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Dedicated to M. Borowiecki on the occasion of his 70th birthday

A NOTE ON BARNETTE'S CONJECTURE

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Abstract

Barnette conjectured that each planar, bipartite, cubic, and 3-connected graph is hamiltonian. We prove that this conjecture is equivalent to the statement that there is a constant $c > 0$ such that each graph G of this class contains a path on at least $c|V(G)|$ vertices.

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We use [4] for terminology and notation not defined here and consider finite simple graphs only.

Barnette's Conjecture, first announced in [1] and later in [6], is part of a series of conjectures stating that all members of certain graph classes contain hamiltonian cycles. In [19], Tait conjectured that all planar, cubic, 3-connected graphs are hamiltonian. If this conjecture would be true then a short and elegant proof of the Four-Colour Theorem would follow. Tait's Conjecture was disproved by Tutte in [20] by constructing a counterexample. In [21], Tutte conjectured that all cubic, 3-connected, bipartite graphs are hamiltonian. Horton showed [9, 5] that also this conjecture is false. A counterexample to the following Barnette's Conjecture would be a simultaneous counterexample to both the Tait's, and Tutte's Conjectures.

A *Barnette graph* is a planar, bipartite, cubic, and 3-connected graph. The famous and longstanding Barnette's Conjecture is the following

Conjecture 1 (Barnette)**.** *Each Barnette graph is hamiltonian.*

There is an interesting survey on conjectures being equivalent to Barnette's Conjecture by Hertel [8]. The list of such equivalences presented here is by far not complete.

In their book on graph coloring problems, Jensen and Toft [10] ask the following question and mention that Barnette's conjecture is true if and only if the answer to this question is affirmative.

Let G be a simple 3-colorable planar graph. Is it possible to partition the vertex set of G into two subsets so that each induces an acyclic subgraph of G?

This is an example of an equivalent formulation of Barnette's Conjecture, where both parts of the proof of the equivalence—necessity and sufficience—are not trivial. The proof is a consequence of the results of Król [15, 16] and Stein [17, 18].

The following equivalent formulations of Barnette's Conjecture strengthen this conjecture in the sense that if such a formulation is true then trivially Barnette's Conjecture is true and the main part of the proof of their equivalence is the converse direction. These results can be found in the papers of Hertel [7, 8] and Kelmans [11, 12, 13, 14].

Theorem 1 (Kelmans)**.** *Barnette's Conjecture holds if and only if for any arbitrary face in a Barnette graph, there is a hamiltonian cycle which passes through any two arbitrary edges on that face.*

Theorem 2 (Kelmans)**.** *Barnette's Conjecture holds if and only if for any arbitrary face in a Barnette graph, and for any arbitrary edges e_1 and e_2 on that face, there is a hamiltonian cycle which passes through e_1 and avoids e_2.*

Theorem 3 (Hertel)**.** *Barnette's Conjecture holds if and only if in any Barnette graph, any arbitrary path of length 3 is part of some hamiltonian cycle.*

In this paper, we formulate a statement concerning the length of a longest path in a Barnette graph which is seemingly weaker than Barnette's Conjecture, i.e. this condition is obviously true if Barnette's Conjecture is true. Actually, Theorem 4 states that both are equivalent.

Theorem 4. *Barnette's Conjecture is true if and only if there is a constant $c > 0$ such that each Barnette graph G contains a path on at least $c|V(G)|$ vertices.*

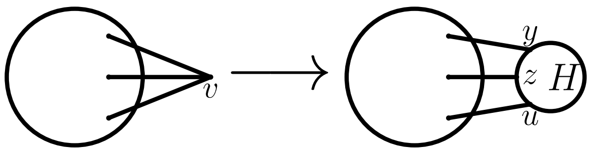
Proof. If Barnette's Conjecture is true then, with $c = 1$, each Barnette graph G contains a path on at least $c|V(G)|$ vertices.

In [2], it is proved that a cubic and 3-connected graph G contains a cycle on at least $\frac{2k}{3}$ vertices if G contains a path on k vertices.

Thus, it is sufficient to show that if there is a constant $c > 0$ such that each Barnette graph G contains a cycle on at least $c|V(G)|$ vertices, then Barnette's Conjecture is true.

For contradiction, we assume that Barnette's Conjecture is not true and let G be a plane embedding of a non-hamiltonian Barnette graph on k vertices. Furthermore, let x be an arbitrary vertex of G, y, z, and u be the neighbors of x in G, and H be the graph obtained from G by removing x. The vertices y, z, and u have degree 2 in H and belong to the boundary of a common face F of H. In the sequel let H be embedded into the plane such that F is the outer face of H.

We define an infinite sequence $\{G_i\}, i \geq 0$ of plane graphs as follows: Let G_0 be an embedding of the graph of the cube and for $i \geq 0$ let G_{i+1} be obtained from G_i by successively replacing each vertex of G_i by a copy of H, where the operation of *replacing a vertex* v *of* G_i *by a copy of* H is shown in the following figure. The inverse operation we denote by *shrinking* H *to* v, i.e. G_i is obtained from G_{i+1} by shrinking all copies of H in G_{i+1}.



It is easy to see that all graphs G_i are Barnette graphs. If $n(G_i)$ denotes the number of vertices of G_i, then $n(G_0) = 8$ and $n(G_{i+1}) = (k-1)n(G_i)$, hence, $n(G_i) = 8 \cdot (k-1)^i$ for $i \geq 0$.

Since H has $k-1$ vertices and because G is non-hamiltonian, each path of H connecting two vertices in $\{y, z, u\}$ contains at most $k-2$ vertices. Consider a longest cycle C of G_{i+1}. If a copy H' of H in G_{i+1} has a non-empty intersection with C, then this intersection is a path of H' connecting two vertices of $\{y, z, u\}$ and, therefore, containing at most $k-2$ vertices of H'. The cycle D obtained from C by shrinking all copies of H in G_{i+1} is a cycle of G_i and the number of vertices of D is the number of copies of H in G_{i+1} having a non-empty intersection with C. If $c(G_i)$ is the number of vertices of a longest cycle of G_i, then $c(G_0) = 8$ and $c(G_{i+1}) = |V(C)| \leq (k-2)|V(D)| \leq (k-2)c(G_i)$, consequently, $c(G_i) \leq 8 \cdot (k-2)^i$ for $i \geq 0$.

Since $\lim\limits_{i\to\infty} \frac{c(G_i)}{n(G_i)} = 0$, there is no constant $c > 0$ such that each Barnette graph G contains a cycle on at least $c|V(G)|$ vertices. ■

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