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SHARP BOUNDS FOR THE NUMBER OF MATCHINGS IN GENERALIZED-THETA-GRAPHS

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Abstract

A generalized-theta-graph is a graph consisting of a pair of end vertices joined by k ($k \geq 3$) internally disjoint paths. We denote the family of all the n-vertex generalized-theta-graphs with k paths between end vertices by Θ_k^n. In this paper, we determine the sharp lower bound and the sharp upper bound for the total number of matchings of generalized-theta-graphs in Θ_k^n. In addition, we characterize the graphs in this class of graphs with respect to the mentioned bounds.

Keywords: generalized-theta-graph, matching, Fibonacci number, Hosoya index.

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1. Introduction

A *matching* of a graph $G = (V(G), E(G))$ is a subset $M \subseteq E(G)$ for which no two distinct edges of M share a common vertex. A *k-matching* is a matching consisting of k edges. The number of k-matchings of G is denoted by $m(G, k)$. It is convenient to set $m(G, 0) = 1$. We denote the number of matchings of a graph G by $t_m(G)$ and define as $\sum_{k=0}^{\lfloor n/2 \rfloor} m(G, k)$, where n is the number of the vertices of G. This invariant is also called the *Hosoya index* in the literature. It is of interest in combinatorial chemistry. It is applied for studying some physico-chemical properties such as entropy and boiling point. This invariant has been extensively studied in some prescribed classes of graphs. Some of the results are as follows.

If G is an n-vertex unicyclic graph then $t_m(G) \leq f(n+1) + f(n-1)$ with equality holding if and only if $G \cong C_n$, where $f(n)$ denotes the nth *Fibonacci number* [8, 2]. In [10] the maximal and the minimal n-vertex graphs with a given clique number with respect to the number of matchings have been characterized. It is shown in [1] that the sharp upper bound for the number of matchings of n-vertex bicyclic graphs is $f(n+1)+f(n-1)+2f(n-3)$ and the extremal graph with respect to this bound has been characterized. In [4] the sharp upper bound and sharp lower bound for the number of matchings of chain hexagonal cacti has been found and the extremal chain hexagonal cacti have been characterized. If G is an n-vertex connected tricyclic graph then $t_m(G) \geq 4n - 6$ [3]; in the same reference the n-vertex connected tricyclic graphs whose number of matchings are $4n - 6$ have been characterized. In [7] the minimal, second minimal, and third minimal number of matchings for fully loaded unicyclic graphs have been found and the fully loaded unicyclic graphs with respect to the bounds have been characterized.

We denote the family of all the n-vertex generalized-theta-graphs with k paths between end vertices by Θ_k^n. In this paper, we determine the sharp lower bound and the sharp upper bound for the number of matchings of generalized-theta-graphs in Θ_k^n. In addition, we characterize the generalized-theta-graphs in Θ_k^n with respect to the mentioned bounds. The rest of the paper is organized as follows. After some preliminaries in Section 2, we present some results in Section 3 for decreasing or increasing the number of matchings of the graph while the graph is remaining in Θ_k^n. In Section 4, we determine the sharp lower bound and the sharp upper bound of the number of matchings of the generalized-theta-graphs in Θ_k^n. In addition, we characterize the generalized-theta-graphs in Θ_k^n with respect to the mentioned bounds in the same section.

2. Preliminaries

In this section, we introduce some preliminaries that we are using throughout the paper. Let $G = (V(G), E(G))$ be a simple connected graph with the vertex set $V(G)$ and the edge set $E(G)$. Let u and v be two adjacent vertices of G; we denote the edge joining these vertices by uv. For any $v \in V(G)$, we denote the neighbors of v by $N_G(v) = \{u|uv \in E(G)\}$. $d_G(v) = |N_G(v)|$ is the degree of v in G. We denote the path on n vertices by P_n. A *multiset* is defined by assuming that for a set $\mathcal{A}$ an element occurs a finite number of times. This number is called the *occurrence number* and is denoted by $OC_{\mathcal{A}}(.)$. The sum of two multisets $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathcal{A} \uplus \mathcal{B}$ and

$$OC_{\mathcal{A}\uplus\mathcal{B}}(.) = OC_{\mathcal{A}}(.) + OC_{\mathcal{B}}(.).$$

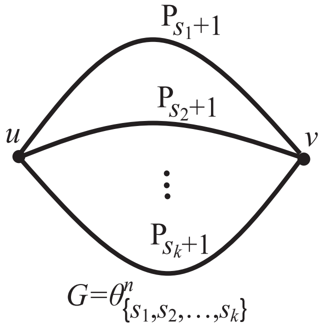

Figure 1. A generalized-theta-graph.

For example, suppose that $\mathcal{A} = \{1,1,2,3,1,2\}$ and $\mathcal{B} = \{1,1,2\}$ then $\mathcal{A} \uplus \mathcal{B} = \{1,1,1,1,1,2,2,2,3\}$. The interested reader is referred to [9] for further details about this topic.

A *generalized-theta-graph* $\theta^n_{\{s_1,s_2,\ldots,s_k\}}$ is an n-vertex graph consisting of a pair of end vertices joined by k ($k \geq 3$) internally disjoint paths of lengths $s_1, \ldots, s_k \geq 1$ (see Figure 1), the set depicted in the subscript is a multiset. Obviously, a generalized-theta-graph can be characterized by its order and the lengths of its internally disjoint paths. By $\mathcal{P}\left(\theta^n_{\{s_1,s_2,\ldots,s_k\}}\right)$, we mean the set of the internally disjoint paths of $\theta^n_{\{s_1,s_2,\ldots,s_k\}}$. By Θ^n_k, we denote the family of all n-vertex generalized-theta-graphs consisting of k paths. Let $G = (V(G), E(G))$ and $G' = (V(G'), E(G'))$ be two graphs such that $V(G) \cap V(G') = \emptyset$. Suppose that $v_1, v_2, \ldots, v_k \in V(G)$ and $v'_1, v'_2, \ldots, v'_k \in V(G')$ ($k \geq 1$). By $G \rhd v_1 = v'_1, v_2 = v'_2, \ldots, v_k = v'_k \lhd G'$, we mean the graph obtained from identifying v_i and v'_i for $i = 1, 2, \ldots, k$. Suppose that $P = v_1 v_2 \ldots v_{k-1} v_k$ is a path, we denote the internal vertices of P by $V[P]$ that means $V[P] = \{v_2, \ldots, v_{k-1}\}$.

We use the following results throughout the paper.

Lemma 1 [6]**.** *If v is a vertex and $e = uv$ is an edge of G, then*

$$t_m(G) = t_m(G-e) + t_m(G - \{u,v\}),$$
$$t_m(G) = t_m(G-v) + \textstyle\sum_{x \in N_G(v)} t_m(G - \{v,x\}).$$

Lemma 2 [5]**.** *If G is a graph with components $G_1, G_2, G_3, \ldots, G_k$ then*

$$t_m(G) = \textstyle\prod_{i=1}^{k} t_m(G_i).$$

Lemma 3 [11]**.** *Let $n = 4s+r$ be a positive integer number, $0 \le r \le 3$ and $s \ge 1$.*
(1) *If $r \in \{0,1\}$, then*

$$\begin{aligned} f(1)f(n+1) &> f(3)f(n-1) > \cdots > f(2s+1)f(2s+r+1) \\ &> f(2s)f(2s+r+2) > f(2s-2)f(2s+r+4) > \cdots \\ &> f(4)f(n-2) > f(2)f(n). \end{aligned}$$

(2) *If $r \in \{2,3\}$, then*

$$\begin{aligned} f(1)f(n+1) &> f(3)f(n-1) > \cdots > f(2s+1)f(2s+r+1) \\ &> f(2s+2)f(2s+r) > f(2s)f(2s+r+2) > \cdots \\ &> f(4)f(n-2) > f(2)f(n). \end{aligned}$$

3. Transformations

In this section, we present some results for increasing or decreasing the number of matchings of generalized-theta-graphs in Θ_k^n. In fact, we present some procedures by which a given graph in Θ_k^n can be transformed to another one in Θ_k^n with larger or smaller number of matchings. We call them the *increasing transformation* or the *decreasing transformation*, respectively.

Let u and v be two adjacent vertices of a graph H. For constructing a simple graph from identifying these vertices on two non-adjacent vertices of C_4 or C_5 we have just one choice. If we use the cycles C_r ($r \ge 6$) then the number of non-isomorphism constructed simple graphs is more than one. The following proposition determines which one has the largest number of matchings and which one has the smallest number of matchings.

Proposition 4. *Let H be a simple graph and $C_r = w_0w_1 \dots w_{r-1}w_0$ be a cycle of length r ($r \ge 6$), where $V(H) \cap V(C_r) = \emptyset$. Suppose that u and v are two adjacent vertices of H. If $G_s := H \rhd u = w_0, v = w_s \lhd C_r$ where $2 \le s \le r-2$, then*
(1) *$t_m(G_s) \le t_m(G_3)$ with equality holding if and only if $s = 3$ or $r - s = 3$,*
(2) *$t_m(G_2) \le t_m(G_s)$ with equality holding if and only if $s = 2$ or $r - s = 2$.*

Proof. We can assume, without loss of generality, that $d_H(u) \ge d_H(v)$. Suppose that $N_H(u) = \{u_i | i = 1, 2, \dots, p\}$ and $N_H(v) - \{u\} = \{v_j | j = 1, 2, \dots, q\}$ where $u_1 = v$. At first we prove the proposition under the condition $d_H(v) = 1$, therefore, $N_H(v) = \{u\}$. Let us denote the vertices obtained from identifying u on w_0 and v on w_s by u' and v', respectively. At first by recursively using the first part of Lemma 1 and deleting the edges in the set $\{u'u_i | i = 1, \dots, p\}$, we have

$$
\begin{aligned}
t_m(G_s) &= t_m(G_s - u'u_1) + t_m(G_s - \{u', u_1\}) \\
&= t_m(G_s - \{u'u_1, u'u_2\}) + t_m(G_s - \{u'u_1, u', u_2\}) + t_m(G_s - \{u', u_1\}) \\
&= \cdots \\
&= t_m\left(G_s - \bigcup_{i=1}^{p} \{u'u_i\}\right) + \sum_{i=1}^{p} t_m(G_s - \{u', u_i\}) \\
&= t_m(H - \{u, v\}) t_m(C_r) \\
&+ \sum_{i=2}^{p} t_m(H - \{u, v, u_i\}) t_m(P_{r-1}) \\
&+ t_m(H - \{u, v\}) t_m(P_{s-1}) t_m(P_{r-s-1}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
t_m(G_s) &= t_m(H - \{u, v\})(f(r+1) + f(r-1)) \\
&+ \sum_{i=2}^{p} t_m(H - \{u, v, u_i\}) f(r) \\
&+ t_m(H - \{u, v\}) f(s) f(r-s).
\end{aligned}
$$

The first two terms of the right hand side of the above equation are fixed and only the last one can be varied, by variation of the value of s. Note that the second term does not exist if $d_H(u) < 2$. Therefore, by Lemma 3, the function $t_m(G_s)$ takes its maximum value for $s = 3$ or $r - s = 3$ and it takes its minimum value for $s = 2$ or $r - s = 2$. It proves the assertion for this case.

Now, suppose that $d_H(v) > 1$. By using a similar method and recursively deleting the edges in the set $\{u'u_1, \ldots, u'u_p\}$ and then recursively deleting the edges in the set $\{v'v_1, \ldots, v'v_q\}$ we have the following relations.

$$
\begin{aligned}
t_m(G_s) &= t_m(G_s - u'u_1) + t_m(G_s - \{u', u_1\}) \\
&= t_m(G_s - \{u'u_1, u'u_2\}) + t_m(G_s - \{u'u_1, u', u_2\}) + t_m(G_s - \{u', u_1\}) = \cdots \\
&= t_m\left(G_s - \bigcup_{i=1}^{p} \{u'u_i\}\right) + \sum_{i=2}^{p} t_m(G_s - \{u', u_i\}) + t_m(G_s - \{u', u_1\}) \\
&= \cdots \\
&= t_m\left(G_s - \bigcup_{i=1}^{p} \bigcup_{j=1}^{q} \{u'u_i, v'v_j\}\right) + \sum_{j=1}^{q} t_m(G_s - \bigcup_{i=1}^{p} \{u'u_i, v', v_j\}) \\
&+ \sum_{i=2}^{p} t_m\left(G_s - \bigcup_{\substack{j=1 \\ v_j \neq u_i}}^{q} \{u', u_i, v'v_j\}\right) \\
&+ \sum_{i=2}^{p} \sum_{\substack{j=1 \\ v_j \neq u_i}}^{q} t_m\left(G_s - \{u', u_i, v', v_j\}\right) + t_m(G_s - \{u', u_1\}) \\
&= t_m(H - \{u, v\}) t_m(C_r) \\
&+ \left(\sum_{j=1}^{q} t_m(H - \{u, v, v_j\}) + \sum_{i=2}^{p} t_m(H - \{u, v, u_i\})\right) t_m(P_{r-1}) \\
&+ \sum_{i=2}^{p} \sum_{\substack{j=1 \\ v_j \neq u_i}}^{q} t_m(H - \{u, v, u_i, v_j\}) t_m(P_{s-1}) t_m(P_{r-s-1}) \\
&+ t_m(H - \{u, v\}) t_m(P_{s-1}) t_m(P_{r-s-1}).
\end{aligned}
$$

Therefore, we have the following equation.

$$\begin{aligned} t_m(G_s) &= t_m(H-\{u,v\})(f(r+1)+f(r-1)) \\ &+ \sum_{j=1}^{q} t_m(H-\{u,v,v_j\})f(r)+\sum_{i=2}^{p} t_m(H-\{u,v,u_i\})f(r) \\ &+ \sum_{i=2}^{p}\sum_{\substack{j=1\\ v_j\neq u_i}}^{q} t_m(H-\{u,v,u_i,v_j\})f(s)f(r-s) \\ &+ t_m(H-\{u,v\})f(s)f(r-s). \end{aligned}$$

We only need to consider the last two terms of the right hand side of the above equation because the other terms are fixed, by variation of the value of s. Note that the third and fourth terms do not exist if $d_H(u)<2$. We conclude by Lemma 3 that the function $t_m(G_s)$ takes its maximum value if we set $s:=3$ or $r-s:=3$ and takes its minimum value if we set $s:=2$ or $r-s:=2$. That means, we prove the assertion for this case too.

■

By the following proposition we determine the extremal graphs obtained from identifying two non-adjacent vertices of a graph on two distinct vertices of a cycle.

Proposition 5. *Let H be a simple graph and $C_r = w_0w_1\dots w_{r-1}w_0$ be a cycle of length r $(r\geq 4)$, where $V(H)\cap V(C_r)=\emptyset$. Let u and v be two non-adjacent and non-isolated vertices of H. Suppose that $G_s := H \rhd u = w_0, v = w_s \lhd C_r$ where $1\leq s\leq r-1$.*

(1) *If $H\cong P_3$, then $t_m(G_s)=f(r+3)$ for all $s=1,\dots,r-1$.*
(2) *If $H\ncong P_3$ and $4\leq r\leq 5$, then $t_m(G_s)\leq t_m(G_1)$ with equality holding if and only if $s=1$ or $s=r-1$.*
(3) *If $H\ncong P_3$, $r\geq 6$, and $s\notin\{1,r-1\}$, then $t_m(G_s)\leq t_m(G_3)<t_m(G_1)=t_m(G_{r-1})$ with equality holding if and only if $s=3$ or $s=r-3$.*
(4) *If $H\ncong P_3$ and $r\geq 4$, then $t_m(G_2)\leq t_m(G_s)$ with equality holding if and only if $s=2$ or $r-s=2$.*

Proof. If $H\cong P_3$ then u and v are the end vertices of H. Let us denote the internal vertex of H by w. It follows that $t_m(G_s)=t_m(G_s-w)+t_m(G_s-\{w,u\})+t_m(G_s-\{w,v\})=t_m(C_r)+t_m(P_{r-1})+t_m(P_{r-1})=f(r-1)+f(r+1)+2f(r)=f(r+3)$.

Now, suppose that $H\ncong P_3$ and $d_H(u)=p$ $(p\geq 1)$ and $d_H(v)=q$ $(q\geq 1)$. Assume that $N_H(u)=\{u_i|i=1,\dots,p\}$ and $N_H(v)=\{v_j|j=1,\dots,q\}$. Let us denote the vertices obtained from identifying u on w_0 and v on w_s by u' and v', respectively. The following relations follow by recursively deleting the edges in $\{u'u_1,\dots,u'u_p\}$ at the first step and then recursively deleting the edges in $\{v'v_1,\dots,v'v_q\}$ at the second step.

$$\begin{aligned}
t_m(G_s) &= t_m(G_s - u'u_1) + t_m(G_s - \{u', u_1\}) \\
&= t_m(G_s - \{u'u_1, u'u_2\}) + t_m(G_s - \{u'u_1, u', u_2\}) \\
&+ t_m(G_s - \{u', u_1\}) = \cdots \\
&= t_m\left(G_s - \bigcup\nolimits_{i=1}^{p}\{u'u_i\}\right) + \sum\nolimits_{i=1}^{p} t_m(G_s - \{u', u_i\}) \\
&= \cdots \\
&= t_m\left(G_s - \bigcup\nolimits_{i=1}^{p}\bigcup\nolimits_{j=1}^{q}\{u'u_i, v'v_j\}\right) + \sum\nolimits_{j=1}^{q} t_m\left(G_s - \bigcup\nolimits_{i=1}^{p}\{u'u_i, v', v_j\}\right) \\
&+ \sum\nolimits_{i=1}^{p} t_m\left(G_s - \bigcup\nolimits^{q}_{\substack{j=1\\ v_j\neq u_i}}\{u', u_i, v'v_j\}\right) \\
&+ \sum\nolimits_{i=1}^{p}\sum\nolimits^{q}_{\substack{j=1\\ v_j\neq u_i}} t_m(G_s - \{u', u_i, v', v_j\}) \\
&= t_m(H - \{u, v\})t_m(C_r) \\
&+ \left(\sum\nolimits_{j=1}^{q} t_m(H - \{u, v, v_j\}) + \sum\nolimits_{i=1}^{p} t_m(H - \{u, v, u_i\})\right) t_m(P_{r-1}) \\
&+ \sum\nolimits_{i=1}^{p}\sum\nolimits^{q}_{\substack{j=1\\ v_j\neq u_i}} t_m(H - \{u, v, u_i, v_j\})t_m(P_{s-1})t_m(P_{r-s-1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
t_m(G_s) &= t_m(H - \{u, v\})(f(r+1) + f(r-1)) + (\sum\nolimits_{j=1}^{q} t_m(H - \{u, v, v_j\}) \\
&+ \sum\nolimits_{i=1}^{p} t_m(H - \{u, v, u_i\}))f(r) \\
&+ \sum\nolimits_{i=1}^{p}\sum\nolimits^{q}_{\substack{j=1\\ v_j\neq u_i}} t_m(H - \{u, v, u_i, v_j\})f(s)f(r-s).
\end{aligned}$$

Analysis similar to that in the proof of the preceding proposition proves the assertion.

■

The following theorem presents a transformation for the increasing number of matchings of a generalized-theta-graph in Θ_k^n.

Theorem 6. *Suppose that $k \geq 3$ and $G \cong \theta^n_{\{s_1, s_2, \ldots, s_k\}}$ is a generalized-theta-graph in Θ_k^n.*

(1) *If $1 \notin \{s_1, s_2, \ldots, s_k\}$, then $t_m\left(\theta^n_{\uplus_{m=1, m\neq i,j}^{k}\{s_m\}\uplus\{1, s_i+s_j-1\}}\right) > t_m(G)$, for all $1 \leq i < j \leq k$.*

(2) *If $1 \in \{s_1, s_2, \ldots, s_k\}$ and for some $1 \leq i < j \leq k$, $s_i, s_j \notin \{1, 3\}$ and $s_i + s_j > 5$, then $t_m\left(\theta^n_{\uplus_{m=1, m\neq i,j}^{k}\{s_m\}\uplus\{3, s_i+s_j-3\}}\right) > t_m(G)$.*

Proof. Suppose that u and v are the end vertices of G. Assume that $k \geq 4$ and P' and P'' are two arbitrary paths of $\mathcal{P}(G)$ of lengths s_i and s_j, respectively, such that $1 \notin \{s_1, s_2\}$. Let $H = G - (V[P'] \cup V[P''])$. Suppose that $P_1 = x_0x_1 \ldots x_{s_i}$ and $P_2 = y_0y_1 \ldots y_{s_j}$ are two paths that are isomorphic with P' and P'' respectively, such that $\{x_0, x_1, \ldots, x_{s_i}\} \cap \{y_0, y_1, \ldots, y_{s_j}\} = \emptyset$. Obviously, $P_1 \rhd x_0 = y_0, x_{s_i} = y_{s_j} \lhd P_2$ is a cycle of length $s_i + s_j$. Let us denote the vertices obtained from identifying x_0 on y_0 and x_{s_i} on y_{s_j} by w and z, respectively. Therefore, $G \cong H \rhd u = w, v = z \lhd C$. Since u and v are non-isolated vertices of H, by using of Propositions 4 and 5 we complete the proof of this case.
Now assume that $k = 3$ and $G \cong \theta^n_{\{s_1,s_2,s_3\}}$. Suppose that P is a shortest path in $\mathcal{P}(G)$ of length s for some $s \in \{s_1, s_2, s_3\}$. Let $H \cong P_{s+1}$ whose end vertices are denoted by w and z. Obviously, $H' := G - V[P]$ is a cycle and $G \cong H \rhd w = u, z = v \lhd H'$. Therefore, if $s = 1$ by Proposition 4 and if $s > 1$ by Proposition 5 the assertion follows for this case too. ■

A transformation for decreasing the number of matchings of generalized-theta-graphs in Θ^n_k is summarized by the following theorem. It can be proved by Propositions 4 and 5 and a similar method shown in the proof of the previous theorem.

Theorem 7. *Suppose that $k \geq 3$ and $G \cong \theta^n_{\{s_1,s_2,\ldots,s_k\}}$ is a generalized-theta-graph in Θ^n_k. If $1 \leq i < j \leq k$ and $2 \notin \{s_i, s_j\}$, then $t_m\left(\theta^n_{\uplus^k_{m=1,m\neq i,j}\{s_m\}\uplus\{2,s_i+s_j-2\}}\right) < t_m(G)$.*

4. Characterizing the Extremal Generalized-theta-graphs in Θ^n_k

In this section, we determine the sharp upper bound and the sharp lower bound for the number of matchings of generalized-theta-graphs in Θ^n_k. In addition, the generalized-theta-graphs with respect to the bounds are characterized. By the following lemma, we calculate the number of matchings of generalized-theta-graphs in Θ^n_k.

Lemma 8. *Let $G = \theta^n_{\{s_1,s_2,\ldots,s_k\}}$ be a generalized-theta-graph in Θ^n_k,*

(1) *If $1 \notin \{s_1, \ldots, s_k\}$, then*
$$t_m(G) = \prod_{i=1}^k f(s_i) + 2\sum_{t=1}^k f(s_t - 1)\prod_{i=1,i\neq t}^k f(s_i)$$
$$+2\sum_{i=1,j<i}^k f(s_i - 1)f(s_j - 1)\prod_{t=1,t\neq i,j}^k f(s_t)$$
$$+\sum_{t=1}^k f(s_t - 2)\prod_{i=1,i\neq t}^k f(s_i).$$

(2) *If $1 \in \{s_1, \ldots, s_k\}$, then*

$$t_m(G) = 2\prod_{i=1}^{k-1} f(s_i) + 2\sum_{t=1}^{k-1} f(s_t-1)\prod_{i=1,i\neq t}^{k-1} f(s_i)$$
$$+2\sum_{i=1,j<i}^{k-1} f(s_i-1)f(s_j-1)\prod_{t=1,t\neq i,j}^{k-1} f(s_t)$$
$$+\sum_{t=1}^{k-1} f(s_t-2)\prod_{i=1,i\neq t}^{k-1} f(s_i).$$

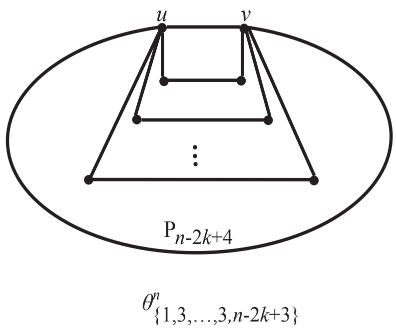

Figure 2. The extremal n-vertex generalized-theta-graph with the maximum number of matchings.

Proof. Let P^i (for $i = 1, \ldots, k$) denote the path of length s_i and u and v be two end vertices of $\theta^n_{\{s_1,s_2,\ldots,s_k\}}$. We denote the vertices adjacent to u and v in P^i by $u_{i,1}$ and $v_{i,1}$, respectively. The second part of Lemma 1 and deleting the vertices u and v, we have the following.

$$\begin{aligned}
t_m(G) &= t_m(G-u) + \sum_{t=1}^{k} t_m(G-\{u,u_{t,1}\}) \\
&= t_m(G-\{u,v\}) + \sum_{t=1}^{k} t_m(G-\{u,v,v_{t,1}\}) \\
&+ \sum_{t=1}^{k} t_m(G-\{u,v,u_{t,1}\}) + \sum_{t=1}^{k}\sum_{\substack{j=1\\ j\neq t}}^{k} t_m(G-\{u,v,u_{t,1},v_{j,1}\}) \\
&+ \sum_{t=1}^{k} t_m(G-\{u,v,u_{t,1},v_{t,1}\}) \\
&= \prod_{i=1}^{k} t_m(P_{s_i-1}) + \sum_{t=1}^{k}\left(t_m(P_{s_t-2})\prod_{\substack{i=1\\ i\neq t}}^{k} t_m(P_{s_i-1})\right) \\
&+ \sum_{t=1}^{k}\left(t_m(P_{s_t-2})\prod_{\substack{i=1\\ i\neq t}}^{k} t_m(P_{s_i-1})\right) \\
&+ 2\sum_{\substack{t=1\\ j<t}}^{k}\left(t_m(P_{s_t-2})t_m(P_{s_j-2})\prod_{\substack{i=1\\ i\neq t,j}}^{k} t_m(P_{s_i-1})\right) \\
&+ \sum_{t=1}^{k}\left(t_m(P_{s_t-3})\prod_{\substack{i=1\\ i\neq t}}^{k} t_m(P_{s_i-1})\right) \\
&= \prod_{i=1}^{k} f(s_i) + 2\sum_{t=1}^{k} f(s_t-1)\prod_{\substack{i=1\\ i\neq t}}^{k} f(s_i)
\end{aligned}$$

$$+ \quad 2\sum_{\substack{t=1\\ j<t}}^{k} f(s_t-1)f(s_j-1)\prod_{\substack{i=1\\ i\neq t,j}}^{k} f(s_i) + \sum_{t=1}^{k}(f(s_t-2))\prod_{\substack{i=1\\ i\neq t}}^{k} f(s_i).$$

Similar arguments apply to the other case of the lemma. ■

The following theorem determines the sharp upper bound for the number of matchings of the generalized-theta-graphs in Θ_k^n. The bound is stated in terms of n (the number of vertices of the graph) and k (the number of internally disjoint paths of the graph). It also shows that $\theta^n_{\{1,3,\ldots,3,n-2k+3\}}$ (see Figure 2) is the unique extremal generalized-theta-graph in Θ_k^n with maximum number of matchings.

Theorem 9. *If G is an arbitrary generalized-theta-graph in Θ_k^n and $n > 2k-2$, then*

$$\begin{aligned} t_m(G) &\leq t_m\left(\theta^n_{\{1,3,\ldots,3,n-2k+3\}}\right) \\ &= \left(2^{k-1}+(k-2)2^{k-2}+(k-2)(k-3)2^{k-3}+(k-2)2^{k-3}\right)f(n-2k+3) \\ &+ \left(2^{k-1}+(k-2)2^{k-2}\right)f(n-2k+2)+2^{k-2}f(n-2k+1), \end{aligned}$$

with equality holding if and only if $G \cong \theta^n_{\{1,3,\ldots,3,n-2k+3\}}$.

Proof. By using recursively Theorem 6 and Lemma 8 the assertion follows. ■

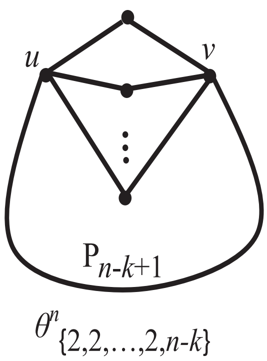

Figure 3. The extremal n-vertex generalized-theta-graph with the minimum number of matchings.

The following theorem determines the sharp lower bound in terms of n (the number of vertices of a graph) and k (the number of internally disjoint paths of a graph). It characterizes the smallest generalized-theta-graph in Θ_k^n with respect to the number of matchings. It shows that $\theta^n_{\{2,2,\ldots,2,n-k\}}$ is the unique generalized-theta-graph in Θ_k^n with minimum number of matchings see (Figure 3).

Theorem 10. *If G is an arbitrary generalized-theta-graph in Θ_k^n and $n \geq k+2$, then*

$$\begin{aligned} t_m(G) &\geq t_m(\theta^n_{\{2,2,\ldots,2,n-k\}}) \\ &= (k^2-k+1)f(n-k)+(k+1)f(n-k-1)+f(n-k-2), \end{aligned}$$

with equality holding if and only if $G \cong \theta^n_{\{2,2,\ldots,2,n-k\}}$.

Proof. By using recursively Theorem 7 and Lemma 8 the assertion follows. ∎

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References

[1] H. Deng, *The largest Hosoya index of $(n, n+1)$-graphs*, Comput. Math. Appl. **56** (2008) 2499–2506.
doi:10.1016/j.camwa.2008.05.020

[2] H. Deng and S. Chen, *The extremal unicyclic graphs with respect to Hosoya index and Merrifield-Simmons index*, MATCH Commun. Math. Comput. Chem. **59** (2008) 171–190.

[3] A. Dolati, M. Haghighat, S. Golalizadeh and M. Safari, *The smallest Hosoya index of connected tricyclic graphs*, MATCH Commun. Math. Comput. Chem. **65** (2011) 57–70.

[4] T. Došlić and F. Måløy, *Chain hexagonal cacti: Matchings and independent sets*, Discrete Math. **310** (2010) 1676–1690.
doi:10.1016/j.disc.2009.11.026

[5] I. Gutman and O.E. Polansky, Mathematical Concepts in Organic Chemistry (Springer-Verlag, Berlin, 1986).

[6] H. Hosoya, *Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons*, Bull. Chem. Soc. Jpn. **44** (1971) 2332–2339.
doi:10.1246/bcsj.44.2332

[7] H. Hua, *Minimizing a class of unicyclic graphs by means of Hosoya index*, Math. Comput. Modelling **48** (2008) 940–948.
doi:10.1016/j.mcm.2007.12.003

[8] J. Ou, *On extremal unicyclic molecular graphs with maximal Hosoya index*, Discrete Appl. Math. **157** (2009) 391–397.
doi:10.1016/j.dam.2008.06.006

[9] A. Syropoulos *Mathematics of multisets*, Multiset Processing, LNCS 2235, C.S. Calude, G. Păun, G. Rozenberg, A. Salomaa (Eds.), (Springer-Verlag, Berlin, 2001) 347–358.
doi:10.1007/3-540-45523-X_17

[10] K. Xu, *On the Hosoya index and the Merrifield-Simmons index of graphs with a given clique number*, Appl. Math. Lett. **23** (2010) 395–398.
doi:10.1016/j.aml.2009.11.005

[11] H. Zhao and X. Li, *On the Fibonacci numbers of trees*, Fibonacci Quart. **44** (2006) 32-38.

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