Discussiones Mathematicae
Graph Theory 33 (2013) 247–260
doi:10.7151/dmgt.1645

4-TRANSITIVE DIGRAPHS I: THE STRUCTURE OF STRONG 4-TRANSITIVE DIGRAPHS

César Hernández-Cruz

Instituto de Matemáticas
Universidad Nacional Autónoma de México
Ciudad Universitaria, México, D.F., C.P. 04510, México

e-mail: cesar@matem.unam.mx

Abstract

Let D be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively. A digraph D is transitive if for every three distinct vertices $u, v, w \in V(D)$, $(u, v), (v, w) \in A(D)$ implies that $(u, w) \in A(D)$. This concept can be generalized as follows: A digraph is k-transitive if for every $u, v \in V(D)$, the existence of a uv-directed path of length k in D implies that $(u, v) \in A(D)$. A very useful structural characterization of transitive digraphs has been known for a long time, and recently, 3-transitive digraphs have been characterized.

In this work, some general structural results are proved for k-transitive digraphs with arbitrary $k \geq 2$. Some of this results are used to characterize the family of 4-transitive digraphs. Also some of the general results remain valid for k-quasi-transitive digraphs considering an additional hypothesis. A conjecture on a structural property of k-transitive digraphs is proposed.

Keywords: digraph, transitive digraph, quasi-transitive digraph, 4-transitive digraph, k-transitive digraph, k-quasi-transitive digraph.

2010 Mathematics Subject Classification: 05C20, 05C75.

1. Introduction

In this work, $D = (V(D), A(D))$ will denote a finite digraph without loops or multiple arcs in the same direction, with vertex set $V(D)$ and arc set $A(D)$. For general concepts and notations we refer the reader to [1], [3] and [6], particularly we will use the notation of [6] for walks, if $W = (x_0, x_1, \ldots, x_n)$ is a walk and $i < j$ then $x_i W x_j$ will denote the subwalk $(x_i, x_{i+1}, \ldots, x_{j-1}, x_j)$ of W. Union of walks will be denoted by concatenation or with $\cup$. For a vertex $v \in V(D)$, we define the

out-neighborhood of v in D as the set $N_D^+(v) = \{u \in V(D) | (v,u) \in A(D)\}$; when there is no possibility of confusion we will omit the subscript D. The elements of $N^+(v)$ are called the *out-neighbors* of v, and the *out-degree* of v, $d_D^+(v)$, is the number of out-neighbors of v. Definitions of *in-neighborhood*, *in-neighbors* and *in-degree* of v are analogously given. We say that a vertex u *reaches a vertex* v in D if a directed uv-directed path (a path with initial vertex u and terminal vertex v) exists in D. An arc $(u,v) \in A(D)$ is called *asymmetrical* (resp. *symmetrical*) if $(v,u) \notin A(D)$ (resp. $(v,u) \in A(D)$).

If D is a digraph and $X, Y \subseteq V(D)$, then an XY-arc is an arc with initial vertex in X and terminal vertex in Y. If $X \cap Y = \varnothing$, then $X \to Y$ will denote that $(x,y) \in A(D)$ for every $x \in X$ and $y \in Y$. Again, if X and Y are disjoint, then $X \Rightarrow Y$ will denote that there are not YX-arcs in D. When $X \to Y$ and $X \Rightarrow Y$, we will simply write $X \mapsto Y$. If D_1, D_2 are subdigraphs of D, we will abuse notation to write $D_1 \to D_2$ or D_1D_2-arc, instead of $V(D_1) \to V(D_2)$ or $V(D_1)V(D_2)$-arc, respectively. Also, if $X = \{v\}$, we will abuse notation to write $v \to Y$ or vY-arc instead of $\{v\} \to Y$ or $\{v\}Y$-arc, respectively. Analogously, if $Y = \{v\}$.

A digraph is *strongly connected* (or *strong*) if for every $u, v \in V(D)$, there exists a uv-directed path, *i.e.*, a directed path with initial vertex u and terminal vertex v. A *strong component* (or *component*) of D is a maximal strong subdigraph of D. The *condensation* of D is the digraph $D^\star$ with $V(D^\star)$ equal to the set of all strong components of D, and $(S,T) \in A(D^\star)$ if and only if there is an ST-arc in D. Clearly $D^\star$ is an acyclic digraph (a digraph without directed cycles), and thus, it has both vertices of out-degree equal to zero and vertices of in-degree equal to zero. A *terminal component* of D is a strong component T of D such that $d_{D^\star}^+(T) = 0$. An *initial component* of D is a strong component S of D such that $d_{D^\star}^-(S) = 0$.

A *biorientation* of the graph G is a digraph D obtained from G by replacing each edge $\{x,y\} \in E(G)$ by either the arc (x,y) or the arc (y,x) or the pair of arcs (x,y) and (y,x). A *semicomplete* digraph is a biorientation of a complete graph. An *orientation* of a graph G is an asymmetrical biorientation of G; thus, an *oriented* graph is an asymmetrical digraph. A tournament is an orientation of a complete graph. An orientation of a digraph D is a maximal asymmetrical subdigraph of D. The *complete orientation* of a graph G is the digraph $\overleftrightarrow{G}$ obtained by replacing each edge $xy \in E(G)$ by the arcs (x,y) and (y,x). A *complete* digraph is a complete biorientation of a complete graph. A digraph D is *cyclically k-partite* if there exists a partition $\{V_0, V_1, \ldots, V_{k-1}\}$ of $V(D)$ such that every arc of D is a V_iV_{i+1}-arc (mod k).

Let D be a digraph with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$ and H_1, $H_2, \ldots,$ H_n be vertex disjoint digraphs. The *composition of digraphs* $D[H_1, H_2, \ldots, H_n]$ is the digraph having $\bigcup_{i=1}^n V(H_i)$ as its vertex set and arc set $\bigcup_{i=1}^n A(H_i) \cup$

$\{(u,v)|u \in V(H_i), v \in V(H_j), (v_i, v_j) \in A(D)\}$. If $D = H[S_1, S_2, \ldots, S_n]$ and none of the digraphs $S_1, \ldots, S_n$ has an arc, then D is an *extension* of H. The dual (or *converse*) of D, $\overleftarrow{D}$ is the digraph with vertex set $V(\overleftarrow{D}) = V(D)$ and such that $(u,v) \in A(\overleftarrow{D})$ if and only if $(v,u) \in A(D)$.

A classical result states that a digraph D is transitive if and only if $D = T[D_1, D_2, \ldots, D_n]$, where D_i is a complete digraph for $1 \leq i \leq n$ and T is an acyclic, transitive digraph. It is clear that $T = D^\star$ and the digraphs D_i are the strong components of D. Using this characterization theorem it can be proved, e.g., that every transitive digraph has a (k,l)-kernel for every pair of integers $k \geq 2$, $l \geq 1$; or that the Laborde-Payan-Xuong conjecture holds for every transitive digraph. Recently, strong 3-transitive digraphs have been characterized in [9]: A strong 3-transitive digraph is either complete, complete bipartite or a directed 3-cycle with none, one or two symmetrical arcs. Also, a thorough description of the interaction between strong components of 3-transitive digraphs has been given.

The families of k-transitive digraphs, along with the k-quasi-transitive digraphs were introduced in [8]. A digraph D is *k-quasi-transitive* if the existence of a uv-directed path of length k implies the existence of $(u,v) \in A(D)$ or $(v,u) \in A(D)$. Clearly, a 2-quasi-transitive digraph is a quasi-transitive digraph in the usual sense. In [8], structural results on k-transitive and k-quasi-transitive digraphs are obtained and used to prove, e.g., that every k-transitive digraph has an n-kernel for $n \geq k$ and that, for even k, every k-quasi-transitive digraph has an n-kernel for $n \geq k+2$. For $k = 2$, Bang-Jensen and Huang proved, in [2], a recursive characterization of quasi-transitive digraphs. For $k = 3$, Galeana-Sánchez, Goldfeder and Urrutia characterized strong 3-quasi-transitive digraphs in [7]; also, the interaction between strong components of a 3-quasi-transitive digraph is completely described by Wang and Wang in [10]. The aforementioned characterization theorems have been used to prove many results concerning these families of digraphs, e.g, that the Laborde-Payan-Xuong is valid for 3-quasi-transitive digraphs; that every 3-quasi-transitive digraph has a 4-kernel; to characterize the 3-transitive digraphs having a kernel; to find a cycle of maximum length in a quasi-transitive digraph in polynomial time, to prove that every quasi-transitive digraph has a (k,l)-kernel for every pair of integers $k \geq 4$, $l \geq 3$ or $k = 3$ and $l = 2$.

The aim of the present work is to prove a characterization theorem for 4-transitive digraphs that, hopefully, will find as many applications as its antecessors. Also, we hope to bring some light on the structure of k-transitive digraphs for arbitrary $k \geq 5$. Let us observe that every digraph is k-transitive for large enough k. So, characterizing k-transitive digraphs for every $k \in \mathbb{Z}^+$ is equivalent to characterizing every existing digraph. In view of this situation, proving that a property holds for every digraph is equivalent to proving that the property holds

for every k-transitive digraph for every $k \geq 2$.

2. Preliminary Results

We begin with a rather trivial observation which will be very useful through this work.

Remark 1. A digraph D is k-(quasi)-transitive if and only if $\overleftarrow{D}$ is k-(quasi)-transitive.

We also need a pair of propositions from [8] in order to prove some of our results not only for k-transitive digraphs, but for k-quasi-transitive digraphs as well.

Proposition 2. *Let $k \geq 2$ be even, D a k-quasi-transitive digraph and $u, v \in V(D)$ such that a uv-directed path exists. Then:*
(1) *If $d(u,v) = k$, then $d(v,u) = 1$.*
(2) *If $d(u,v) = k+1$, then $d(v,u) \leq k+1$.*
(3) *If $d(u,v) \geq k+2$, then $d(v,u) = 1$.*

Proposition 3. *Let $k \geq 3$ be odd, D a k-quasi-transitive digraph and $u, v \in V(D)$ such that a uv-directed path exists. Then:*
(1) *If $d(u,v) = k$, then $d(v,u) = 1$.*
(2) *If $d(u,v) = k+1$, then $d(v,u) \leq k+1$.*
(3) *If $d(u,v) \geq k+2$ is odd, then $d(v,u) = 1$.*
(4) *If $d(u,v) \geq k+3$ is even, then $d(v,u) \leq 2$.*

The proofs of Proposition 4 and Corollaries 5, 6 and 7, are almost the same for k-transitive and k-quasi-transitive digraphs. Only the proofs for k-quasi-transitive digraphs will be written, but it is clear that the same arguments can be followed for the k-transitive case when the reachability conditions (appearing between parentheses in the proposition and corollaries) are dropped.

Proposition 4. *Let $k \geq 2$ be an integer, D a k-transitive (k-quasi-transitive) digraph and $C = (v_0, v_1, \ldots, v_{n-1}, v_0)$ a directed cycle in D with $n \geq k$. If $v \in V(D) \setminus V(C)$ is such that $(v, v_0) \in A(D)$ (and $v \Rightarrow C$), then $v \to S = \{v_i \big| i \in (k-1)\mathbb{Z}_n\}$.*

Proof. Let us observe that if $(v, x_1, x_2, \ldots, x_k)$ is a directed path in D with $x_k \in V(C)$, by the k-quasi-transitivity of D, $(v, x_k) \in A(D)$ or $(x_k, v) \in A(D)$. But, since $v \Rightarrow C$, we have that $(x_k, v) \notin A(D)$. Thus, $(v, x_k) \in A(D)$. Let us assume without loss of generality that $V(C) = \mathbb{Z}_n$, and hence $C = (0, 1, \ldots, n-1, 0)$. It can be derived inductively that $(v, m(k-1)) \in A(D)$ for $0 \leq m \leq n-1$. We have

by hypothesis that $(v,0) \in A(D)$. For the inductive step it suffices to consider the arc $(v, m(k-1)) \in A(D)$ and the directed path $(v, m(k-1), m(k-1)+1, \ldots, m(k-1)+(k-1))$ of length k in D. Again, by the k-quasi-transitivity of D and $v \Rightarrow C$ we can conclude that $(v, m(k-1)+(k-1)) = (v, (m+1)(k-1)) \in A(D)$. So, $(v,x) \in A(D)$ for every $x \in (k-1)\mathbb{Z}_n$. ■

Some corollaries can be obtained from Proposition 4.

Corollary 5. *Let $k \geq 2$ be an integer, D a k-transitive (k-quasi-transitive) digraph and C an n-cycle with $n \geq k$ and $(n, k-1) = 1$. If $v \in V(D) \setminus V(C)$ is such that a vC-arc exists in D (and $v \Rightarrow C$), then $v \to C$.*

Proof. Let us recall that $(n, k-1) = 1$ implies $(k-1)\mathbb{Z}_n = \mathbb{Z}_n$. The result is then clear from Proposition 4. ■

Corollary 6. *Let $k \geq 2$ be an integer, D a k-transitive (k-quasi-transitive) digraph and $C = (v_0, v_1, \ldots, v_{n-1}, v_0)$ a directed cycle in D with $n \geq k$. If $v \in V(D) \setminus V(C)$, $(v, x_1, \ldots, x_{m-1}, v_0)$ is a vv_0-directed path in D (and C does not reach v in D), then $v \to S = \{v_i | i \in (k-1)\mathbb{Z}_n + (k-m)\}$.*

Proof. Since C does not reach v in D, it is clear from Propositions 2 and 3 that $d(v,C) \leq k-1$. So, we may choose $d(v,C) = m \leq k-1$. Consider the directed path $(v, x_1, \ldots, x_{m-1}, v_0, \ldots, v_{k-m})$ of length k in D. It follows from the k-quasi-transitivity and the fact that $(v_{k-m}, v) \notin A(D)$, that $(v, v_{k-m}) \in A(D)$. In virtue of Proposition 4, we can conclude that $(v, v_{r(k-1)+(k-m)}) \in A(D)$ for every $0 \leq r \leq n \pmod n$. ■

Corollary 7. *Let $k \geq 2$ be an integer, D a k-transitive (k-quasi-transitive) digraph and C an n-cycle with $n \geq k$ and $(n, k-1) = 1$. If $v \in V(D) \setminus V(C)$ is such that a vC-directed path exists in D (and C does not reach v in D), then $v \to C$.*

Proof. It follows directly from Corollaries 5 and 6. ■

The following proposition and its corollary will be useful to describe the interaction between strong components of a k-quasi-transitive digraph.

Proposition 8. *Let $k-1$ be a prime, D a k-transitive (k-quasi-transitive) digraph and $C_1 = (v_0, v_1, \ldots, v_{k-2}, v_0)$ a $(k-1)$-cycle in D.*

(1) *If $2 \leq n \leq k-2$ and $C_2 = (u_0, u_1, \ldots, u_{n-1}, u_0)$ is an n-cycle in $D - C_1$ such that $(u_0, v_0) \in A(D)$ (and $C_2 \Rightarrow C_1$), then $C_2 \to C_1$.*

(2) *If $C_2 = (u_0, u_1, \ldots, u_{k-2}, u_0)$ is a $(k-1)$-cycle in $D - C_1$ such that $(u_0, v_0) \in A(D)$ (and $C_2 \Rightarrow C_1$), then $(u_i, v_i) \in A(D)$ for every $0 \leq i \leq k-2$.*

Proof. Let $C_2=(u_0,u_1,\ldots,u_{n-1},u_0)$ be an n-cycle in $D-C_1$ such that $(u_0,v_0)\in A(D)$ and $2\leq n\leq k-2$. Clearly, since D is k-quasi-transitive, $(u_i,u_{i+1},\ldots,u_0,v_0,v_1,\ldots,v_{k-(n-i+1)})$ is a directed path of length k in D and $C_2\Rightarrow C_1$, then $(u_i,v_{k-(n-i+1)})\in A(D)$ for $i\in\{1,2,\ldots,n-1\}$. It will suffice to prove that $u_0\to C_1$, a similar argument can be used to prove that $u_i\to C_1$ for each $1\leq i\leq n-1$.

We will prove by induction on m that $(u_0,v_{-mn})\in A(D)\pmod{(k-1)}$ for $0\leq m\leq k-2$. If $m<k-2$ and $(u_0,v_{-mn})\in A(D)$, then $(u_1,\ldots,u_0,v_{-mn},\ldots,v_{-mn+(k-n)})$ is a directed path of length k in D. But D is k-quasi-transitive and $C_2\Rightarrow C_1$, so $(u_1,v_{-mn+(k-1)-(n-1)})\equiv(u_1,v_{-mn-(n-1)})\in A(D)\pmod{(k-1)}$. Thus, $(u_0,u_1,v_{-mn-n+1},\ldots,v_{-mn-n+1+(k-2)})\pmod{(k-1)}$ is a directed path of length k in D and, by the k-quasi-transitivity and the fact that $C_2\Rightarrow C_1$, $(u_0,v_{-mn-n+1+(k-2)})\in A(D)\pmod{(k-1)}$. But $-mn-n+1+(k-2)=-mn-n+1+(k-1)-1\equiv -mn-n+1-1=-mn-n=-(m+1)n\pmod{(k-1)}$. Thus, $(u_0,v_{-(m+1)n})\in A(D)$. The desired result follows from the Principle of Mathematical Induction and the fact that $(k-1,-n)=1$, and then, $-n$ generates $\mathbb{Z}_{k-1}$.

For (2) we can observe that, if $(u_i,v_i)\in A(D)$, then $(u_{i+1}C_2u_i)\cup(u_i,v_i,v_{i+1})$ is a directed path of length k in D. But, D is k-quasi-transitive and $C_2\Rightarrow C_1$, so $(u_{i+1},v_{i+1})\in A(D)$. It follows inductively that $(u_i,v_i)\in A(D)$ for every $0\leq i\leq k-2$. ■

Corollary 9. *Let $k-1$ be a prime, D a k-transitive (k-quasi-transitive) digraph and $C_1=(v_0,v_1,\ldots,v_{k-2},v_0)$ a $(k-1)$-cycle in D.*

(1) *If $2\leq n\leq k-2$ and $C_2=(u_0,u_1,\ldots,u_{n-1},u_0)$ is an n-cycle in $D-C_1$ such that C_2 reaches C_1 (and C_1 does not reach C_2) in D, then $C_2\to C_1$.*

(2) *If $C_2=(u_0,u_1,\ldots,u_{k-2},u_0)$ is a $(k-1)$-cycle in $D-C_1$ such that u_0 reaches v_0 with a directed path of length $m\leq k-1$ (and C_1 does not reach C_2) in D, then $(u_i,v_{i+(k-m)})\in A(D)(\mathrm{mod}(k-1))$ for every $0\leq i\leq k-2$.*

Proof. It is straightforward from Proposition 8. ■

We finalize this section with a proposition, intending to give a general idea of the structure of a k-transitive digraph more than serving as a tool to prove the characterization of 4-transitive digraphs.

Proposition 10. *Let $k\geq 2$ be an integer, D a k-transitive digraph and $C=(v_0,v_1,\ldots,v_{n-1},v_0)$ an n-cycle in D such that $n\geq k+2$ and $d=(n,k-1)=k-1$. If $H=D[V(C)]$, then H contains a d-cycle extension as a spanning subdigraph. Moreover, $\{V_i\}_{i=1}^{d}$ is the cyclical partition of $V(H)$, where $V_i=\{v_j|j\equiv i\pmod d\}$.*

Proof. Once again, we may assume without loss of generality that $V(C) = V(H) = \mathbb{Z}_n$. So, $C = (0, 1, \ldots, n-1, 0)$. It suffices to prove, by induction on m, that $(i, i + m(k-1) + 1) \in A(D)$ for every $0 \leq m < \frac{n}{d}$. For $m = 0$, we have that $(i, i+1) \in A(C) \subseteq A(D)$. So, let us assume that $(i, i + m(k-1) + 1) \in A(D)$ for some $m < \left(\frac{n}{d} - 1\right)$, then $(i, i+m(k-1)+1, i+m(k-1)+2, \ldots, i+m(k-1)+k)$ is a directed path in D of length k. Hence, $(i, i + m(k-1) + k) = (i, i + (m + 1)(k-1) + 1) \in A(D)$, because D is k-transitive. Thus, we have proved that $i \to \left\{(i+1) + md \middle| 0 \leq m < \frac{n}{d}\right\} = \left\{j \in \mathbb{Z}_n \middle| j \equiv i + 1 (\text{mod } d)\right\} = V_{i+1}$. Therefore, if $x \in V_i$, then $x \to V_{i+1}$ for $0 \leq i < k-1$ or $x \to V_0$ for $i = k-1$. ■

3. 4-transitive Digraphs

The results of this section are directed to the characterization theorem.

Proposition 11. *Let D be a 4-transitive digraph and $C = (v_0, v_1, \ldots, v_{n-1}, v_0)$ an n-cycle in D such that $n \geq 7$ and $(n, 3) = 1$. Then $D[V(C)] \cong \overleftrightarrow{K_n}$.*

Proof. We will assume without loss of generality that $V(C) = \mathbb{Z}_n$, then $C = (0, 1, \ldots, n-1, 0)$. Clearly, $(n-1, 0, 1, 2, 3)$ is a directed path of length 4 in D, thus $(n-1, 3) \in A(D)$. But also, since $n \geq 7$, $C' = 3C(n-1) \cup (n-1, 3)$ is a directed cycle of length $n - 3 \geq 4$. Let us observe that $n - 3 \equiv n$ (mod 3), thus, $(n-3, 3) = 1$. We can use Corollary 7 and Remark 1 to conclude that $i \to C'$ and $C' \to i$ for every $i \in \{0, 1, 2\}$. Then, $(0, 3, 4, 5, 2)$ is a directed path of length 4 in D, hence, the 4-transitivity of D implies that $(0, 2) \in A(D)$. We have proved that $(0, i) \in A(D)$ for every $i \in V(C) \setminus \{0\}$. By the symmetries of C we can conclude that $D[V(C)] \cong \overleftrightarrow{K_n}$. ■

Proposition 12. *If D is a 4-transitive digraph and S a strong component of D containing a directed n-cycle such that $n \geq 7$ and $(n, 3) = 1$, then S is a complete digraph.*

Proof. Let C be an n-cycle such that $n \geq 7$ and $(n, 3) = 1$, contained in a strong component S of D. By Proposition 11, $D[V(C)]$ is a complete digraph. Also, in virtue of Corollary 7, Remark 1 and the fact that S is strong, for every $v \in V(S) \setminus V(C)$, it can be observed that $v \to V(C)$ and $V(C) \to v$. Let $x, y, z \in V(C)$ be arbitrarily chosen, then, for every $u, v \in V(S) \setminus V(C)$, we have that (u, x, y, z, v) is a directed path of length 4 in D. Since D is 4-transitive, we can conclude that $(u, v) \in A(D)$. Thus, S is a complete digraph. ■

Proposition 13. *If D is a 4-transitive digraph and S a strong component of D with $|V(S)| \geq 6$ and containing a directed 5-cycle, then S is a complete digraph.*

Proof. Let $C = (0, 1, \ldots, 4, 0)$ be a 5-cycle of S. Since $(5, 3) = 1$, using again Corollary 7, Remark 1 and the fact that S is strong, for every $v \in V(S) \setminus V(C)$, it can be observed that $v \to V(C)$ and $V(C) \to v$. If $|V(S)| \geq 7$, then we can consider $u, v \in V(S) \setminus V(C)$, and the directed path $(u, 0, 1, 2, v)$ of length 4 in D. Thus, $(u, v) \in A(D)$ and $(0, u, v, 1, 2, \ldots, 4, 0)$ is a 7-cycle in S, then, by Proposition 12, S is a complete digraph. So, let us suppose that $|V(S)| = 6$. We may consider the 5-cycle $C_i = (v, i+1, i+2, i+3, i+4, v)$ for every $i \in \mathbb{Z}_5$. Clearly, $i \notin V(C_i)$ and $(v, i), (i, v) \in A(D)$. We can derive from Corollary 5 and Remark 1 that $i \to C_i$ and $C_i \to i$ for each $i \in \mathbb{Z}_5$. Thus, S is a complete digraph. ■

Proposition 14. *If D is a 4-transitive digraph and S a strong component of D with $|V(S)| \geq 5$ and containing a directed 4-cycle, then S is a complete digraph.*

Proof. Let $C = (0, 1, 2, 3, 0)$ be a 4-cycle of S. Since $(4, 3) = 1$, in virtue of Corollary 7, Remark 1 and the fact that S is strong, for every $v \in V(S) \setminus V(C)$, it can be observed that $v \to V(C)$ and $V(C) \to v$. Since $|V(S)| \geq 5$, there is at least one $v \in V(S) \setminus V(C)$, so, $(0, v, 1, 2, 3, 0)$ is a 5-cycle in S. If $|V(S)| \geq 6$, by Proposition 13, the desired result follows. If $|V(S)| = 5$, for each $i \in \mathbb{Z}_4$, $C_i = (v, i+1, i+2, i+3, v)$ is a directed 4-cycle. Since $(i, v), (v, i) \in A(D)$ for $i \in \mathbb{Z}_4$, by Corollary 5 and Remark 1 we can conclude that $i \to C_i$ and $C_i \to i$. The desired conclusion is then reached. ■

Now, the previous propositions of this section can be condensed in the following lemma.

Lemma 15. *Let D be a strong 4-transitive digraph with $|V(D)| \geq 5$ and C an n-cycle of D such that $(n, 3) = 1$. If $n \neq 2$ and D is not a symmetrical 5-cycle, then D is a complete digraph.*

Proof. Since $n \neq 2$ and $(n, 3) = 1$, then $n \geq 4$. If $n \neq 5$, then the result follows from Propositions 12 and 14. If $n = 5$, and $|V(D)| \geq 6$, the result follows from Proposition 13. If $n = 5$ and $|V(D)| = 5$, since every 5-cycle in a 4-transitive digraph is symmetrical and D is not a symmetrical 5-cycle, C must have at least one diagonal. But, for each diagonal of C, a 4-cycle exists in D. So, again, the result follows from Proposition 14. ■

To prove our following lemma we will need a pair of well known theorems. The following theorem is a classic characterization of strong cyclically k-partite digraphs. A proof of this result can be found in [5].

Theorem 16. *Let D be a strong digraph. Then D is cyclically k-partite if and only if every directed cycle of D has length $\equiv 0 \pmod{k}$.*

The following theorem was proved by Boesch and Tindell in [4]. It is a generalization of the classic theorem due to Robbins stating that a graph G admits a strong orientation if and only if G is 2-edge-connected. Let us recall that an orientation of a digraph D is a maximal asymmetrical subdigraph of D.

Theorem 17. *A digraph D admits a strong orientation if and only if D is strong and its underlying graph $UG(D)$ is 2-edge-connected.*

With these results in mind, we are now able to prove some new lemmas.

Lemma 18. *Let D be a strong 4-transitive digraph such that every directed cycle has length $\equiv 0 \pmod 3$. Then D is a 3-cycle extension.*

Proof. If D does not have symmetrical arcs (2-cycles), then D is a cyclically 3-partite digraph by Theorem 16. But D is strong, so, for every pair of distinct vertices $v_i \in V_i$, $v_{i+1} \in V_{i+1}$ (mod 3), we have that a $v_i v_{i+1}$-directed path exists in D. Since D is 4-transitive, $d(v_i, v_{i+1}) \leq 3$, and D is cyclically 3-partite, so $d(v_i, v_{i+1}) = 1$. Then D is a 3-cycle extension. ■

Lemma 19. *Let D be a strong 4-transitive digraph such that every directed cycle has length $\equiv 0 \pmod 3$, except for the symmetrical arcs. If D has circumference ≥ 3, has symmetrical arcs and $UG(D)$ is 2-edge-connected, then D has a 3-cycle extension with cyclical partition $\{V_0, V_1, V_2\}$ as a spanning subdigraph. Moreover, D has circumference 3, and for every symmetrical arc $(v_i, v_{i+1}) \in A(D)$, with $v_j \in V_j$ for $j \in \{i, i+1\} \pmod 3$, $|V_i| = 1$ or $|V_{i+1}| = 1$.*

Proof. If $UG(D)$ is 2-edge-connected, in virtue of Theorem 17 we may find a strong orientation H of D. Since D has symmetrical arcs, then H is an asymmetrical proper subdigraph of D. Thus, it follows from Theorem 16 that H is a cyclically 3-partite spanning subdigraph of D.

We affirm that H is 4-transitive. Let $(u_1, u_2, u_3, u_4, u_5)$ be a directed path in H. Since D is 4-transitive, $(u_1, u_5) \in A(D)$. Let us recall that every asymmetrical arc of D is also in H, and for every symmetrical arc $(x, y) \in A(D)$, either $(x, y) \in A(H)$ or $(y, x) \in A(H)$. Hence, if $(u_1, u_5) \notin A(H)$, then $(u_5, u_1) \in A(H)$. But H is cyclically 3-partite and, without loss of generality, we may assume that $u_1 \in V_0$, thus $u_5 \in V_1$ and it cannot be the case that $(u_5, u_1) \in A(H)$. Then, H is 4-transitive and cyclically 3-partite. It follows from Lemma 18 that H is a 3-cycle extension.

Let us assume without loss of generality that (v_2, v_0) is a symmetrical arc of D with $v_i \in V_i$ for $i \in \{0, 2\}$ and suppose that $|V_i| \geq 2$, for each $i \in \{0, 2\}$. We may consider $v_0 \neq v_0' \in V_0$ and arbitrary vertices $v_1 \in V_1$, $v_2 \neq v_2' \in V_2$. But then, $(v_2, v_0', v_1, v_2', v_0, v_2)$ is a 5-cycle in D, contrary to our assumption. So, $|V_i| = 1$, for some $i \in \{0, 2\}$. ■

Lemma 20. *Let D be a strong 4-transitive digraph such that every directed cycle has length $\equiv 0 \pmod 3$, except maybe for the symmetrical arcs. If D has circumference ≥ 3, has symmetrical arcs, $UG(D)$ is not 2-edge-connected and $\{S_1, S_2, \ldots, S_n\}$ are the vertex sets of the maximal 2-edge-connected subgraphs of $UG(D)$, then $S_i = \{u_i\}$ for every $2 \leq i \leq n$, $D[S_1]$ has a 3-cycle extension with cyclical partition $\{V_0, V_1, V_2\}$ as a spanning subdigraph and there is a vertex (without loss of generality) $v_0 \in V_0$ such that $(v_0, u_j), (u_j, v_0) \in A(D)$ for every $2 \leq j \leq n$. Moreover, D has circumference 3, $|V_0| = 1$ and $D[S_1]$ has the structure described in Lemma 18 or Lemma 19.*

Proof. We affirm that the circumference of D is exactly 3. Suppose that a cycle C of length greater than 3 exists in D and $v \in V(D) \setminus V(C)$ is an arbitrarily chosen vertex. Recalling that D is strong and in virtue of Corollary 6, there are at least two different Cv-arcs. Clearly, all edges of $UG(D)$ corresponding with these arcs or with the arcs of C are contained in some cycle of $UG(D)$ and thus, are not bridges. In this way, for every pair of vertices $u, v \in V(D) \setminus V(C)$, we also found a uv-path in $UG(D)$ passing through C and not using the edge uv. Hence $UG(D)$ has no bridges, which results in a contradiction.

Since the circumference of D is 3, we can consider a 3-cycle C in D. Let S_1 be the vertex set of the maximal 2-edge-connected subgraph of $UG(D)$ containing C. It is easy to observe that $D[S_1]$ is strong. Let $u, v \in S_1$, since D is strong, there is a uv-directed path P in D. If (x, y) is a bridge of $UG(D)$ and (x, y) is an arc of P, then, it follows from the maximality of $UG(D)[S_1]$ that (y, x) must also be an arc of P, contradicting that P is a path. We can conclude that $V(P) \subseteq S_1$ and hence, $D[S_1]$ is strong. But, being $D[S_1]$ strong, 4-transitive and with $UG(D[S_1])$ 2-edge-connected, depending on the existence of symmetrical arcs in $D[S_1]$ we can apply Lemma 18 or Lemma 19 to $D[S_1]$. Either way, $D[S_1]$ has a 3-cycle extension as a spanning subdigraph with cyclical partition $\{V_0, V_1, V_2\}$ and the structure described in Lemma 18 or Lemma 19. Moreover, recalling that the circumference of D is 3, we can assume without loss of generality that $V_0 = \{v_0\}$, otherwise, a cycle of length greater than 3 would exist in D.

Let $\{S_1, \ldots, S_n\}$ be the vertex sets of the maximal 2-edge-connected subgraphs of $UG(D)$. If $i \neq 1$, it can be proved that $D[S_i]$ is strong with the same argument used to prove that $D[S_1]$ is strong. If $|S_i| > 1$ for some $2 \leq i \leq n$, then $D[S_i]$ must contain a directed cycle of length 2 or 3. But, since D is strong, it would follow from Corollary 9 that at least two different S_1S_i-arcs exist in D, contradicting that S_1 and S_i are the vertex maximal 2-edge connected subgraphs of $UG(D)$. Hence, $S_i = \{u_i\}$ for every $2 \leq i \leq n$.

Now, let P be a u_iS_1-directed path of minimum length for $2 < i \leq n$. If $\ell(P) > 1$, then $\ell(P) = 2$ or $\ell(P) = 3$. If $\ell(P) = 2$, then there exists u_j such that $1 < i \neq j \leq n$ and $P = (u_i, u_j, v)$ for some $v \in S_1$. We can assume without loss of generality that $v \in V_1$ and $(v, v_2), (v_2, v_0) \in A(D)$ for some $v_2 \in V_2$ and

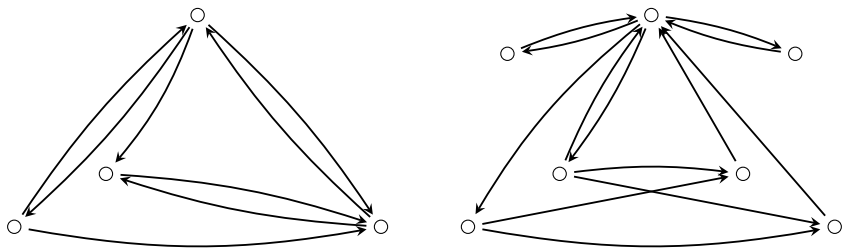

Figure 1. Digraphs of the families described in Lemmas 19 and 20.

$v_0 \in V_0$. Hence, (u_i, u_j, v, v_2, v_0) is a directed path of length 4 in D, and the 4-transitivity of D implies that $(u_i, v_0) \in A(D)$, contradicting the minimality of P. An analogous reasoning can be followed in the case that $\ell(P) = 3$, so it must be the case that $d(u_i, S_1) = 1$. By means of dualization (Remark 1), it can be shown that $d(S_1, u_i) = 1$ for every $2 \leq i \leq n$. Since $u_i \in S_i \neq S_1$, for every $2 \leq i \leq n$, and each S_j, $1 \leq j \leq n$ is the vertex set of a maximal 2-edge connected subdigraph of $UG(D)$, there must exist a single $v \in S_1$ such that $(u_i, v), (v, u_i) \in A(D)$.

We have already assumed, without loss of generality, that $|V_0| = 1$. Let us suppose that $|V_1| \geq 2$ and $(v_1, u_i), (u_i, v_1) \in A(D)$ for some $2 \leq i \leq n$ and $v_1 \in V_1$. We can consider $v_1' \in V_1$, $v_2 \in V_2$ and $(v_1', v_2, v_0, v_1, u_i)$ a directed path in D. Therefore, $(v_1', u_i) \in A(D)$, contradicting that $v_1 u_i \in E(UG(D))$ is a bridge between S_1 and S_i. Thus, if a $V_j S_i$-arc exists in D, it must be the case that $|V_j| = 1$. Now, let us suppose that $V_1 = \{v_1\}$ and $(v_0, u_i), (u_i, v_0), (v_1, u_j), (u_j, v_1) \in A(D)$ for $2 \leq i \neq j \leq n$ and $v_2 \in V_2$. We can consider the directed path $(u_j, v_1, v_2, v_0, u_i)$ of length 4 in D. The 4-transitivity of D implies that $(u_j, u_i) \in A(D)$. Again, $(u_j, v_1, v_2, v_0, u_i, u_j)$ is a cycle in $UG(D)$, contradicting that S_i and S_j were different maximal 2-edge-connected subgraphs of $UG(D)$. Thence, there exists a unique pair $(u_i, v_j), (v_j, u_i) \in A(D)$ for every $2 \leq i \leq n$ and for a unique $0 \leq j \leq 2$ such that $V_j = \{v_j\}$ and (u_i, v_j), $(v_j, u_i) \in A(D)$ for every $2 \leq i \leq n$. We can assume without loss of generality that $j = 0$. ■

Examples of digraphs described in Lemmas 19 and 20 are illustrated in Figure 1.

Definition. A *double star* is a tree of diameter three. It consists of an edge and two (non-empty) bouquets of pendant edges added to the end vertices of this edge. We denote by $D_{n,m}$ a double star with bouquets consisting of n and m pendant edges respectively (see Figure 2).

The final lemma, before the characterization theorem, deals with strong 4-transitive digraph with circumference 2. As we will see, there are only two possibilities for such digraphs.

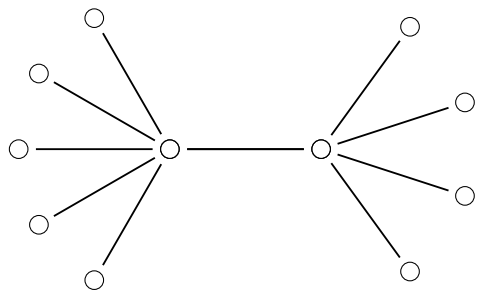

Figure 2. A double star $D_{5,4}$.

Lemma 21. *Let D be a strong 4-transitive digraph with circumference 2. Then D is either a complete biorientation of the star $K_{1,n}$, $n \geq 1$ or a complete biorientation of the double star $D_{n,m}$.*

Proof. If D is a strong digraph with circumference 2, then D is bipartite and every arc of D is symmetrical. The former because D is strong and every directed cycle has even length. The latter because, otherwise, if $(u,v) \in A(D)$ and $(v,u) \notin A(D)$, then $(u,v) \cup P$ is a directed cycle of length ≥ 3, where P is a vu-directed path of minimum length. Thus, $UG(D)$ is a tree and $D = \overleftrightarrow{UG(D)}$. Besides, by the 4-transitivity, D (and therefore $UG(D)$) has diameter ≤ 3. There is only one tree of diameter 0 (K_1), but $\overleftrightarrow{K_1}$ does not have circumference 2. Only one tree of diameter 1 exists ($K_2 \cong K_{1,1}$). The only trees of diameter 2 are the stars $K_{1,n}$ with $n \geq 3$. Finally, the only trees of diameter 3 are the double stars $D_{n,m}$. ■

And finally, the characterization theorem.

Theorem 22. *Let D be a strong 4-transitive digraph. Then exactly one of the following possibilities holds.*

(1) *D is a complete digraph.*

(2) *D is a 3-cycle extension.*

(3) *D has circumference 3, a 3-cycle extension as a spanning subdigraph with cyclical partition $\{V_0, V_1, V_2\}$, at least one symmetrical arc exists in D and for every symmetrical arc $(v_i, v_{i+1}) \in A(D)$, with $v_j \in V_j$ for $j \in \{i, i+1\} \pmod 3$, $|V_i| = 1$ or $|V_{i+1}| = 1$.*

(4) *D has circumference 3, $UG(D)$ is not 2-edge-connected and $\{S_1, S_2, \ldots, S_n\}$ are the vertex sets of the maximal 2-edge connected subgraphs of $UG(D)$, with $S_i = \{u_i\}$ for every $2 \leq i \leq n$ and such that $D[S_1]$ has a 3-cycle extension with cyclical partition $\{V_0, V_1, V_2\}$ as a spanning subdigraph. A vertex $v_0 \in V_0$ (without loss of generality) exists such that $(v_0, u_j), (u_j, v_0) \in A(D)$ for every*

$2 \leq j \leq n$. *Also* $|V_0| = 1$ *and* $D[S_1]$ *has the structure described in* (1) *or* (2), *depending on the existence of symmetrical arcs.*

(5) D *is a symmetrical* 5*-cycle.*

(6) D *is a complete biorientation of the star* $K_{1,n}$, $n \geq 3$.

(7) D *is a complete biorientation of the double star* $D_{n,m}$.

(8) D *is a strong digraph of order less than or equal to* 4 *not included in the previous families.*

Proof. Let us assume that D is not a symmetrical 5-cycle. If $|V(D)| \geq 5$ and D contains an n-cycle with $n \geq 4$ and $(n, 3) = 1$, it follows from Lemma 15 that D is complete. Suppose every cycle of D has length $\equiv 0 \pmod 3$. If there is no symmetrical arc, then we can conclude from Lemma 18 that D is a 3-cycle extension. If D has cycles of length 2 and 3 then, accordingly to Lemmas 19 and 20, D has the structure described in (3) or (4) of this theorem, depending on the edge connectivity of $UG(D)$. If D has circumference 2 we can conclude from Lemma 21 that D is a complete biorientation of the star $K_{1,n}$ with $n \geq 3$ or a complete biorientation of the double star $D_{n,m}$.

We have now covered the cases when the circumference of D is 2, when cycles of length 2 and 3 exists, when every cycle has length $\equiv 0 \pmod 3$, when D is a symmetrical 5-cycle and when $|V(D)| \geq 5$ and D contains an n-cycle with $n \geq 4$ and $(n, 3) = 1$. The only remaining case is that D is a strong digraph of order ≤ 4 not included in the families described above.

Since the cases are exhaustive, the desired characterization is obtained. ■

4. Conclusions

The family of strong 4-transitive digraphs has been characterized in Theorem 22. Although some aspects of the interaction between strong components of a non-strong 4-transitive digraph can be deduced from Corollaries 6, 7 and 9, a thorough study considering the result of Theorem 22 will represent very valuable information on this family of digraphs.

As a matter of fact, a sequel of this article is in preparation, where the aforementioned study of non-strong 4-transitive digraphs will be done. The results will be used to prove some very nice properties of this family of digraphs.

As another line of research, the results of Section 2, along with the known structures of strong transitive and 3-transitive digraphs, bring to our attention the following conjecture.

Conjecture 23. *Let* $k - 1$ *be a prime and* D *a strong* k*-transitive digraph. If* $|V(D)| \geq k + 1$, D *contains an* n*-cycle with* $n \geq k$, $(n, k - 1) = 1$ *and* D *is not a symmetrical* $(k + 1)$*-cycle, then* D *is a complete digraph.*

Acknowledgments

The author wishes to express his deepest gratitude to an Anonymous Referee for her/his very careful examination of the present work. Also the author is very thankful for many accurate and pertinent suggestions that helped to improve substantially the present paper.

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Received 25 May 2011
Revised 22 February 2012
Accepted 23 February 2012