Discussiones Mathematicae

# WHEN IS AN INCOMPLETE $3 \times n$ LATIN RECTANGLE COMPLETABLE? 

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#### Abstract

We use the concept of an availability matrix, introduced in Euler [7], to describe the family of all minimal incomplete $3 \times n$ latin rectangles that are not completable. We also present a complete description of minimal incomplete such latin squares of order 4.


Keywords: incomplete latin rectangle, completability, solution space enumeration, branch-and-bound.

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## 1. Introduction and Basic Results

An $n \times n$ array $L$ each cell of which contains exactly one symbol $i \in N=\{1, \ldots, n\}$ such that each symbol occurs in each row and in each column exactly once is a latin square (of order $n$ ). If we replace "exactly" by "at most" and if not all cells

[^0]are filled, we obtain an incomplete latin square, and if for $r, s \in N$ the non-empty cells of an incomplete latin square form a rectangle of $r$ rows and $s$ columns, we will speak of an $r \times s$ latin rectangle. Given $n \in \mathbb{N}$ the problem of characterizing those incomplete latin squares that are completable (to a latin square of the same order), is an open question. There are however partial results: Evans' conjecture [8] (proved by Smetaniuk [13], and independently by Andersen and Hilton [2]) states that an incomplete latin square containing at most $n-1$ filled cells is always completable. A similar result is due to Hall [10], who showed that no condition is required to complete an $r \times n$ latin rectangle. A third result due to Ryser [12] states that an $r \times s$ latin rectangle can be completed if and only if each symbol $n \in N$ appears at least $r+s-n$ times. By using the concept of an availability matrix the first author has shown in [7] how further such results can be obtained for the completability of (one or more) incomplete rows of specific structure.

The result presented in this paper is of different a nature. Let $E_{n}=\left\{e_{i j k}\right.$ : $1 \leq i, j, k \leq n\}$ be an arbitrary set of $n^{3}$ elements. Call an $n \times n$ array $L$ feasible, if each cell of $L$ contains a symbol $k$ at most once. Obviously, we can identify the selection of the element $e_{i j k}$ with the appearance of symbol $k$ in cell $i j$, and hereby obtain a 1-1-relation between subsets of $E_{n}$ and feasible arrays over $N$. For convenience, we will make no real distinction between a feasible array and its corresponding subset. In particular, any latin square corresponds to a specific $n^{2}$-element subset of $E_{n}$, and the system $\mathcal{B}_{n}$ of all these sets constitutes a clutter, say, of bases, a notion well known from matroid theory. Any such clutter induces a (unique) clutter $\mathcal{C}_{n}$ of circuits, i.e., subsets of $E_{n}$ that are not contained in any member of $\mathcal{B}_{n}$ and that are minimal with respect to this property. As a consequence, the complete knowledge of $\mathcal{C}_{n}$ would answer the completability question in the following sense: an incomplete latin square can be completed if and only if it does not contain any circuit.

In 1985 (see Euler et al. [6]), we have initiated the study of $\mathcal{C}_{n}$ by considering circuits arising from two distinct symbols in one cell or two identical symbols in one row or in one column, that we call elementary, and others arising from particular latin rectangles. Our motivation was the application of linear programming techniques to solve the planar 3 -dimensional assignment problem $(P)$, the solutions of which correspond to the latin squares of the given order. Observe that $(P)$ also contains our completability question, shown by Colbourn [5] to be NP-complete, as a special case. In this context, circuits are useful for providing facet-defining inequalities for associated polyhedra. For surveys on 3-dimensional assignment problems we refer the reader to Burkard et al. [4] and Spieksma [14].

The main objective of this paper is to study the clutter of circuits associated with the collection of all $r \times n$ latin rectangles for given $r$. We will limit ourselves to non-elementary circuits, i.e., the collection $\mathcal{C}_{n}^{r}$ of all those incomplete $r \times n$ latin rectangles, that are not completable and minimal with respect to this property.

A complete answer for all $r \in\{1, \ldots, n\}$ would provide necessary and sufficient conditions for the completability of any incomplete latin square. In the following, we will fully answer this question for $r=3$. Just observe that by Hall's theorem [10], $\mathcal{C}_{n}^{r}$ is a subfamily of $\mathcal{C}_{n}$, and our result therefore contributes to a better knowledge of $\mathcal{C}_{n}$. We also point to the work of Brankovic et al. [3], who studied circuits under the name of premature partial latin squares, and to the recent work of Adams et al. [1] for which the knowledge of circuits could open a different approach. We also give a complete description of $\mathcal{C}_{n}$ for $n=4$ that we have obtained by computer calculations. We just mention that generating the family of circuits associated with a clutter of bases is a special case of transversal hypergraph generation (as for instance studied by Khachiyan et al. [11]), which has many applications in combinatorics and computer science and whose exact complexity status is still open. We refer to Hagen [9] for recent results on this topic.

The basis of our analysis is the following theorem:
Theorem 1 (Frobenius-König). $A(0,1)$-matrix $A$ of size $n \times n$ contains $n 1^{\prime}$ s no two of which lie in the same row or column if and only if $A$ does not contain $a 0$-submatrix of size $u \times v$ such that $u+v=n+1$.

It is the application of this theorem to a very special matrix that will lead us to a complete description of the family $\mathcal{C}_{n}^{3}$.

Definition 2. Let $L$ be an incomplete latin square the $m$-th row of which contains $0<t<n$ empty cells. Moreover, let $S(m)$ denote the set of symbols not appearing in that row and $J(m)$ the set of column indices of its empty cells. The availability matrix $A(L, m)$ is the $t \times t$ matrix obtained from the $n \times n$ matrix $A$ by deleting rows $A_{i}$ for $i \in N \backslash S(m)$ and columns $A^{j}$ for $j \in N \backslash J(m)$, and with an element $A_{i}^{j}(L, m), i \in S(m), j \in J(m)$ marked with an asterisk as "non-available" if and only if symbol $i$ appears in column $j$ of $L$.

$$
A=\left[\begin{array}{ccc}
1 & \ldots & 1 \\
2 & \ldots & 2 \\
\vdots & & \vdots \\
n & \ldots & n
\end{array}\right]
$$

What is the use of $A(L, m)$ ?

- If it is possible to select within this matrix $t$ available elements, one per row and one per column, then row $m$ is completable;
- if, however, this is not possible, by Frobenius-König's Theorem $1, A(L, m)$ has to contain a $p \times q$ submatrix of non-available elements such that $p+q=t+1$ (see Figure 1 for an illustration of that case).
In case of an $r \times n$-latin rectangle $L$ we hereby obtain necessary and sufficient conditions for the completability of a new type of incomplete latin square (that


Figure 1. An incomplete, non-completable latin square and its availability matrix with non-available elements marked by a $" *$.
can be checked in polynomial time by solving an assignment or bipartite matching problem over $A(L, m))$ :

Theorem 3. Let $L$ be an $r \times n$ latin rectangle with $0<t<n$ empty cells in row $r+1$. Then $L$ is completable if and only if any subset $I$ of $S(r+1)$ is contained in at most $t-|I|$ of the columns $L^{j}, j \in J(r+1)$.

## 2. A Complete Description of $\mathcal{C}_{n}^{3}$

Before turning to three rows we just mention that the case $r=1$ is obvious: an incomplete latin row is always completable, i.e., $\mathcal{C}_{n}^{1}$ is empty, and for the case $r=2$ the family $\mathcal{C}_{n}^{r}$ for $n \geq 3$ is fully represented by the two types of circuits illustrated in Figure 2 (up to row- and column interchangements, and for any symbol $i \in N)$.


| $N\{\{i\}$ |  |
| :---: | :--- |
| $N\{i\}$ |  |

Figure 2. The 2 types of circuits for $r=2$.
As to $r=3$ we start with those circuits that arise from the non-completability of a single row, so-called 1-row-circuits. Applying Theorem 3 we come up with 4 different types as illustrated in Figure 3 (again for $n \geq 3$ and throughout the paper, up to row- and column interchangements, and for any distinct $i, j, k \in N)$. We just remark that this family is well understood for any $r \in\{1, \ldots, n\}$; a description (in its conjugate form rows $\leftrightarrow$ symbols) has already been given in Euler et al. [6].

Now let us turn to 2 -row-circuits, i.e., those incomplete $3 \times n$ latin rectangles, which are not completable and minimal with respect to this property, which do not contain any 1 -row-circuit but which contain two rows, say row 1 and row 2 ,


Figure 3. The 4 types of 1-row-circuits for $r=3$.
that are not completable. Clearly, the second circuit depicted in Figure 2 is a first such 2-row-circuit.

To describe the others, we consider the availability matrices $A_{1}$ and $A_{2}$ of rows 1 and 2 , and observe the following: first, both matrices must have a line (i.e., row or column) in common, since otherwise the two rows would be completable; second, a common line can contain at most one non-available symbol, since an asterisk can only arise from a symbol in row 3 . Therefore, a forbidden submatrix within $A_{1}$, engendered by the completion of row 2 (or vice versa) can only be of size $2 \times 2,2 \times 1$ or $1 \times 2$. In the first case, $A_{2}$ cannot be of size $1 \times 1$ only: a symbol $k$, already appearing in row 2 , is marked as non-available for row 1 , and thus also for row 2 and its empty cell. Therefore, we can delete symbol $k$ from row 2 , a contradiction to minimality. We are thus lead to a second 2-row-circuit, depicted as type 2 in Figure 4, for which we also indicate the 2 availability matrices and the way to complete row 1 after deletion of an arbitrary symbol $l$. For $n \geq 6$, row 3 is then always completable due to Theorem 3. The proof for row 2 is similar, and for row 3 it is straightforward.

As to forbidden submatrices of size $2 \times 1$ or $1 \times 2$ we come up with 3 possibilities, types 3 to 5 in Figure 4, illustrated in a similar way as type 2. We just mention that types 1,3 and 4 exist for $n \geq 3$, type 2 for $n \geq 3$ and $n \neq 4$, and type 5 for $n \geq 4$. We also ask for the following properties:

- in type $3, i$ and $j$ have to be different from $k$ but $i$ may be equal to $j$;
- in type 4 , the empty cells in rows 1 and 2 , not in the last column, may appear in a same column;
- in type $5, i, j$ and $k$ must all be different ( $i=k$ would contradict minimality). Finally, if one of the availability matrices is of size $1 \times 1$ and the other of size $2 \times 2$, we obtain a contradiction to minimality. Our 5 types of 2 -row-circuits are therefore exhaustive.

We now come to 3 -row-circuits, i.e., those incomplete $3 \times n$ latin rectangles, which are not completable, minimal with respect to this property and which do

type 2

type 3

type 4

type 5


| i | $\mathrm{N} \backslash\{i, j, \mathrm{l}\}$ | j | i |
| :---: | :---: | :---: | :---: |
| k | $\mathrm{N} \backslash\{\mathrm{j}, \mathrm{k}\}$ |  | j |
| j |  | i |  |

Figure 4. The 4 remaining types of 2-row-circuits, their availability matrices and completability for one element deleted in row 1 .
not contain any 1- or 2-row-circuit. Figure 5 exhibits 5 different types. The first of them exists for $n \geq 5$ and the remaining for $n \geq 4$.

As before, the following conditions are required:

- all symbols $i, j, k, l \in N$ have to be different with the exception of type 5 , in which we may have $l=j$ (if $l=j$ or $l=k$ in type 4 for instance, then $C$ would properly contain a 2 -row-circuit);
- in type 2 , the first 3 empty cells in rows $1,2,3$ have to be in different columns;
- in type 4 , the first 2 empty cells in rows 1,2 may appear in a same column;
- in type 5 , the first empty cell in row 3 must not appear in the column containing $k$ or $i$.
To show that this list is exhaustive, let us consider a 3-row-circuit $C$ together with

type 1

type 2

type 3

| $N \backslash\{i, j, k\}$ |  |  | $j$ |  |
| :--- | :--- | :--- | :--- | :--- |
|  | $N \backslash\{i, j, k\}$ |  | $k$ |  |
| $\neq j$ | $N \backslash\{i, j\}$ | $\neq k$ |  |  |

type 4

| $N \backslash\{i, j, k\}$ | $k$ |  |  |
| :---: | :---: | :---: | :---: |
| $N \backslash\{i, k, l\}$ |  | $i$ |  |
|  | $N \backslash\{i, k\}$ | $\neq 1$ | $\neq j$ |

type 5


| $N \backslash\{i, j, k\}$ |  | $j$ | k | i |
| :---: | :---: | :---: | :---: | :---: |
| i | $\mathrm{N} \backslash\{\mathrm{i}, \mathrm{j}, \mathrm{l}\}$ | j | l |  |
| $\mathrm{N} \backslash\{\mathrm{i}, \mathrm{j}\}$ |  |  | i | j |

$$
\left.\left[\begin{array}{l}
\mathrm{i} \\
\mathrm{j} \\
\mathrm{i} \\
\mathrm{i} \\
\mathrm{j}
\end{array}\left[\begin{array}{l}
\mathrm{i} \\
\mathrm{j}
\end{array}\right]\right] \begin{array}{l}
\mathrm{i} \\
\mathrm{j}
\end{array}\right]
$$



$$
\left[\begin{array}{ll}
k & k^{*} \\
i & {\left[\begin{array}{ll}
i \\
l
\end{array}\right]\left[\begin{array}{ll}
j^{*} & j \\
i \\
i &
\end{array}\right]}
\end{array}\right]
$$

| $k$ | $N \backslash\{i, j, k, m\}$ | $i$ | $j$ | $m$ |
| :---: | :---: | :---: | :---: | :---: |
| $j$ | $N \backslash\{i, j, k\}$ |  | $i$ | $k$ |
| $N \backslash\{i, l\}$ |  |  |  | i |



| i | $N \backslash\{i, j, k, m\}$ | k | j | m |
| :---: | :---: | :---: | :---: | :---: |
|  | $\mathrm{N} \backslash\{\mathrm{i}, \mathrm{k}$, | 1 | i | k |
| k | $N \backslash\{i, k\}$ |  |  |  |

Figure 5. All types of 3-row-circuits.
the associated availability matrices $A_{1}, A_{2}$ and $A_{3}$. We claim that $A_{3}$ is of size at most $3 \times 3$, i.e., there are at most 3 empty cells in row 3 of $C$. By assumption the first two rows of $C$ are completable. By Theorem 3 and for any such completion, there must be a set $I \subseteq S(3)$ which is contained in $t-|I|+1$ columns $C^{j}$, for $j \in J(3)$. Such an $I$ can only appear in rows 1 and 2 , and therefore, $t-|I|+1 \leq 2$ and $|I| \in\{1,2\}$, which implies $t \leq 3$. By symmetry, any other of the 3 rows of $C$ cannot have more than 3 empty cells.

Can there be more than one row with exactly 3 empty cells? For an answer to this question let row 3 be empty in columns $C^{n-2}, C^{n-1}$ and $C^{n}$, and let $I=\{i, j\}$. The only ways to make $I$ appear twice, say in columns $C^{n-1}$ and $C^{n}$, are the following:

1. symbol $i$ is in cells $1, n-1$ and $2, n$, and symbol $j$ is in cells $1, n$ and $2, n-1$, but then $C$ would contain a 1 -row-circuit (type 4 in Figure 3);
2. symbol $i$ is in cell $1, n-1$, and symbol $j$ is in cell $1, n$, the cells $2, n-1$ and $2, n$ being empty, and row 2 containing all symbols from $N \backslash\{i, j\}$, but then $C$ would contain a 2 -row-circuit (type 2 in Figure 4);
3. rows 1 and 2 both contain $N \backslash\{i, j\}$ in their first $n-2$ columns the remaining cells being empty.

Only case 3 applies and, therefore, row 3 is the only row with 3 empty cells. Obviously, $C$ induces a unique 3 -row-circuit (type 1 in Figure 5).

We are left with the situation that $A_{1}, A_{2}$ and $A_{3}$ are all of size at most $2 \times 2$. Since any of the 3 matrices has to share at least one line with one of the others, the possibility of size $1 \times 1$ for all 3 contradicts non-completability. The only possible case in which two of the matrices are of size $1 \times 1$ is illustrated in Figure 6.


| N\{i\} |  |
| :---: | :---: |
| $\mathrm{N}\{$ ji\} |  |
| $N\{i, j\}$ |  |

Figure 6
If symbols $i$ and $j$ are in cells $2, n-1$ and $1, n-1$, respectively, $C$ properly contains a 1-row-circuit of type 3 , and if both symbols appear in a same column $C^{j}, j \in\{1, \ldots, n-2\}, C$ properly contains a 3 -row-circuit of type 1 . If only symbol $i$ (or symbol $j$, respectively) appears in column $n-1$, then $C$ contains a 2 -row-circuit of type 3 . And finally, if symbols $i$ and $j$ are in different columns $C^{j}, j \in\{1, \ldots, n-2\}, C$ properly contains a 3 -row-circuit of type 2 . Altogether, $C$ cannot represent a circuit.

All other cases, in which one of the 3 matrices is of size $1 \times 1$, are depicted in Figure 7 (up to symmetries). They can be treated along the same line leading to the same conclusion that $C$ cannot be a circuit.


Figure 7
We still have to treat the case that all 3 matrices are of size $2 \times 2$. It turns out that any induced 3 -row-circuit is among those presented in Figure 5.

If two of the matrices, say $A_{1}$ and $A_{2}$, coincide, i.e., for row 1 and row 2 we have $S(1)=S(2)$ and $J(1)=J(2), C$ either contains a 3 -row-circuit of type 1 or it is completable.

If $A_{1}$ and $A_{2}$ just coincide in one column, the assumption on the completability of any pair of rows gives us a 3 -row-circuit of type 2 or 3 , or we obtain completability of $C$.

If finally, $A_{1}$ and $A_{2}$ coincide in just one cell, let $\alpha$ denote the number of cells that $A_{3}$ can have in common with the superposition of $A_{1}$ and $A_{2}$. In case that $\alpha=1$ the only way to make $C$ non-completable leads to a 3 -row-circuit of type 5 or, by symmetry, of type 4 , for $\alpha=2$ it is such a circuit of type 4 (with the first two empty cells in rows 1,2 appearing in a same column) or, by symmetry, of type 5 with $l=j$, and for $\alpha=3$ the assumption on the completability of any pair of rows always implies completability of $C$.

Altogether, we have shown:
Theorem 4. An incomplete $3 \times n$ latin rectangle is completable if and only if it does not contain any 1-row-circuit of type 1 to 4, 2-row-circuit of type 1 to 5 , or 3 -row-circuit of type 1 to 5 .

We just remark that by interchanging the role of column indices and symbols (so-called conjugacy), the number of types in each class can be reduced further to 3 , 4 and 3 for 1 -row, 2 -row and 3 -row-circuits, respectively.

To conclude this section we would like to show how most of the circuits arise as special cases of more general descriptions.
i) Let $L$ be an $r \times s$ latin rectangle of order $n$ such that $r+s=n+1, r \geq 2$, symbol $k$ not occurring in $L$ and the other members all appearing at least two times, in rows $1, \ldots, r$ and columns $1, \ldots, s$. By Ryser's theorem [12] $L$ cannot be completed, but the removal of any symbol leads to completability.
$L$ therefore represents a member of $\mathcal{C}_{n}$ covering in particular type 3 of the 3-row-circuits.
ii) Let $L$ be as above with the difference, that all symbols are from $\{1, \ldots, s\}$ and that $n-s-1$ arbitrary symbols have been removed. Again, by [12] $L$ cannot be completed but the removal of any symbol leads to completability, i.e., $L$ represents a member of $\mathcal{C}_{n}$ (already described in Euler et al. [6]) and covering type 1 of the 2 -row-circuits as well as type 1 of the 3 -row-circuits.
iii) Let $L$ be an $r \times s$ latin rectangle over $N$ and let symbol $k$ appear $l$ times in rows $r+1, \ldots, n$ and columns $s+1, \ldots, n$, at most once in a same row or column. In Euler et al. [6] we have shown that this incomplete latin square can be completed if and only if each symbol from $N \backslash\{k\}$ appears at least $r+s-n$ times in $L$ and symbol $k$ appears at least $r+s+l-n$ times in $L$. If $r+s=n-l+1$ and symbol $k$ does not appear in $L$ we obtain a member of $\mathcal{C}_{n}$ in general form covering in particular any 2 -row-circuit of type 3 . By conjugacy rows $\leftrightarrow$ symbols all types of 1 -row-circuits are members of $\mathcal{C}_{n}$, too.
iv) The last case to be treated will be type 2 of the 2-row-circuits. Again we start with an $r \times s$ latin rectangle $L$ as given in ii). Since $n-s-1=r-2$, we may choose one of the (at least) 2 rows, say row $r$, containing $s$ symbols. We remove these symbols from row $r$ and place symbols $s+1, \ldots, n$ (arbitrarily) in columns $s+1, \ldots, n$ of that same row. We obtain an incomplete latin square $L^{\prime}$ which is not completable, since any completion would have to contain symbols $1, \ldots, s$ in the first $s$ columns of row $r$. Removing any symbol from the first $r-1$ rows leads to completability, and moving any symbol in row $r$ to a cell within columns $1, \ldots, s$ also leads to completability, as in ii). Again, $L^{\prime}$ is shown to be a member of $\mathcal{C}_{n}$ covering type 2 of the 2-row-circuits.
v) Finally, type 4 of the 3 -row-circuits and type 5 of the 2 -row-circuits have already been studied in Euler [7] with respect to a general form. This was, however, only possible in the context of circulant latin rectangles.

## 3. Complete Descriptions of $\mathcal{C}_{n}$ for Small $n$

This section is on the generation of a complete description of $\mathcal{C}_{n}$ for small $n$ on a computer. For $n=3$ such a description consisting only of 1 - and 2-rowcircuits has already been given in Euler et al. [6]. Up to row- column- and symbol interchangements (so-called isotopy) and the exchange of row- column- or
symbol indices (i.e., conjugacy) we are left with two types which are represented in Figure 8.


Figure 8. All types of circuits for $n=3$.
As to $n=4$ we have generated the associated family of circuits by means of a computer program, whose basic idea is traversing the solution space with the application of heuristics (like branch-and-bound technique and prediction of properties).

In view of the previous results it is sufficient to exhibit those $4 \times 4$ arrays that contain all 4 symbols and no empty row or column. Up to isotopy and conjugacy we come up with 2 types, both 2 -row-circuits with respect to rows 1 and 2. Together with their availability matrices they are represented in Figure 9.


$$
\left.\left[\begin{array}{lll}
2^{*} & 2 & 2 \\
3^{*} & 3^{*} \\
4 & 4 & 3 \\
4^{*}
\end{array}\right] \begin{array}{l}
3 \\
4
\end{array}\right]
$$

Figure 9. The 2 remaining types of circuits for $n=4$.
These results allow us to present a complete description of $\mathcal{C}_{4}$ :

Theorem 5. An incomplete latin square of order 4 is completable if and only if, up to isotopy and conjugacy, it does not contain any of the following arrays as a subarray:


| 1 |  |  |  |
| :--- | :--- | :--- | :--- |
|  | 2 | 3 |  |
|  | 3 | 2 |  |
|  |  |  |  |


| 1 | 2 | 3 |  |
| :--- | :--- | :--- | :--- |
| 3 | 1 | 2 |  |
|  |  |  |  |
|  |  |  |  |


| 1 |  |  |  |
| :--- | :--- | :--- | :--- |
|  | 2 | 3 |  |
|  | 3 | 4 |  |
|  |  |  |  |


| 1 | 2 |  |  |
| :--- | :--- | :--- | :--- |
|  | 3 | 1 |  |
| 4 |  |  | 3 |
|  |  |  |  |


| 1 | 2 |  |  |
| :--- | :--- | :--- | :--- |
| 2 |  |  |  |
|  |  | 3 |  |
|  |  |  | 4 |


| 1 |  |  |  |
| :--- | :--- | :--- | :--- |
|  | 2 | 1 |  |
|  |  | 3 | 4 |
|  | 3 |  |  |

2-row-circuits

| 1 |  | 2 |  |
| :--- | :--- | :--- | :--- |
|  | 1 |  | 3 |
| 3 | 2 |  |  |
|  |  |  |  |

3-row-circuit

Figure 10. All types of circuits for $n=4$.

## 4. Conclusion and Future Work

A first direction for future research could be a complete characterization of 2-row-circuits. Also, to obtain a complete description of $\mathcal{C}_{n}$ via an extension of this work to $r \times n$ latin rectangles for any $r$, the idea arises of generalizing 1-row, 2 -row and 3 -row-circuits to $m$-row-circuits for $m>3$. For this a detailed study of our computational results should be helpful.

In this respect our results for a complete description of $\mathcal{C}_{5}$ are of much a wider scope than those for $\mathcal{C}_{4}$. They are currently being analysed.

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