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THE S-PACKING CHROMATIC NUMBER OF A GRAPH

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Abstract

Let $S = (a_1, a_2, \ldots)$ be an infinite nondecreasing sequence of positive integers. An S-*packing* k-*coloring* of a graph G is a mapping from $V(G)$ to $\{1, 2, \ldots, k\}$ such that vertices with color i have pairwise distance greater than a_i, and the S-*packing chromatic number* $\chi_S(G)$ of G is the smallest integer k such that G has an S-packing k-coloring. This concept generalizes the concept of proper coloring (when $S = (1, 1, 1, \ldots)$) and broadcast coloring (when $S = (1, 2, 3, 4, \ldots)$). In this paper, we consider bounds on the parameter and its relationship with other parameters. We characterize the graphs with $\chi_S = 2$ and determine χ_S for several common families of graphs. We examine χ_S for the infinite path and give some exact values and asymptotic bounds. Finally we consider complexity questions, especially about recognizing graphs with $\chi_S = 3$.

Keywords: graph, coloring, packing, broadcast chromatic number.

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1. Introduction

In a proper coloring of a graph, any two vertices of the same color must be at distance more than one apart. In this paper we consider strengthening this condition. One natural way is to insist that any two vertices of the same color must be some distance d apart. In this paper we propose to treat the colors differently. That is, for each color i there is a value a_i such that any two vertices of color i are distance more than a_i apart. This generalizes ordinary colorings and some other coloring parameters such as the broadcast chromatic number.

Let $G = (V, E)$ be a simple undirected graph. A set $X \subseteq V(G)$ is called an i-*packing* if vertices of X have pairwise distance greater than i. For a positive integer k, a *packing k-coloring* of G is a mapping f from $V(G)$ to $\{1, 2, \ldots, k\}$ such that vertices with color i form an i-packing. The *packing chromatic number* $\chi_\rho(G)$ of G is the smallest positive integer k such that G has a packing k-coloring.

The concept of the packing chromatic number was introduced in [6] under the name *broadcast chromatic number.* The term packing chromatic number was introduced later by Brešar et al. [1]. There have been several papers written on this concept, especially on the values for grids and lattices [1, 2, 4, 6, 8], and the computational complexity of the parameter [3, 6].

We consider here a more general concept, which was mentioned in [6] but not explored there. Let $S = (a_1, a_2, \ldots)$ be an infinite nondecreasing sequence of positive integers. An *S-packing k-coloring* of a graph G is a mapping f from $V(G)$ to $\{1, 2, \ldots, k\}$ such that vertices with color i have pairwise distance greater than a_i (thus vertices of color i form an a_i-packing of G). It is also called an *$(a_1, a_2, \ldots, a_k)$-packing coloring* of G. For example, a $(1, 1, 1, 1)$-packing coloring is just a usual 4-coloring, a $(2, 2, 2, 2)$-packing coloring is a 4-coloring of the square of the graph, and a $(1, 2, 3, 4)$-packing coloring is a packing 4-coloring as defined earlier. The *S-packing chromatic number* $\chi_S(G)$ of G is the smallest integer k such that G has an S-packing k-coloring. For example, for the 4-cycle, $\chi_S(C_4) = 3$ for $S = (1, 2, 3, 4, \ldots)$, while $\chi_S(C_4) = 4$ for $S = (2, 2, 2, 2, \ldots)$. If no such coloring of G exists for any positive integer k (e.g. for the infinite complete graph), then we say $\chi_\rho(G) = \infty$.

In this paper, we consider bounds on the parameter and its relationship with other parameters. We characterize the graphs with S-packing chromatic number two and determine the S-packing chromatic number of several common families of graphs. The emphasis of the paper is on the S-packing chromatic number of the infinite path, for which some exact values and asymptotic bounds are given, and the complexity of recognizing graphs with S-packing chromatic number three.

Throughout this paper, we shall use the following notation: $d(u, v)$ for the distance between vertices u and v, $\operatorname{diam}(G)$ for $\max_{u,v \in V(G)} d(u, v)$, $\alpha(G)$ for the independence number, $\beta(G)$ for the vertex cover number, and $\chi(G)$ for the chromatic number of G. Also, unless specified otherwise, every sequence is an infinite nondecreasing sequence of positive integers; S is always used to denote such a sequence, and a_i denotes the i^{th} term of S. For concepts and notation not defined here, see [9].

2. Observations

Observation 1. *Let $S_a = (a_1, a_2, \ldots)$ and $S_b = (b_1, b_2, \ldots)$. If $\chi_{S_a}(G) = k$ and $b_i \le a_i$ for $i = 1, 2, 3, \ldots, k$, then $\chi_{S_b}(G) \le k$.*

Proof. An $(a_1, a_2, \ldots, a_k)$-packing coloring of G is also a $(b_1, b_2, \ldots, b_k)$-packing coloring of G. ■

Observation 2. *If G_2 is a subgraph of G_1, then $\chi_S(G_2) \leq \chi_S(G_1)$ for every sequence S.*

Proof. The distance between two vertices u and v in G_2 is at least the distance between them in G_1. Thus, any S-packing k-coloring of G_1 when restricted to $V(G_2)$ is also a S-packing k-coloring of G_2. ■

Observation 3. *Let $S = (a_1, a_2, \ldots)$ and let G be a finite graph of order n.*
(1) $1 \leq \chi_S(G) \leq n$.
(2) $\chi_S(G) = 1$ *if and only if G has no edges.*
(3) $\chi_S(G) = n$ *if and only if G is connected and $a_1 \geq \operatorname{diam}(G)$.*

Proof. Parts (1) and (2) are immediate, so we prove only (3). If $a_1 \geq \operatorname{diam}(G)$, then since S is nondecreasing, $a_i \geq \operatorname{diam}(G)$ for each i, and so each color can be used at most once. On the other hand, if $a_1 < \operatorname{diam}(G)$, we can give color 1 to at least two vertices in G, and so $\chi_S(G) \leq n - 1$. ■

For example, $\chi_S(K_n) = n$ for the complete graph K_n.

We finish this section by characterizing the graphs with $\chi_S(G) = 2$.

Proposition 4. *Let $S = (a_1, a_2, \ldots)$ and let G be a nonempty connected graph.*
(1) *If $a_1 = a_2 = 1$, then $\chi_S(G) = 2$ if and only if G is bipartite.*
(2) *If $a_1 = 1 < a_2$, then $\chi_S(G) = 2$ if and only if G is a star.*
(3) *If $a_1 > 1$, then $\chi_S(G) = 2$ if and only if G is K_2.*

Proof. (1) Assume that $a_1 = a_2 = 1$. If G is bipartite, one can give color 1 to the vertices in one partite set and color 2 to the other vertices, and so $\chi_S(G) \leq 2$. On the other hand, if $\chi_S(G) = 2$, then both the set of vertices with color 1 and the set of vertices with color 2 are independent sets. Hence G is bipartite.

(2) Assume that $a_1 = 1$ and $a_2 \geq 2$. For the star $K_{1,m}$, we can give the central vertex color 2 and every other vertex color 1, and so $\chi_S(K_{1,m}) = 2$. On the other hand, assume $\chi_S(G) = 2$. Then by (1), G is bipartite. Furthermore, G cannot contain P_4, since it is easy to see that under these conditions, $\chi_S(P_4) = 3$. If G has maximum degree 1, then $G = K_2$, a star. Otherwise, let v be a vertex of degree at least 2, with neighbors w_1 and w_2. Since G is bipartite, w_1 and w_2 are nonadjacent; since G has no P_4, neither w_i has another neighbor. Hence G is a star with center v.

(3) Assume that $a_1 \geq 2$ and $\chi_S(G) = 2$. Then by the previous discussion, G is a star. Since $a_1 \geq 2$, each color can be used at most once. Hence G is K_2. ■

3. Graphs of Diameter Two

In this section we show that it is easy to determine $\chi_S(G)$ for graphs of diameter 2 such as complete bipartite graphs, wheels, and Rooks graphs.

The following proposition extends a result in [6].

Proposition 5. *Let $S = (1, a_2, a_3, \ldots)$. For every graph G without isolates, $\chi_S(G) \le \beta(G) + 1$, with equality if $a_2 \ge \mathrm{diam}(G)$.*

Proof. Let A be a maximum independent set in G. We can give color 1 to every vertex in A, and a distinct color to every other vertex. Thus, $\chi_S(G) \le 1 + |V(G)| - \alpha(G) = \beta(G) + 1$. If $a_2 \ge \mathrm{diam}(G)$, then no two vertices can receive the same color i for any $i \ge 2$. So the above coloring is optimal and equality holds. ■

In fact, one can generalize the above to provide a "formula" for the S-packing chromatic number of graphs of diameter two.

Proposition 6. *Let $S = (a_1, a_2, \ldots)$ and let G be a graph of diameter two and order n. If exactly k of the a_i are 1, then $\chi_S(G) = n - \alpha(G, k) + \min(k, \chi(G))$, where $\alpha(G, k)$ is the maximum number of vertices of G that can be properly colored using k colors.*

Corollary 7. *Let $S = (a_1, a_2, \ldots)$. For the complete bipartite graph $K_{m,n}$ with $m \le n$,*

$$\chi_S(K_{m,n}) = \begin{cases} 2, & \text{if } a_1 = a_2 = 1, \\ m+1, & \text{if } a_1 = 1 < a_2, \\ m+n, & \text{if } a_1 > 1. \end{cases}$$

Proof. This follows from Proposition 4, Proposition 5, and Observation 3. Alternatively, note that $\alpha(K_{m,n}, 1) = n$. ■

Corollary 8. *Let $S = (a_1, a_2, \ldots)$. Let $n \ge 4$. For the wheel W_n formed by taking a C_{n-1} and adding one vertex adjacent to all the vertices in the cycle,*

$$\chi_S(W_n) = \begin{cases} 3, & \text{if } a_1 = a_2 = 1 \text{ and } n \text{ is odd}, \\ 4, & \text{if } a_1 = a_2 = 1 \text{ and } n \text{ is even}, \\ \lfloor n/2 \rfloor + 2, & \text{if } a_1 = 1 < a_2, \\ n, & \text{if } a_1 > 1. \end{cases}$$

Proof. This follows from the fact that $\alpha(W_n, 1) = \lfloor (n-1)/2 \rfloor$ while $\alpha(W_n, 2) = n - 1$ or $n - 2$ depending on the parity of $n - 1$. ■

If G has diameter two and has $\chi(G) = n/\alpha(G)$, then the formula for $\chi_S(G)$ is particularly simple.

Proposition 9. *Let $S = (a_1, a_2, \ldots)$. If graph G of order n has diameter 2 and $\chi(G) = s = n/\alpha(G)$, then*

$$\chi_S(G) = \begin{cases} s, & \text{if } a_s = 1, \\ n - (\alpha(G) - 1)k, & \text{if } a_1 = \cdots = a_k = 1 \text{ and } a_{k+1} > 1 \text{ for some } 1 \le k < s, \\ n, & \text{if } a_1 > 1. \end{cases}$$

Proof. The cases where $a_s = 1$ or $a_1 > 1$ follow as before. So assume $a_1 = \cdots = a_k = 1$ and $a_{k+1} \ge 2$ for some $1 \le k < s$. Then each of colors 1 through k can be used at most $\alpha(G)$ times. By the condition, G has a proper coloring with s colors where each color is used exactly $\alpha(G)$ times. So we can use colors 1 though k to color $k\,\alpha(G)$ vertices. By the diameter hypothesis, the remaining vertices must have distinct colors. ■

For example, the Rooks graph $K_r \square K_s$ (where $\square$ denotes the Cartesian product) for $r \le s$ is covered by this theorem.

4. THE INFINITE PATH

The *two-way infinite path*, written P_∞, is the graph with the integer set $\mathbb{Z}$ as the vertex set, such that two vertices are adjacent if and only if they correspond to consecutive integers. Determining χ_S for P_∞ is surprisingly difficult.

To obtain lower bounds, a key observation is that at most $1/(a_i + 1)$ of the vertices can receive color i (Fiala et al. [4], Goddard et al. [6]):

Proposition 10. *Let $S = (a_1, a_2, \ldots)$. For the infinite path P_∞, if $\chi_S(P_\infty) \le k$, then $\sum_{i=1}^{k} \frac{1}{a_i+1} \ge 1$.*

Corollary 11. *If $\sum_{i=1}^{\infty} 1/(a_i + 1) \le 1$, then $\chi_S(P_\infty) = \infty$.*

Proof. The result is immediate from the above proposition if the sum is strictly less than 1. But even if the sum is 1, every partial sum is less than 1. ■

Remark: The converse of this corollary is not true. For example, consider $S = (1, 2, 4, 8, \ldots)$. Then $\sum_{i=1}^{\infty} \frac{1}{a_i+1} > 1$, but we will prove that $\chi_S(P_\infty) = \infty$ in Proposition 18.

The following claim is immediate (or follows from Proposition 4):

Observation 12. *Let $S = (a_1, a_2, \ldots)$. Then $\chi_S(P_\infty) = 2$ if and only if $a_1 = a_2 = 1$.*

A *periodic coloring* is one where there exists an integer p such that vertices a and $a + p$ receive the same color for all a. We will use square brackets to indicate such a coloring: for example, $[1, 2, 3]$ to mean the periodic coloring $\ldots, 1, 2, 3, 1, 2, 3, 1, \ldots$.

Proposition 13. *Let* $S = (a_1, a_2, \ldots)$. *Then* $\chi_S(P_\infty) = 3$ *if and only if* (a_1, a_2, a_3) *is one of* $(1, 2, 3)$, $(1, 3, 3)$, *or* $(2, 2, 2)$.

Proof. Assume that $\chi_S(P_\infty) = 3$. By the previous proposition, $a_2 \geq 2$. Further, by Proposition 10, we have $\sum_{i=1}^{3} 1/(a_i + 1) \geq 1$. Thus the possible values for (a_1, a_2, a_3) are $(1, 2, 3)$, $(1, 2, 4)$, $(1, 2, 5)$, $(1, 3, 3)$, and $(2, 2, 2)$.

The periodic pattern $[1, 2, 3]$ provides a $(2, 2, 2)$-packing coloring. The periodic pattern $[1, 2, 1, 3]$ provides a $(1, 3, 3)$- and hence a $(1, 2, 3)$-packing coloring.

It remains to show that there is no $(1, 2, 4)$-packing coloring of P_∞. But this is easy to see. Consider a vertex with color 3. Then the next vertex either has color 1, in which case the sequence continues 2, 1 and we encounter a problem, or has color 2, in which case the sequence continues 1 and we again encounter a problem. ■

It is easy to see that if the a_i in S are bounded by n, then $\chi_S(P_\infty) \leq n + 1$, by the periodic pattern $[1, 2, \ldots, n + 1]$. So a necessary condition for $\chi_S(P_\infty) = \infty$ is that the a_i are unbounded. So this raises the question of how fast the growth must be. We do not know the answer.

We consider first some arithmetic sequences.

Proposition 14. $\chi_S(P_\infty) = 6$ *if* $S = (2, 3, 4, 5, 6, \ldots)$.

Proof. The upper bound is proved by the periodic coloring $[1, 2, 3, 1, 4, 2, 5, 6]$. The lower bound can be proved by several paragraphs of case analysis, or by computer search (the longest path that can be S-packing 5-colored is P_{34}). We omit the proof. (A similar result is given in [7].) ■

In general we have the result.

Proposition 15 [6]**.** *Let* $S = (a, a + 1, a + 2, \ldots)$. *Then* $(e - 1)a \leq \chi_S(P_\infty) \leq 2a + 3$.

Note that the result is stated differently in [6], since there they measure the maximum color used, not the total number of colors used.

Proposition 16. *Let S be an arithmetic progression. Then* $\chi_S(P_\infty)$ *is finite.*

Proof. Let $S = S(a, d) = (a, a + d, a + 2d, a + 3d, \ldots)$. The proof is by induction on d. The case $d = 1$ is given by Proposition 15. So assume $d \geq 2$.

Now, let $d' = \lceil d/2 \rceil$ and let $S' = S(\lfloor a/2 \rfloor, d')$. By the induction hypothesis, $\chi_{S'}(P_\infty)$ is finite; say it is k. Then let $S'' = S(\lfloor (a + kd)/2 \rfloor, d')$. By the induction hypothesis, $\chi_{S''}(P_\infty)$ is finite. Now, partition P_∞ into the even- and odd-numbered vertices. Color the even-numbered vertices using the coloring corresponding to S'. Color the odd-numbered ones using the coloring corresponding to S'' and adding k throughout. (This doubles the distance between vertices of the same color.) The result is an $S(a,d)$-packing coloring using a finite number of colors. ■

We can also provide a lower bound on the chromatic number.

Proposition 17. *Let $S = (a, a+d, a+2d, a+3d, \ldots)$. Then*

$$\chi_S(P_\infty) \geq \begin{cases} (e^d - 1)(a - d + 1)/d, & \text{if } a \geq d; \\ ((a+1)e^{\frac{d}{1+1/a}} - (a - d + 1))/d, & \text{otherwise.} \end{cases}$$

Proof. Suppose that $\chi_S(P_\infty) = k$. Then we have $\sum_{i=1}^{k} 1/(a_i + 1) \geq 1$. Let $\alpha = a - d + 1$. If $a \geq d$, then

$$1 \leq \sum_{i=1}^{k} \frac{1}{a + (i-1)d + 1} \leq \int_0^k \frac{1}{\alpha + xd}\, dx = \frac{(\ln(\alpha + kd) - ln(\alpha))}{d}.$$

Otherwise,

$$1 \leq \sum_{i=1}^{k} \frac{1}{a + (i-1)d + 1} \leq \frac{1}{a+1} + \int_1^k \frac{1}{\alpha + xd}\, dx = \frac{1}{a+1} + \frac{(\ln(\alpha + kd) - ln(\alpha + d))}{d}.$$

In each case the result follows by solving for k. ■

We consider next a geometric sequence.

Proposition 18. *Let $S = (1, 2, 4, 8, \ldots)$. Then $\chi_S(P_\infty) = \infty$.*

Proof. We show by induction on k that $\chi_S(P_{2^k}) > k$, where P_m is the path on m vertices. If $k = 1$, then P_2 requires 2 colors; so the claim is true for $k = 1$. Assume that the claim is true for k, and suppose that $\chi_S(P_{2^{k+1}}) \leq k+1$. By the distance constraint, color $k+1$ can be used at most once. Thus some subpath of length at least 2^k must be colored with colors 1 up to k. By the induction hypothesis this cannot be done, a contradiction. ■

However, if the growth of S does not continue, then we get the following result:

Proposition 19. *Let $S = (a_1, a_2, \ldots)$, where $a_i = 2^i - 1$ for $i = 1, 2, \ldots, k$, and $a_{k+1} = 2^k - 1$. Then $\chi_S(P_\infty) = k + 1$.*

Proof. Since $\sum_{i=1}^{k} \frac{1}{a_i+1} = 1 - 1/2^k$, by Proposition 10, $\chi_S(P_\infty) \geq k+1$. On the other hand, let us give color 1 to the integers of the form $2m+1$, color 2 to the integers of the form $4m+2$, color 3 to the integers of the form $8m+4$, and so on, with color k to the integers of the form $2^k m + 2^{k-1}$, and color $k+1$ to the integers of the form $2^k m$. Since each integer belongs to precisely one of these classes, and the distance constraints are satisfied, we have an S-packing coloring for P_∞. For example, for $k = 3$ we have the periodic coloring $[1, 2, 1, 3, 1, 2, 1, 4]$. ■

5. The Complexity of S-packing 3-colorability

We saw earlier that there is a simple characterization of graphs with $\chi_S(G) = 2$. Determining the broadcast chromatic number is NP-complete for general graphs [6]. Fiala and Golovach [3] showed that the problem remains NP-complete for trees. In contrast, they showed that if S is bounded by a constant or if S is growing double-exponentially, then the S-packing chromatic number can be computed in polynomial time for tree and indeed for graphs of bounded treewidth.

In [6] it was proved that it is NP-complete to decide whether $\chi_\rho(G) \leq 4$. And yet, they showed that there is a polynomial-time algorithm to determine if $\chi_\rho(G) \leq 3$. In this section we consider the following problem: Given a graph G and three colors with associated distance constraints (a_1, a_2, a_3), does G have an (a_1, a_2, a_3)-packing coloring? This problem is of course NP-hard for $a_1 = a_2 = a_3 = 1$, but it turns out to be trivial if the a_i are large. So the question is where does the change occur?

The first result is that when $a_1 \geq 2$, the problem is easy.

Proposition 20 [6]**.** *Let G be a connected graph. Then G has an $(2,2,2)$-packing coloring if and only if it is a path of any length, or a cycle of length a multiple of* 3.

From this one can deduce the following:

Corollary 21. *If $a_1 \geq 2$ and $a_3 > 2$ and connected graph G has an (a_1, a_2, a_3)-packing, then G has at most* 5 *vertices.*

So consider $a_1 = 1$.

Proposition 22. *Let $4 \leq a_2 \leq a_3$ and let G be a connected graph. Then G has an $(1, a_2, a_3)$-packing coloring if and only if $\beta(G) \leq 2$.*

Proof. Assume that G is $(1, a_2, a_3)$-colorable. Suppose two vertices receive color 2; then they must be at least distance 5 apart. But it is easy to see that one cannot color the vertices on the path between them. That is, both color 2 and color 3 are used at most once. It follows that G has a vertex cover of at most two vertices. Conversely, if G has a vertex cover of two vertices, then give one of these color 2 and one color 3 and give the remaining vertices of G color 1. ■

The graphs that have a $(1,2,3)$-packing coloring were characterized in [6]. Building on this, one can prove the following result. The 1-subdivision of a graph is obtained by subdividing each edge once; an almost 1-subdivision is obtained by subdividing all edges except one.

Proposition 23. *Let G be a connected graph. Then*

(1) *G has a $(1, 3, 3)$-packing coloring if and only if $\beta(G) \leq 2$ or G is a subgraph of the 1-subdivision of a bipartite multigraph.*

(2) *For $a_3 \geq 4$, G has a $(1, 2, a_3)$-packing coloring if and only if G is a subgraph of a 1-subdivision or almost 1-subdivision of a bipartite multigraph with a dominating vertex.*

(3) *For $a_3 \geq 4$, G has a $(1, 3, a_3)$-packing coloring if and only if $\beta(G) \leq 2$ or G is a subgraph of the 1-subdivision of a bipartite multigraph with a dominating vertex.*

Proof. The proof is similar to the one in [6] for $(1, 2, 3)$-packing coloring, so we give just a sketch here.

(1) If there is an edge joining a vertex of color 2 and a vertex of color 3, then the two vertices form a vertex cover, because their other neighbors are colored 1, and there is no color available for any new neighbor of these neighbors. So assume there is no edge joining vertices of color 2 and 3. Then the set of vertices of these colors is an independent set. Further, vertices of color 1 have degree at most 2, and every cycle has length a multiple of 4.

(2) There is at most one vertex of color 3, since it is impossible to color a path between two such vertices. (Note that a graph with $\beta(G) \leq 2$ is a 1-subdivision or almost 1-subdivision of a 2-vertex multigraph.)

(3) This follows from (1) and (2). ∎

In [6] it was shown that $(1, 1, 2)$-packing coloring is NP-hard. Using a similar approach, we can show that:

Proposition 24. *$(1, 1, a_3)$-packing coloring is NP-hard for all a_3.*

Proof. The proof is almost the same as the proof in [6], which is a reduction from 3-colorability. In that proof, ones takes the input graph G and forms graph G' by replacing each edge uv with a gadget shown in Figure 1. The result is that a proper 3-coloring of G can be extended to a $(1, 1, 2)$-packing coloring of G', while any $(1, 1, 2)$-packing coloring of G' when restricted to G is a proper 3-coloring.

To prove the result for $a_3 > 2$, simply modify the gadget by adding $a_3 - 2$ vertices on the paths joining each of u and v to each of the pentagons. The argument is otherwise identical. ∎

The question of $(1, 2, 2)$-packing was left as an open problem in [6]. We now show that that is hard too.

Proposition 25. *$(1, 2, 2)$-packing coloring is NP-hard.*

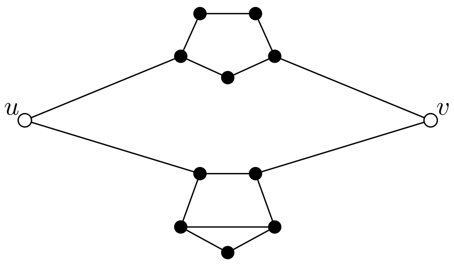

Figure 1. The gadget that replaced an edge in the reduction to $(1,1,2)$-packing coloring.

Proof. We reduce NAE3SAT to $(1,2,2)$-packing coloring. NAE3SAT is the problem of given a boolean formula in conjunctive normal form with 3 literals per clause, is there a truth value of the variables such that each clause has both a true and a false literal. This problem is known to be NP-complete ([5]).

In the reduction we will construct a graph and consider a $(1,2,2)$-packing coloring. Let us use x for the color that is a 1-packing, and T and F for the colors that are 2-packings. We will call such a $(1,2,2)$-packing coloring a *valid* coloring. Let us call a vertex of degree 3 or more *large*. Note that a large vertex cannot be colored x. In the construction certain vertices are designated as large; this is achieved by adding as many leaf neighbors as necessary.

Given a boolean formula ϕ, we construct a graph G_ϕ as follows. For each variable in ϕ, say x, introduce a P_3 such that the two ends of P_3 are large and labeled x and $\bar{x}$. Further, let H denote the graph consisting in a cycle C_{18} with vertices numbered 1 up to 18, such that vertices 1, 3, 7, 13, and 17 are large; call the vertices numbered 1, 7, and 13 the *ports*. Then, for each clause, introduce a copy of H and join the ports to the three vertices corresponding to the literals by paths of length 2 in some order. This construction is illustrated in Figure 2.

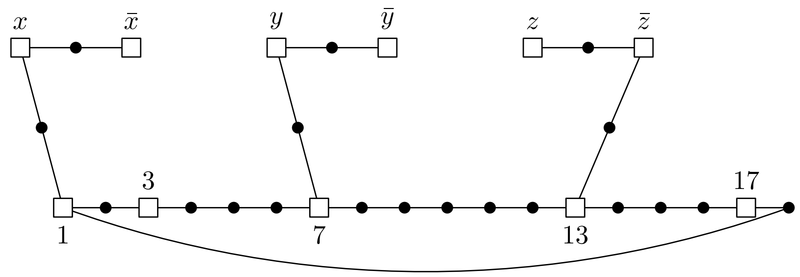

Figure 2. A possible construction of G_ϕ where $\phi = x \vee y \vee \bar{z}$ (square is large vertex).

Observation 26. (1) *There is no valid coloring of H such that the three ports receive the same color.*

(2) *Any coloring of the ports with F, T such that the three ports do not receive the same color can be extended to a valid coloring of H.*

Proof. (1) Suppose there is a valid coloring with ports 1, 7, 13 all T. Start at vertex 1. Since vertices 3 and 17 are both large, they must both be F. It follows that both 5 and 15 are x, and both 6 and 14 are F. From this it follows that 8 and 12 are both x, and thus 9 and 11 are both F, which is a contradiction.

(2) Let P be a P_7 with one end F and one end T. Then there is a unique valid coloring of P—F, x, T, x, F, x, T. Note that the 3rd vertex does not have color x. Let Q be a P_7 with vertices 1, 3, and 7 large and vertices 1 and 7 the same color, say T. Then there is a unique valid coloring of Q—T, x, F, T, x, F, T. Our H consists of two copies of P and one copy of Q fitted together by identifying their ends appropriately. It follows that there is a valid coloring of H. □

We now verify the desired property of the reduction. If ϕ has an NAE3SAT assignment, then for the variable gadgets, give the central vertex color x, and the literals the color T or F depending on whether they are true or false in the assignment. Give color x to the vertices connecting the variable gadgets to the copies of H. Give the ports the opposite color to their literal. And by the above claim, we can extend this to a valid coloring.

Conversely, if there is a valid coloring of G_ϕ, then in each variable gadget, one literal is colored T and one F. Further, since the ports receive the opposite color to their literals, by the above claim, in each H the three ports do not all receive the same color and hence the corresponding literals are not all the same truth-value. So we can assign the variables in ϕ the truth value given by the coloring of their literals. That is, ϕ has an NAE3SAT assignment if and only if G_ϕ has a valid coloring. ■

6. Conclusion

We conclude this paper by listing a few open problems:

1. Determine how fast S must grow for $\chi_S(P_\infty)$ to be infinite.

2. Determine the S-packing chromatic numbers of grids.

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