mathcal{I} : \text{each component of } G \text{ has at most } k+1 \text{ vertices}\},\\
\mathcal{S}_k &= \{G \in \mathcal{I} : \Delta(G) \le k\},\\
\mathcal{D}_k &= \{G \in \mathcal{I} : \delta(H) \le k \text{ for each } H \subseteq G\},\\
\mathcal{I}_k &= \{G \in \mathcal{I} : G \text{ contains no } K_{k+2}\},
\end{aligned}$$

where $\Delta(G)$ is the maximum degree and $\delta(G)$ the minimum degree of a graph $G = (V(G), E(G))$.

A *total coloring* of a graph G is a coloring of the vertices and edges (together called the *elements* of G) such that all pairs of adjacent or incident elements obtain distinct colors. The minimum number of colors of a total coloring of G is called the *total chromatic number* $\chi''(G)$ of G.

Since a vertex of degree $\Delta(G)$ and the incident edges must have pairwise different colors, we have $\chi''(G) \geq \Delta(G)+1$. The total coloring conjecture says that $\chi''(G) \leq \Delta(G)+2$ for every graph G [4, 21]. Therefore, the truth of the total coloring conjecture would imply that $\chi''(G)$ attains one of two possible values for every graph G.

So far, the total coloring conjecture is proved for some classes of graphs, e. g., for complete graphs, for bipartite graphs, for complete multipartite graphs [22], for graphs G with $\Delta(G) \geq 3|V(G)|/4$ [9] or $\Delta(G) \leq 5$ [13], and for planar graphs G with $\Delta(G) \neq 6$ [5, 10, 17]. It was proved by Molloy and Reed [15, 16] that there exists a constant c, $2 \leq c \leq 500$, such that $\chi''(G) \leq \Delta(G)+c$ for any graph G.

A *total* (r,s)*-coloring* of G is an assignment of s-element subsets of $\mathbb{Z}_r$ to the vertices and edges of G such that every two adjacent or incident elements of $V(G) \cup E(G)$ are colored with disjoint subsets. The *fractional total chromatic number* of G, denoted by $\chi''_f(G)$, is defined as

$$\chi''_f(G) = \inf\left\{\frac{r}{s} : G \text{ has a total } (r,s)\text{-coloring}\right\}.$$

The fractional version of the total coloring conjecture was proved by Kilakos and Reed ([12]; see also [18], pp. 87–95): $\chi''_f(G) \leq \Delta(G)+2$ for any graph G.

A total coloring requires that for each color i the set of all vertices colored by i is an independent vertex set, i.e., that the subgraph induced by vertices with color i has property $\mathcal{O}$, the set of all edges colored by i is an independent edge set, i.e., that the subgraph induced by edges with color i has property $\mathcal{O}_1$, and incident vertices and edges are colored differently. By using the class of hereditary properties there is a natural generalization of total colorings. We obtain a $(\mathcal{P},\mathcal{Q})$*-total coloring* by replacing the property $\mathcal{O}$ in the definition of a total coloring by any other hereditary graph property $\mathcal{P}$ and the property $\mathcal{O}_1$ by any other hereditary graph property $\mathcal{Q} \supseteq \mathcal{O}_1$. A $(\mathcal{P},\mathcal{Q})$*-total* k*-coloring* is a $(\mathcal{P},\mathcal{Q})$-total coloring with k colors and the $(\mathcal{P},\mathcal{Q})$*-total chromatic number* of G, denoted by $\chi''_{\mathcal{P},\mathcal{Q}}(G)$, is the minimum number k of colors of a $(\mathcal{P},\mathcal{Q})$-total k-coloring of G.

Let $r,s \in \mathbb{N}$, $r \geq s$, and $\mathcal{P} \supseteq \mathcal{O}$ and $\mathcal{Q} \supseteq \mathcal{O}_1$ be two additive and hereditary graph properties. A $(\mathcal{P},\mathcal{Q})$*-total* (r,s)*-coloring* of a graph G is a coloring of the vertices and edges of G by s-element subsets of $\mathbb{Z}_r$ such that for each color i, $0 \leq i \leq r-1$, the vertices colored by subsets containing i induce a subgraph of G with property $\mathcal{P}$, the edges colored by subsets containing i induce a subgraph of G with property $\mathcal{Q}$, and color sets of incident vertices and edges are disjoint.

The *fractional* $(\mathcal{P},\mathcal{Q})$*-total chromatic number* $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ of G is defined by

$$\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \inf\left\{\frac{r}{s} : G \text{ has a } (\mathcal{P},\mathcal{Q})\text{-total } (r,s)\text{-coloring}\right\}.$$

$(\mathcal{P},\mathcal{Q})$-total colorings were introduced in [7] and $(\mathcal{P},\mathcal{Q})$-total (r,s)-colorings in [11] where first results can be found. For example, it was shown in [11] that the following definition of the fractional $(\mathcal{P},\mathcal{Q})$-total chromatic number is equivalent to the one given above.

Let $G=(V,E)$ be a simple graph. A $(\mathcal{P},\mathcal{Q})$-*total independent set* $T = V_T \cup E_T \subseteq V \cup E$ is the union of a set V_T of vertices and a set E_T of edges of G such that for the graphs induced by the sets V_T and E_T it holds that $G[V_T] \in \mathcal{P}$, $G[E_T] \in \mathcal{Q}$, and $G[V_T]$ and $G[E_T]$ are disjoint.

Let $\mathcal{T}_{\mathcal{P},\mathcal{Q}}$ be the set of all $(\mathcal{P},\mathcal{Q})$-total independent sets of G.

A *fractional* $(\mathcal{P},\mathcal{Q})$-*total coloring* of G is a mapping $\varphi : \mathcal{T}_{\mathcal{P},\mathcal{Q}} \to [0,1]$ such that

$$\sum\nolimits_{T\in\mathcal{T}_{\mathcal{P},\mathcal{Q}};\, x\in T} \varphi(T) \geq 1 \qquad \text{for all } x \in V \cup E \tag{1}$$

where $[0,1]$ is the closed real interval from 0 to 1.

The *fractional* $(\mathcal{P},\mathcal{Q})$-*total chromatic number* $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ of G is the solution of the linear program (1) with objective function

$$\sum\nolimits_{T\in\mathcal{T}_{\mathcal{P},\mathcal{Q}}} \varphi(T) \to \min .$$

In the standard definition of total colorings any color of the given color set may be assigned to any element. On the other hand, in choosability problems the availability of colors is restricted for each element. This condition is usually given by a *list* $L(x)$ of admissible colors for every element $x \in V \cup E$.

Let $G=(V,E)$ be a simple graph and let $L(x)$ be a set of admissible colors for every element $x \in V \cup E$. The graph G is called $(\mathcal{P},\mathcal{Q})$-*total* (a,b)-*list colorable* if for each list assignment L with $|L(x)| = a$ for all $x \in V \cup E$ it is possible to choose a subset $C(x) \subseteq L(x)$ with $|C(x)| = b$ for all $x \in V \cup E$ such that the set T_i which is defined by $T_i = \{x \in V \cup E :\ i \in C(x)\}$ belongs to $\mathcal{T}_{\mathcal{P},\mathcal{Q}}$ for every color i.

The $(\mathcal{P},\mathcal{Q})$-*choice ratio* $\mathrm{chr}_{\mathcal{P},\mathcal{Q}}(G)$ of G is defined by

$$\mathrm{chr}_{\mathcal{P},\mathcal{Q}}(G) = \inf\left\{\frac{a}{b} : G \text{ is } (\mathcal{P},\mathcal{Q})\text{-total } (a,b)\text{-list colorable}\right\}.$$

We will show that the $(\mathcal{P},\mathcal{Q})$-choice ratio of G coincides with the fractional $(\mathcal{P},\mathcal{Q})$-total chromatic number of G. This is related to results in [2] and [14] where analogous statements for fractional vertex list colorings and corresponding concepts in hypergraphs are proved.

Let us have a closer look to the results on hypergraphs. In [14] the authors consider hypergraphs $\mathcal{H} = (X,\mathcal{F})$ with finite vertex set X and hyperedge set $\mathcal{F}$. A hypergraph $\mathcal{H} = (X,\mathcal{F})$ is called (a,b)-*list colorable* if, for every list assignment L with $|L(x)| = a$ for all $x \in X$, it is possible to choose subsets $C(x) \subseteq L(x)$ with

$|C(x)| = b$ such that $\cap_{x\in F} C(x) = \emptyset$ holds for all $F \in \mathcal{F}$. *The choice ratio* of a hypergraph is defined by

$$\mathrm{chr}(\mathcal{H}) := \inf\left\{\tfrac{a}{b} : \mathcal{H} \text{ is } (a,b)\text{-list colorable}\right\}.$$

For a graph $G = (V, E)$ and a pair $\mathcal{P}, \mathcal{Q}$ of additive and hereditary properties construct a hypergraph $\mathcal{H} = (X, \mathcal{F})$ in the following way. Set $X = V \cup E$ and $F \in \mathcal{F}$ if and only if $F \notin \mathcal{T}_{\mathcal{P},\mathcal{Q}}$ of G but every subset of F belongs to it. This construction implies that for every $B \subseteq V \cup E$ with $B \notin \mathcal{T}_{\mathcal{P},\mathcal{Q}}$ there is an $F \in \mathcal{F}$ with $F \subseteq B$. Thus, we have $\mathrm{chr}_{\mathcal{P},\mathcal{Q}}(G) = \mathrm{chr}(\mathcal{H})$. Moreover, define $\mathcal{A} = \{A \subseteq X :$ there is no $F \in \mathcal{F}$ with $F \subseteq A\}$.

A *fractional coloring* of $\mathcal{H}$ is a mapping $\varphi : \mathcal{A} \to [0, 1]$ such that

$$\sum\nolimits_{A\in\mathcal{A};\, x\in A} \varphi(A) \geq 1 \qquad \text{for all } x \in X. \tag{2}$$

The *fractional chromatic number* $\chi_f(\mathcal{H})$ of $\mathcal{H}$ is the solution of the linear program (2) with objective function

$$\sum\nolimits_{A\in\mathcal{A}} \varphi(A) \to \min.$$

It follows that $\chi_f(\mathcal{H}) = \chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ since $\mathcal{A}$ in $\mathcal{H}$ corresponds with $\mathcal{T}_{\mathcal{P},\mathcal{Q}}$ in the underlying graph G. In [14] it is proved that the choice ratio $\mathrm{chr}(\mathcal{H})$ for any hypergraph $\mathcal{H}$ equals its fractional chromatic number $\chi_f(\mathcal{H})$ which immediately implies $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \mathrm{chr}_{\mathcal{P},\mathcal{Q}}(\mathrm{G})$.

Nevertheless we will give another proof in Section 2. We follow the ideas of [1], [2], and [14]. However we apply Steinitz' Lemma directly, not using derived results on hypergraph partitions.

Moreover, we present some new bounds for the $(\mathcal{P},\mathcal{Q})$-total chromatic number of G and therefore also for the $(\mathcal{P},\mathcal{Q})$-choice ratio of G.

2. General Results

Let G be a simple graph and let φ^* be a fractional $(\mathcal{P},\mathcal{Q})$-total coloring of G realizing $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$. Furthermore, let $\mathcal{T}^* = \{T_1, \ldots, T_s\} \subseteq \mathcal{T}_{\mathcal{P},\mathcal{Q}}$ be the set of all $(\mathcal{P},\mathcal{Q})$-total independent sets of G with $\varphi^*(T_i) > 0$.

Recall that $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ is the solution of the above linear optimization problem (1). Thus there are $p_1, \ldots, p_s, q$ such that $\varphi^*(T_i) = \frac{p_i}{q}$ for all $i = 1, \ldots, s$ and $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \frac{p}{q}$ with $p = \sum_{i=1}^{s} p_i$.

Let $\mathcal{S}$ be a multiset of sets T_i such that each $T_i \in \mathcal{T}^*$ occurs exactly p_i times in $\mathcal{S}$. Therefore, $\mathcal{S}$ contains exactly p sets and every $x \in V(G) \cup E(G)$ belongs to at least q of these sets since $\sum_{T_i\in\mathcal{T}^*;\, x\in T_i} \varphi^*(T_i) \geq 1$ for all $x \in V \cup E$. Thus we have the following observation.

Proposition 1. *Let G be a simple graph with $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \frac{a}{b}$. Then there are p and q with $\frac{a}{b} = \frac{p}{q}$ and a multiset $\mathcal{S}$ of p not necessarily distinct sets which are members of $\mathcal{T}_{\mathcal{P},\mathcal{Q}}$ such that each $x \in V(G) \cup E(G)$ belongs to at least q of these sets.*

Theorem 2. *For every simple graph G it holds that $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) \leq \mathrm{chr}_{\mathcal{P},\mathcal{Q}}(G)$.*

Proof. Suppose that G is $(\mathcal{P},\mathcal{Q})$-total (a,b)-list colorable and let $L(x)$ be a list assignment with identical lists $L(x) = \{1,\ldots,a\}$ for all $x \in V \cup E$. Thus we can choose color sets $C(x)$ with $|C(x)| = b$ for every $x \in V \cup E$ such that for every color $i \in \{1,\ldots,a\}$ we have $T_i = \{x \in V \cup E : i \in C(x)\} \in \mathcal{T}_{\mathcal{P},\mathcal{Q}}$.

Let $\mathcal{S} = \{T_1,\ldots,T_a\}$ be the multiset of all (not necessarily distinct) sets T_i. Note that every $x \in V \cup E$ belongs to exactly b of them.

For a set $T \in \mathcal{T}_{\mathcal{P},\mathcal{Q}}$ let $a(T)$ be the number of occurrences of T in $\mathcal{S}$. Define $\varphi(T) = \frac{a(T)}{b}$. We obtain $\sum_{T\in\mathcal{T}_{\mathcal{P},\mathcal{Q}};\, x\in T} \varphi(T) = 1$ for all $x \in V \cup E$ and $\sum_{T\in\mathcal{T}_{\mathcal{P},\mathcal{Q}}} \varphi(T) = \frac{a}{b}$. Thus φ is a fractional $(\mathcal{P},\mathcal{Q})$-total coloring of G and therefore $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) \leq \frac{a}{b}$. ■

Theorem 3. *For every pair of additive and hereditary graph properties $\mathcal{P} \supseteq \mathcal{O}$ and $\mathcal{Q} \supseteq \mathcal{O}_1$ and every simple graph G with $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \frac{a}{b}$ there is an integer M such that G is $(M, \frac{b\cdot M}{a})$-list $(\mathcal{P},\mathcal{Q})$-total colorable.*

Proof. Let G be a graph with $|V(G)| = n$, $|E(G)| = m$, and $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \frac{a}{b}$. Furthermore, let p and q be integers according to Proposition 1.

We will prove the statement of the theorem for very huge M. Set $M = p\cdot k\cdot s$ where $k = \lceil \frac{1}{2}(2n+2m+1)^{n+m} \rceil$ and $s = \mathrm{lcm}\{2,\ldots,k\}$ is the least common multiple of $2,\ldots,k$.

Assume that there is a list assignment with $|L(x)| = M$ for all $x \in V \cup E$ and let $\mathcal{L} = \bigcup_{x\in V\cup E} L(x)$ be the set of all colors occurring in the lists.

Claim. There is a partition $\mathcal{L} = L_1 \cup \cdots \cup L_p$ such that $|L(x) \cap L_i| = \frac{M}{p}$ for all $x \in V \cup E$ and all $i = 1,\ldots,p$.

Proof. First, we show that there is a partition of $\mathcal{L}$ into subsets D_j such that for every D_j there is a $d_j \leq k$ with $|L(x) \cap D_j| = d_j$ for all $x \in V \cup E$. From this partition we construct the required partition unifying appropriate sets D_j to an L_i. The first part of the proof is similar to a proof given in [1] for the existence of proper regular spanning subhypergraphs. The second part is analogous to a proof given in [2].

Denote the elements of $V \cup E$ by $x_1,\ldots,x_{n+m}$. For every color $i \in \mathcal{L}$ define a 0-1 vector $\underline{v}_i \in \mathbb{Z}^{n+m}$ with $\underline{v}_i(j) = 1$ if $i \in L(x_j)$ and $\underline{v}_i(j) = 0$ otherwise. Let $\underline{w} \in \mathbb{Z}^{n+m}$ be the vector with $\underline{w}(j) = -1$ for all j. Consider the multiset $\mathcal{Z} \subseteq \mathbb{Z}^{n+m}$ consisting of the vectors $\underline{v}_i$ for all $i \in \mathcal{L}$ and M times the vector $\underline{w}$. Denote the elements of $\mathcal{Z}$ by $\underline{z}_1,\ldots,\underline{z}_\ell$.

Clearly, $\ell = |\mathcal{Z}| \geq 2M \geq 2k$ since there are at least M colors in $\mathcal{L}$ and M vectors $\underline{w}$. Moreover, $\sum_{i=1}^{\ell} \underline{z}_i = \underline{0}$ since for every component (corresponding to an $x \in V \cup E$) the value $+1$ occurs exactly M times and also -1 occurs exactly M times. Additionally, we have for the sup-norm $||\underline{z}_i|| = \max\{|\underline{z}_i(1)|, \ldots, |\underline{z}_i(n+m)|\} = 1$ for all $\underline{z}_i \in \mathcal{Z}$.

Now, we apply the following lemma due to Steinitz with an improved bound of Sevastyanov [19].

Lemma 4 (Steinitz, see [1], [3]). *Let $\mathcal{Z}$ be any normed $(n+m)$-dimensional space. Suppose that $\underline{z}_1, \ldots, \underline{z}_\ell \in \mathcal{Z}$, $||\underline{z}_i|| \leq 1$, and $\sum_{i=1}^{\ell} \underline{z}_i = \underline{0}$. Then there is a permutation π of $1, 2, \ldots, \ell$ such that $||\sum_{i=1}^{j} \underline{z}_{\pi(i)}|| \leq n+m$ holds for all $1 \leq j \leq \ell$.*

Thus we may assume without loss of generality that $||\sum_{i=1}^{j} \underline{z}_i|| \leq n+m$ holds for all $1 \leq j \leq \ell$. Note that there are at least $2k = (2n+2m+1)^{n+m} + 1$ such sums $S_j = \sum_{i=1}^{j} \underline{z}_i$ since $\ell \geq 2k$. On the other hand, there are at most $(2n+2m+1)^{n+m} < 2k$ vectors in $\mathbb{Z}^{n+m}$ with sup-norm at most $n+m$. Hence, there are j_1 and j_2 with $j_1 < j_2 \leq j_1 + 2k - 1$ such that $S_{j_1} = S_{j_2}$. Now consider the difference of these sums $S_{j_2} - S_{j_1} = \sum_{i=j_1+1}^{j_2} \underline{z}_i = \underline{0}$ and denote the set of vectors occurring in this difference by $\mathcal{Z}_1$, that is, $\mathcal{Z}_1 = \{\underline{z}_{j_1+1}, \ldots, \underline{z}_{j_2}\}$. Clearly, $|\mathcal{Z}_1| \leq 2k-1$ and at most $k-1$ of these vectors can have negative entries. Thus, there are $d_1 \leq k-1$ vectors $\underline{w}$ in $\mathcal{Z}_1$, and the remaining vectors with nonnegative entries are of type $\underline{v}_i$. It follows that $\sum_{\underline{v}_i \in \mathcal{Z}_1} \underline{v}_i = (d_1, d_1, \ldots, d_1)$.

Remember that every vector $\underline{v}_i$ corresponds to a color $i \in \mathcal{L}$. Define $D_1 = \{i \in \mathcal{L} : \underline{v}_i \in \mathcal{Z}_1\}$. According to the construction we have $|L(x) \cap D_1| = d_1 \leq k-1$ for all $x \in V \cup E$. Delete the colors of D_1 from all lists $L(x)$ (and therefore from $\mathcal{L}$) and the vectors of $\mathcal{Z}_1$ from $\mathcal{Z}$. Then continue to construct sets D_j, $j \geq 2$, in an analogous way as long as the length of the reduced lists is at least k. Note that the sum of the remaining vectors of $\mathcal{Z}$ is also $\underline{0}$ and therefore at least k of them are vectors $\underline{w}$ which implies that there are altogether at least $2k$ vectors in this remaining set. In the last step the set D_u consists of the remaining colors $\mathcal{L} \setminus (D_1 \cup D_2 \cup \cdots \cup D_{u-1}) = D_u$. Clearly, $|L(x) \cap D_u| = d_u < k$.

Thus we have a partition of $\mathcal{L}$ into subsets D_j such that $|L(x) \cap D_j| = d_j \leq k$ for all $x \in V \cup E$ and all j. Moreover, the sum of all d_js is equal to M. Now we construct the required partition $\mathcal{L} = L_1 \cup \cdots \cup L_p$ such that $|L(x) \cap L_i| = \frac{M}{p}$ for all $x \in V \cup E$ and all $i = 1, \ldots, p$.

Remember that $s = \mathrm{lcm}\{2, \ldots, k\}$ is the least common multiple of $2, \ldots, k$. If there is a t such that at least $\frac{s}{t}$ of the d_js are equal to t, then collect exactly $\frac{s}{t}$ of the corresponding indices in an index set I_r. Note that for every such index set we have $\sum_{j \in I_r} d_j = s$. If it is not possible to build a further index set of this kind then the sum of the remaining d_js is smaller than $\sum_{t=1}^{k} \frac{s}{t} \cdot t = k \cdot s = \frac{M}{p}$.

It follows that we have at least $k(p-1)$ index sets I_r since the sum of all d_js corresponding to these sets must be greater than $M - \frac{M}{p} = s \cdot k \cdot (p-1)$. Unify k of these index sets at a time to a new set I_i and define $L_i = \bigcup_{j \in I_i} D_j$. Note that we obtain $p-1$ sets L_i where the corresponding sum of the d_js is $\sum_{j \in I_i} d_j = k \cdot s = \frac{M}{p}$ for every i. Finally, L_p is defined to be the union of the remaining sets D_j not used in $L_1, \ldots, L_{p-1}$. Clearly, the sum of the corresponding d_js is $M - (p-1)\frac{M}{p} = \frac{M}{p}$.

Thus we have the required partition and the claim is proved. □

Remember that the multiset $\mathcal{S} = \{T_1, \ldots, T_p\}$ of Proposition 1 consists of p sets of $\mathcal{T}_{\mathcal{P},\mathcal{Q}}$. Define

$$C(x) = \bigcup_{T_i \in \mathcal{S};\, x \in T_i} L_i \cap L(x).$$

It follows that $|C(x)| \geq q \cdot \frac{M}{p}$ since x belongs to at least q of the sets in $\mathcal{S}$, $|L_i \cap L(x)| = \frac{M}{p}$ for all $i = 1, \ldots, p$ and all $x \in V \cup E$, and $L_i \cap L_j = \emptyset$ if $i \neq j$. Moreover, we obtain for every color j that $R_j = \{x \in V \cup E : j \in C(x)\}$ is a subset of some $T_i \in \mathcal{T}_{\mathcal{P},\mathcal{Q}}$.

Since $\frac{q}{p} = \frac{b}{a}$ it follows that G is $(M, \frac{b \cdot M}{a})$-list $(\mathcal{P}, \mathcal{Q})$-total colorable by choosing appropriate subsets $C(x)$ with $|C(x)| = q \cdot \frac{M}{p}$. ■

Theorem 3 gives immediately $\mathrm{chr}_{\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ and together with Theorem 2 we obtain the following statement.

Theorem 5. *If G is a simple graph and $\mathcal{P} \supseteq \mathcal{O}$ and $\mathcal{Q} \supseteq \mathcal{O}_1$ are additive and hereditary graph properties, then $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \mathrm{chr}_{\mathcal{P},\mathcal{Q}}(G)$.*

Note that the theorem gives the equality of these two parameters. Nevertheless, it may happen that a graph has a $(\mathcal{P}, \mathcal{Q})$-total (a, b)-coloring but it is not $(\mathcal{P}, \mathcal{Q})$-total (a, b)-list colorable.

Note that in the proof of Theorem 3 it is not sufficient to choose a very big M but M also has to fulfill some divisibility conditions.

In the next section we will obtain some bounds and exact values for $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ for some properties $\mathcal{P}$ and $\mathcal{Q}$ and some classes of graphs. The proofs also allow to fix a and b such that the graphs in question have a $(\mathcal{P}, \mathcal{Q})$-total (a, b)-coloring. Thus we also have bounds and exact values for $\mathrm{chr}_{\mathcal{P},\mathcal{Q}}(\mathrm{G})$ but so far we are not able to say whether there are a and b which are smaller than those given in the proof of Theorem 3 such that G is $(\mathcal{P}, \mathcal{Q})$-total (a, b)-list colorable.

3. Bounds for $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ for Specific Properties

Remember that $\mathcal{D}_1$ is the property of a graph to be acyclic. In [11] it is proved that $\chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(K_n) = \frac{n+m}{n-1}$. The following bound is based on the ideas of the proof of this result.

Theorem 6. *Let G be a simple connected graph on $n \geq 2$ vertices and m edges. Then*

$$\chi''_{f,\mathcal{P},\mathcal{Q}}(G) \geq \frac{n+m}{n-1}$$

if $G \notin \mathcal{P}$ and $\mathcal{O}_1 \subseteq \mathcal{Q} \subseteq \mathcal{D}_1$.

Proof. We show that if G has a $(\mathcal{P},\mathcal{Q})$-total (r,s)-coloring then $\frac{r}{s} \geq \frac{n+m}{n-1}$.

Assume that a fixed color is used for the coloring of i vertices. We have $i < n$ since $G \notin \mathcal{P}$. Then the same color can be used for at most $n-i-1$ edges building an acyclic graph of property $\mathcal{Q}$ on the remaining $n-i$ vertices. Thus each of the r available colors can be used for the coloring of at most $n-1$ elements. On the other hand, every vertex and every edge should get s colors and we obtain $r(n-1) \geq (n+m)s$. ■

For example, $\mathcal{Q}$ can be chosen as set $\mathcal{S} = \{H \in \mathcal{I} : H \text{ consists of stars}\}$ or as set $\mathcal{L} = \mathcal{D}_1 \cap \mathcal{S}_2 = \{H \in \mathcal{I} : H \text{ consists of paths}\}$.

Note that $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = 1$ if and only if $G \in \mathcal{O}$, and $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = 2$ if and only if $G \in (\mathcal{P} \cap \mathcal{Q}) \setminus \mathcal{O}$, and $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) > 2$ if and only if $G \notin \mathcal{P} \cap \mathcal{Q}$ (see [11]).

Theorem 7. *Let G be a simple connected graph on $n \geq 3$ vertices. Then*

$$\chi''_{f,\mathcal{P},\mathcal{Q}}(G) \leq \frac{n(n+1)}{2(n-1)}$$

if $\mathcal{P} \supseteq \mathcal{O}_1$ and $\mathcal{Q} \supseteq \mathcal{S}$.

Proof. We will use the definition of $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ by a linear program mentioned in the Introduction.

Let G be a simple graph with vertex set $\{v_1,\ldots,v_n\}$. We define two types of $(\mathcal{P},\mathcal{Q})$-total independent sets $T = V_T \cup E_T \subseteq V \cup E$.

Let E_i be the set of all edges of G which are incident to v_i and $E_{i;j,k}$ the set of all edges of G incident to v_i but not incident to v_j and to v_k.

Define $\mathcal{T}_1 = \{E_i : i = 1,\ldots,n\}$ and $\mathcal{T}_2 = \{\{v_j,v_k\} \cup E_{i;j,k} : j = 1,\ldots,n-1,\ k = j+1,\ldots,n,\ i = 1,\ldots,n,\ i \neq j, i \neq k\}$. It may happen that a set T occurs more than once in $\mathcal{T}_2$ (see Example). Thus $\mathcal{T}_2$ can be a multiset.

Obviously, $|\mathcal{T}_1| = n$ and every edge of $E(G)$ occurs in exactly two sets of $\mathcal{T}_1$. Moreover, $|\mathcal{T}_2| = \binom{n}{2}(n-2)$ where every vertex of $V(G)$ occurs in $(n-1)(n-2)$ of these sets and every edge occurs in $\binom{n-2}{2} \cdot 2 = (n-2)(n-3)$ of them.
Assign the following weights to the sets T of G:

$$\varphi(T) = \begin{cases} \frac{1}{n-1} & \text{if } T \in \mathcal{T}_1, \\ \frac{t}{(n-1)(n-2)} & \text{if } T \text{ occurs } t \text{ times in } \mathcal{T}_2, \\ 0 & \text{otherwise.} \end{cases}$$

Since every vertex v_i occurs in $(n-1)(n-2)$ sets of the multiset $\mathcal{T}_2$ it holds that $\sum_{T \in \mathcal{T}_1 \cup \mathcal{T}_2;\, v_i \in T} \varphi(T) = 1$ for all i. Furthermore,

$\sum_{T\in\mathcal{T}_1\cup\mathcal{T}_2;\, e\in T}\varphi(T) = \frac{2}{n-1} + \frac{(n-2)(n-3)}{(n-1)(n-2)} = 1.$

Finally we have $\sum_{T\in\mathcal{T}_1\cup\mathcal{T}_2}\varphi(T) = \frac{n}{n-1} + \frac{n(n-1)(n-2)}{2(n-1)(n-2)} = \frac{n(n+1)}{2(n-1)}$ proving the statement of the theorem. ■

If G is a complete graph on n vertices then the lower and upper bounds from Theorems 6 and 7 coincide. Therefore, we have the following result which generalizes the result of [11] mentioned at the beginning of this section.

Theorem 8. *If $\mathcal{P}\supseteq\mathcal{O}_1$, and $K_n\notin\mathcal{P}$, and $\mathcal{S}\subseteq\mathcal{Q}\subseteq\mathcal{D}_1$ or $\mathcal{L}\subseteq\mathcal{Q}\subseteq\mathcal{D}_1$, and $n\geq 3$, then*

$$\chi''_{f,\mathcal{P},\mathcal{Q}}(K_n) = \frac{n(n+1)}{2(n-1)}.$$

Proof. The lower bound is an implication of Theorem 6 since $\mathcal{Q}\subseteq\mathcal{D}_1$ and the upper bound for $\mathcal{S}\subseteq\mathcal{Q}$ is an implication of Theorem 7.

Let $\mathcal{P}\supseteq\mathcal{O}_1$, $K_n\notin\mathcal{P}$ ($n\geq 3$), and $\mathcal{Q}=\mathcal{L}$. Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n in a cyclic order.

If n is odd then define $(\mathcal{P},\mathcal{L})$-total independent sets $T_i = \{v_i, v_{i+\lfloor n/2\rfloor}\}\cup\{v_{i-1}v_{i+1}, v_{i+1}v_{i-2}, v_{i-2}v_{i+2}, v_{i+2}v_{i-3}, \ldots, v_{i+\lfloor n/2\rfloor-1}v_{i-\lfloor n/2\rfloor}\}$ and $T'_i = \{v_iv_{i+1}, v_{i+1}v_{i+2}, \ldots, v_{i-2}v_{i-1}\}$, $i=1,\ldots,n$, where the indices are considered modulo n. Let $\mathcal{T}_1 = \{T_1,\ldots,T_n\}$ and $\mathcal{T}_2 = \{T'_1,\ldots,T'_n\}$.

Assign the weights $\varphi(T)=\frac{1}{2}$ if $T\in\mathcal{T}_1$, $\varphi(T)=\frac{1}{n-1}$ if $T\in\mathcal{T}_2$, and $\varphi(T)=0$ to the remaining $(\mathcal{P},\mathcal{L})$-total independent sets T of K_n.

Define the distance $d(e)$ of an edge $e=v_jv_i$, $j>i$, by $d(e)=\min\{j-i, n-(j-i)\}$. Each vertex of K_n as well as each edge of distance $2,\ldots,\lfloor n/2\rfloor$ occurs in exactly two sets of $\mathcal{T}_1$ and each edge of distance 1 occurs in exactly $n-1$ sets of $\mathcal{T}_2$. Therefore, $\sum_{T\in\mathcal{T}_1\cup\mathcal{T}_2;\, v_i\in T}\varphi(T) = 2\cdot\frac{1}{2} = 1$ for $i=1,\ldots,n$, $\sum_{T\in\mathcal{T}_1\cup\mathcal{T}_2;\, e\in T,\, d(e)>1}\varphi(T) = 2\cdot\frac{1}{2} = 1$, $\sum_{T\in\mathcal{T}_1\cup\mathcal{T}_2;\, e\in T,\, d(e)=1}\varphi(T) = (n-1)\frac{1}{n-1} = 1$, and finally $\sum_{T\in\mathcal{T}_1\cup\mathcal{T}_2}\varphi(T) = n\cdot\frac{1}{2} + n\frac{1}{n-1} = \frac{n(n+1)}{2(n-1)}$.

If n is even an analogous construction can be given which concludes the proof of the upper bound $\frac{n(n+1)}{2(n-1)}$ for $\chi''_{f,\mathcal{P},\mathcal{L}}(K_n)$ and therefore also for $\chi''_{f,\mathcal{P},\mathcal{Q}}(K_n)$ for $\mathcal{Q}\supseteq\mathcal{L}$. ■

Observe that, for example, $\mathcal{P}\in\{\mathcal{O}_k,\mathcal{D}_k,\mathcal{S}_k,\mathcal{I}_k : 1\leq k\leq n-2\}$ fulfills the condition of the theorem.

If $\mathcal{P}=\mathcal{O}$ then $\chi''_{f,\mathcal{O},\mathcal{Q}}(K_n)=n$ if n is odd or if $n\geq 4$ is even and $\mathcal{O}_1\subset\mathcal{Q}$ and $\chi''_{f,\mathcal{O},\mathcal{Q}}(K_n)=n+1$ if $n=2$ or if $n\geq 4$ is even and $\mathcal{Q}=\mathcal{O}_1$ (the proof runs analogously to that in [11] for the nonfractional case).

Example: $\chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(K_4) = \frac{20}{6} = \frac{10}{3}$.

In fact, K_4 has a $(\mathcal{D}_1,\mathcal{D}_1)$-total $(10,3)$-coloring. Figure 1 shows all different sets from $\mathcal{T}_1\cup\mathcal{T}_2$. Note that $E_{i;j,k} = E_{\ell;j,k}$ for all 4-element sets $\{i,\ell,j,k\}$. Thus

every set of $\mathcal{T}_2$ occurs twice there. Consequently, $\varphi(T)=\frac{1}{3}$ for each of the sets in Figure 1. If we would like to construct a $(\mathcal{D}_1,\mathcal{D}_1)$-total $(10s,3s)$-coloring from these sets we have to assign s of $10\cdot s$ colors to every set. In Figure 1 we assign exactly one color to every set.

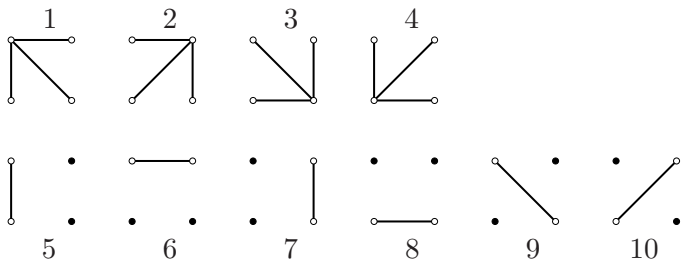

Figure 1. $\mathcal{T}_1\cup\mathcal{T}_2$ for K_4.

Figure 2 shows the resulting $(\mathcal{D}_1,\mathcal{D}_1)$-total $(10,3)$-coloring of K_4.

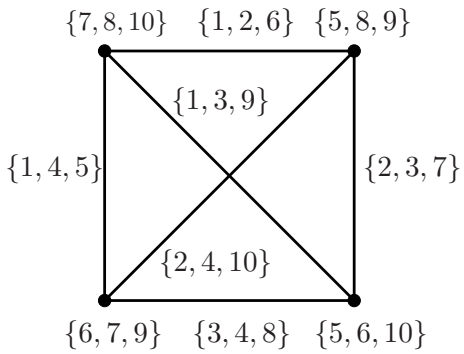

Figure 2. A $(\mathcal{D}_1,\mathcal{D}_1)$-total $(10,3)$-coloring of K_4.

Theorem 9. *Let C_n be a cycle on $n\geq 3$ vertices. Then*

$$\chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(C_n)=\frac{2n}{n-1}=2+\frac{2}{n-1}.$$

Proof. We obtain $\chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(C_n)\geq\frac{2n}{n-1}$ immediately from Theorem 6.

Let $v_1,v_2,\ldots,v_n$ be the vertices of C_n in cyclic order. If n is odd then set $h=\frac{n-1}{2}$ and define $T_i=V_i\cup E_i$ with $V_i=\{v_i,\ldots,v_{i+h-1}\}$ and $E_i=\{v_{i+h}v_{i+h+1},\ldots,v_{i+2h-1}v_{i+2h}\}$, $i=1,2,\ldots,n$, where the indices are taken modulo n. Thus each T_i is a set of h consecutive vertices and h consecutive edges. Moreover, we have exactly n sets T_i and every vertex and every edge occurs in h of them.

Set $\varphi(T_i) = \frac{1}{h} = \frac{2}{n-1}$. We obtain $\sum_{T_i; v \in T_i} \varphi(T_i) = 1$ for all $v \in V(C_n)$ and $\sum_{T_i;\, e \in T_i} \varphi(T_i) = 1$ for all $e \in E(C_n)$.

Moreover, $\sum_{i=1}^{n} \varphi(T_i) = \frac{2n}{n-1}$ and therefore $\chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(C_n) \leq \frac{2n}{n-1}$ as required.

If n is even then set $h = \frac{n}{2}$ and define n sets T_i as sets of h consecutive vertices each and $h-1$ consecutive edges where the end-vertices of the edges are disjoint from the chosen consecutive vertices. Moreover, define n sets S_i consisting of sets of $h-1$ consecutive vertices each and of h consecutive edges where the end-vertices of the edges are disjoint from the chosen consecutive vertices. Note that each vertex and each edge occurs in $2h-1 = n-1$ of the the sets S_i and T_i.

Set $\varphi(T_i) = \varphi(S_i) = \frac{1}{n-1}$.

Obviously, we have $\sum_{T_i;\, v \in T_i} \varphi(T_i) + \sum_{S_i;\, v \in S_i} \varphi(S_i) = 1$ for all $v \in V(C_n)$ and $\sum_{T_i;\, e \in T_i} \varphi(T_i) + \sum_{S_i;\, e \in S_i} \varphi(S_i) = 1$ for all $e \in E(C_n)$.

Moreover, $\sum_{i=1}^{n} (\varphi(T_i) + \varphi(S_i)) = \frac{2n}{n-1}$ which completes the proof of the theorem. ■

Note that the property $\mathcal{D}_1$ in Theorem 9 can be replaced by any properties $\mathcal{P}$ and $\mathcal{Q}$ with $\mathcal{L} = \mathcal{S}_2 \cap \mathcal{D}_1 \subseteq \mathcal{P}, \mathcal{Q} \subseteq \mathcal{D}_1$.

Using similar constructions it should be possible to determine the fractional $(\mathcal{D}_1, \mathcal{D}_1)$-total chromatic number of some other classes of graphs.

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