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# COLORING SOME FINITE SETS IN $\mathbb{R}^{n}$ 

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#### Abstract

This note relates to bounds on the chromatic number $\chi\left(\mathbb{R}^{n}\right)$ of the Euclidean space, which is the minimum number of colors needed to color all the points in $\mathbb{R}^{n}$ so that any two points at the distance 1 receive different colors. In [6] a sequence of graphs $G_{n}$ in $\mathbb{R}^{n}$ was introduced showing that $\chi\left(\mathbb{R}^{n}\right) \geq \chi\left(G_{n}\right) \geq(1+o(1)) \frac{n^{2}}{6}$. For many years, this bound has been remaining the best known bound for the chromatic numbers of some lowdimensional spaces. Here we prove that $\chi\left(G_{n}\right) \sim \frac{n^{2}}{6}$ and find an exact formula for the chromatic number in the case of $n=2^{k}$ and $n=2^{k}-1$.


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## 1. INTRODUCTION

In this note, we study the classical chromatic number $\chi\left(\mathbb{R}^{n}\right)$ of the Euclidean space. The quantity $\chi\left(\mathbb{R}^{n}\right)$ is the minimum number of colors needed to color all the points in $\mathbb{R}^{n}$ so that any two points at a given distance $a$ receive different colors. By a well-known compactness result of Erdős and de Bruijn (see [1]), the value of $\chi\left(\mathbb{R}^{n}\right)$ is equal to the chromatic number of a finite distance graph $G=(V, E)$, where $V \subset \mathbb{R}^{n}$ and $E=\{\{\mathbf{x}, \mathbf{y}\}:|\mathbf{x}-\mathbf{y}|=a\}$.

Now we know that

$$
(1.239 \ldots+o(1))^{n} \leq \chi\left(\mathbb{R}^{n}\right) \leq(3+o(1))^{n}
$$

where the lower bound is due to the third author of this paper (see [8]) and the upper bound is due to Larman and Rogers (see [6]). Also, in [3] one can find an up-to-date table of estimates obtained for the dimensions $n \leq 12$.

It is worth noting that the linear bound $\chi\left(\mathbb{R}^{n}\right) \geq n+2$ is quite simple, and the first superlinear bound was discovered by Larman, Rogers, Erdős, and Sós in [6]. They considered a family of graphs $G_{n}=\left(V_{n}, E_{n}\right)$ with

$$
\begin{gathered}
V_{n}=\left\{\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right): x_{i} \in\{0,1\}, x_{1}+\cdots+x_{n}=3\right\} \\
E_{n}=\{\{\mathbf{x}, \mathbf{y}\}:|\mathbf{x}-\mathbf{y}|=2\}
\end{gathered}
$$

In other words, the vertices of $G_{n}$ are all the 3 -subsets of the set $[n]=\{1, \ldots, n\}$ and two vertices $A, B$ are connected with an edge iff $|A \cap B|=1$. Larman and Rogers [6] used an earlier result by Zs. Nagy who proved the following theorem.

Theorem 1 [6]. Let $s$ and $t \leq 3$ be nonnegative integers and let $n=4 s+t$. Then

$$
\alpha\left(G_{n}\right)= \begin{cases}n, & \text { if } t=0 \\ n-1, & \text { if } t=1 \\ n-2, & \text { if } t=2 \text { or } t=3\end{cases}
$$

The standard inequality $\chi\left(G_{n}\right) \geq \frac{\left|V_{n}\right|}{\alpha\left(G_{n}\right)}$ combined with the above theorem gives an obvious corollary.

Corollary 2 [6]. Let $s$ and $t \leq 3$ be nonnegative integers and let $n=4 s+t$. Then

$$
\chi\left(G_{n}\right) \geq \begin{cases}\frac{(n-1)(n-2)}{6}, & \text { if } t=0, \\ \frac{n(n-2)}{6}, & \text { if } t=1, \\ \frac{n(n-1)}{6}, & \text { if } t=2 \text { or } t=3\end{cases}
$$

The bounds from the corollary are applied to estimate from below the chromatic numbers $\chi\left(\mathbb{R}^{n-1}\right)$, since the vertices of $G_{n}$ lie in the hyperplane $x_{1}+\cdots+x_{n}=$ 3. Now all these bounds are surpassed due to the consideration of some other distance graphs (see [3]). However, it could happen that actually $\chi\left(G_{n}\right)$ is much bigger than the ratio $\frac{\left|V_{n}\right|}{\alpha\left(G_{n}\right)}$. It turns out that this is not the case, and the main result of this note is as follows.

Theorem 3. If $n=2^{k}$ for some integer $k \geq 2$, then

$$
\chi\left(G_{n}\right)=\frac{(n-1)(n-2)}{6} .
$$

Additionally, if $n=2^{k}-1$ for some integer $k \geq 2$, then

$$
\chi\left(G_{n}\right)=\frac{n(n-1)}{6} .
$$

Finally, there is a constant $c$ such that for every $n$,

$$
\chi\left(G_{n}\right) \leq \frac{(n-1)(n-2)}{6}+c n .
$$

Our proof yields that $c \leq 5.5$. With some more work we could prove that $c \leq 4.5$. On the other hand, since $n(n-1) / 6-(n-1)(n-2) / 6=(n-1) / 3$, we have $c \geq 1 / 3$.

In the next section, we prove Theorem 3.

## 2. Proof of Theorem 3

Easily, $\chi\left(G_{3}\right)=1, \quad \chi\left(G_{4}\right)=1, \quad \chi\left(G_{5}\right)=3$.
Let $f(n):=\frac{(n-1)(n-2)}{6}$. We show by induction on $k$ that $\chi\left(G_{2^{k}}\right)=f\left(2^{k}\right)$. For $k=2$ it is trivial. Assume that for some $k$ we established the induction hypothesis. Partition the set $[n]=\left[2^{k+1}\right]$ into the equal parts $A_{1}=\left[\frac{n}{2}\right], A_{2}=[n] \backslash\left[\frac{n}{2}\right]$ of size $2^{k}$. Denote by $U_{1}$ and $U_{2}$ the sets of vertices of $G=G_{2^{k+1}}$ lying in the sets
$A_{1}$ and $A_{2}$ respectively. By the induction assumption, each of the induced subgraphs $G\left[U_{1}\right]$ and $G\left[U_{2}\right]$ can be properly colored with at most $f\left(2^{k}\right)$ colors. Cover all pairs of the $2 k$ elements of $A_{1}$ with disjoint perfect matchings $N_{1}, \ldots, N_{2^{k}-1}$ and all pairs of the $2^{k}$ elements of $A_{2}$ with matchings $M_{1}, \ldots, M_{2^{k}-1}$. We form a color class $C(i, j)$ for $1 \leq i \leq 2^{k}-1,1 \leq j \leq 2^{k-1}$ as follows. Consider the matchings $N_{i}, M_{i}$ and assume that the edges are $\left\{u_{1}, u_{2}\right\},\left\{u_{3}, u_{4}\right\}, \ldots$ in $N_{i}$ and $\left\{v_{1}, v_{2}\right\},\left\{v_{3}, v_{4}\right\}, \ldots$ in $M_{i}$. For $j=1, \ldots, 2^{k-1}$ let $D(i, j)$ denote the following set of 4 -tuples (indices are considered modulo $2^{k}$ ):

$$
\left\{u_{1}, u_{2}, v_{2 j-1}, v_{2 j}\right\},\left\{u_{3}, u_{4}, v_{2 j+1}, v_{2 j+2}\right\}, \ldots,\left\{u_{2^{k}-1}, u_{2^{k}}, v_{2 j-3}, v_{2 j-2}\right\} .
$$

For $i=1, \ldots, 2^{k}-1$ and $j=1, \ldots, 2^{k-1}$, the color class $C(i, j)$ is formed by the collection of triples contained in the members of $D(i, j)$. The intersection sizes are all 0 or 2 , so the triples in $C(i, j)$ form an independent set in $G$. Moreover, each triple is contained in a member of some $D(i, j)$. The total number of used colors is

$$
2^{k-1}\left(2^{k}-1\right)+f\left(2^{k}\right)=2^{2 k-1}-2^{k-1}+\frac{\left(2^{k}-1\right)\left(2^{k}-2\right)}{6}=f\left(2^{k+1}\right)
$$

This proves the first statement of the theorem. Since $\chi\left(G_{n}\right) \leq \chi\left(G_{n+1}\right)$, this together with Corollary 2 also implies the statement of the theorem for $n=2^{k}-1$.

It remains to show that there exists a constant $c$ such that $\chi\left(G_{n}\right) \leq \frac{n^{2}}{6}+c n$ for every $n$. Consider our coloring in steps.

Step 1: Let $n=4 s_{1}+t_{1}$ where $t_{1} \leq 3$. First, color all triples containing the elements $4 s_{1}+1, \ldots, 4 s_{1}+t_{1}$ with at most $t_{1}(n-1)<3 n$ colors. Now consider the set $\left[4 s_{1}\right]$ and all the triples in this set. Partition $\left[4 s_{1}\right]$ into $A_{1}=\left[2 s_{1}\right]$ and $A_{2}=\left[4 s_{1}\right]-\left[2 s_{1}\right]$ and color the triples intersecting both $A_{1}$ and $A_{2}$ with $s_{1}\left(2 s_{1}-1\right)<\frac{n}{4}\left(\frac{n}{2}-1\right)$ colors as above.

Step 2: Since the triples contained in $A_{1}$ are disjoint from the triples contained in $A_{2}$, we will use for coloring the triples contained in $A_{2}$ the same colors and the same procedure as for the triples contained in $A_{1}$. Consider $A_{1}$. Let $n_{1}=\left|A_{1}\right|=$ $2 s_{1}=4 s_{2}+t_{2}$ where $t_{2} \leq 3$. Since $2 s_{1}$ is even, $t_{2} \leq 2$. By construction, $n_{1} \leq \frac{n}{2}$. Similarly to Step 1, color all triples containing the elements $4 s_{2}+1, \ldots, 4 s_{2}+t_{2}$ with at most $t_{2}\left(n_{1}-1\right)<2 n_{1}$ colors. Partition [ $4 s_{2}$ ] into $A_{1,1}=\left[2 s_{2}\right.$ ] and $A_{1,2}=\left[4 s_{2}\right]-\left[2 s_{2}\right]$ and color the triples intersecting both $A_{1,1}$ and $A_{1,2}$ with at most $\frac{n}{8}\left(\frac{n}{4}-1\right)$ new colors.

Step i (for $i \geq 3$ ): If $2 s_{i-1} \leq 2$, then Stop. Otherwise, repeat Step 2 with [ $\left.2 s_{i-1}\right]$ in place of $\left[2 s_{1}\right]$.

Altogether, we use at most $\left(3 n+\frac{n}{4}\left(\frac{n}{2}-1\right)\right)+\left(\frac{2 n}{2}+\frac{n}{8}\left(\frac{n}{4}-1\right)\right)+\left(\frac{2 n}{4}+\frac{n}{16}\left(\frac{n}{8}-\right.\right.$ 1)) $+\cdots<5 n+\frac{n^{2}}{8} \cdot \frac{4}{3}=\frac{n^{2}}{6}+5 n=\frac{(n-1)(n-2)}{6}+5.5 n-1 / 3$ colors. The theorem is proved.

## 3. Discussion

As we have already said, the constant 5 in the bound $\chi\left(G_{n}\right) \leq \frac{n^{2}}{6}+5 n$ is not the best possible and can be improved. However, to find the exact value of the chromatic number is still interesting. For example, we know that $\chi\left(\mathbb{R}^{12}\right) \geq 27$ (see [3]). At the same time, $\chi\left(G_{13}\right) \geq\left\lceil\left[\begin{array}{c}\binom{13}{3} \\ 12\end{array}\right\rceil=24\right.$ (due to Corollary 2), and the proof of Theorem 3 applied for $n=13$ yields a bound $\chi\left(G_{13}\right) \leq 31$.

It would be quite interesting to study more general graphs. Let $G(n, r, s)$ be the graph whose set of vertices consists of all the $r$-subsets of the set $[n]$ and whose set of edges is formed by all possible pairs of vertices $A, B$ with $|A \cap B|=s$. Larman proved in [5] that

$$
\chi\left(\mathbb{R}^{n}\right) \geq \chi(G(n, 5,2)) \geq \frac{\binom{n}{5}}{\alpha(G(n, 5,2))} \geq(1+o(1)) \frac{\binom{n}{5}}{1485 n^{2}} \sim \frac{n^{3}}{178200}
$$

Thus, the main result of Larman was in finding the bound $\alpha(G(n, 5,2)) \leq(1+$ $o(1)) 1485 n^{2}$. However, the so-called linear algebra method ([2], see also [8]) can be directly applied here to show that $\alpha(G(n, 5,2)) \leq(1+o(1))\binom{n}{2} \sim \frac{n^{2}}{2}$. This substantially improves Larman's estimate and gives $\chi(G(n, 5,2)) \geq(1+$ $o(1) \frac{n^{3}}{60}$. We do not know any further improvements on this result. On the other hand, observe that for any 3 -set $A$, the collection of 5 -sets containing $A$ forms an independent set in $G(n, 5,2)$, yielding $\chi(G(n, 5,2)) \leq\binom{ n}{3} \sim \frac{n^{3}}{6}$. It is plausible that $\chi(G(n, 5,2)) \sim c n^{3}$ with a constant $c \in[1 / 60,1 / 6]$, but this constant is not yet found and even no better bounds for $c$ have been published.

Furthermore, the graphs $G(n, 5,3)$ have been studied, since the best known lower bound $\chi\left(\mathbb{R}^{9}\right) \geq 21$ is due to the fact that $\chi(G(10,5,3))=21$ (see [4]). No related results concerning the case of $n \rightarrow \infty$ have apparently been published.

Although for combinatorial geometry small values of $n$ are of greater interest, we see that the consideration of graphs $G(n, r, s)$ with small $r, s$ and growing $n$ is of its intrinsic interest, too. So assume that $r, s$ are fixed and $n \rightarrow \infty$. We have

$$
\chi(G(n, r, s)) \leq \min \left\{O\left(n^{r-s}\right), O\left(n^{s+1}\right)\right\} .
$$

The first bound follows from Brooks' theorem, since the maximum degree of $G(n, r, s)$ is

$$
\binom{r}{s}\binom{n-r}{r-s}=(1+o(1)) \frac{r!}{s!(r-s)!(r-s)!} n^{r-s} .
$$

The second bound is a simple generalization of the above-mentioned bound $\chi(G(n, 5,2)) \leq(1+o(1)) n^{3} / 6$.

Note that the second bound can be somewhat improved. Assume $s<r / 2$, so $q:=\lceil(r-1) / s\rceil$ is at least 2. Assuming that $q$ divides $n$, partition $[n]$ into $q$
equal classes, $A_{1}, \ldots, A_{q}$. Let $\mathcal{C}$ be the family of $(s+1)$-sets that are subsets of some $A_{i}$. For each $B \in \mathcal{C}$, the $r$-sets containing $B$ form an independent set in $G(n, r, s)$, and by the pigeonhole principle every $r$-set contains such $B$, hence

$$
\chi(G(n, r, s)) \leq|\mathcal{C}|=q\binom{n / q}{s+1}=(1+o(1)) \frac{n^{s+1}}{q^{s}(s+1)!} .
$$

In particular, $\chi(G(n, 5,2)) \leq(1+o(1)) \frac{n^{3}}{24}$, which improves the previous bound $\frac{n^{3}}{6}$.

It is worthwhile to look at the construction in Section 2 from a different point of view. For $n=2^{k}$ we constructed a 4 -uniform hypergraph $\mathcal{H}$ with the property that every 3 -subset of vertices is covered exactly once. Note that $e(\mathcal{H})=$ $\binom{n}{3} / 4$. Then we decomposed $E(\mathcal{H})$ into $\binom{n}{3}$ perfect matchings. Each matching gives a color class of our coloring. Note that instead of providing the explicit decomposition, we could have used a classical theorem of Pippenger and Spencer [7], which claims the existence of $(1+o(1))\binom{n}{3}$ covering matchings.

This motivates the following possible approach to the case $r=2 s+1$. The discussion here is not a proof, it is just a sketch of a possible way to generalize our argument. Assume that we managed to construct an $(r+s)$-uniform hypergraph $\mathcal{H}$ that covers every $r$-set exactly once. Then $e(\mathcal{H})=\binom{n}{r} /\binom{r+s}{s}$. Assume that $\mathcal{H}$ can be decomposed into $t$ hypergraphs, $\mathcal{N}_{1}, \ldots, \mathcal{N}_{t}$, such that for every $i$ and every $A, B \in \mathcal{N}_{i}$ we have $|A \cap B| \leq s-1$. Then the $r$-sets covered by sets in $\mathcal{N}_{i}$ form an independent set, yielding $\chi(G(n, r, s)) \leq t$. Probably a generalization of the theorem of Pippenger and Spencer [7] would give $t \leq(1+o(1))\binom{n}{r} /\binom{n}{s}=(1+$ $o(1))(s!/ r!) n^{r-s}$. This bound, if true, would be asymptotically best possible, since the already mentioned linear algebra method (see $[2,8])$ ensures that $\alpha(G(n, 2 s+$ $1, s)) \leq(1+o(1))\binom{n}{s}$ and so $\chi(G(n, 2 s+1, s)) \geq(1+o(1))\binom{n}{r} /\binom{n}{s}$, provided $s+1$ is a prime power. In particular, we would get $\chi(G(n, 5,2)) \sim \frac{n^{2}}{60}$.

The case of simultaneously growing $n, r, s$ has also been studied. Namely, $r \sim r^{\prime} n$ and $s \sim s^{\prime} n$ with any $r^{\prime} \in(0,1)$ and $s^{\prime} \in\left(0, r^{\prime}\right)$ have been considered. This is due to the fact that the first exponential estimate to the quantity $\chi\left(\mathbb{R}^{n}\right)$, $\chi\left(\mathbb{R}^{n}\right) \geq(1.207 \cdots+o(1))^{n}$, was obtained by Frankl and Wilson in [2] with the help of some graphs $G(n, r, s)$ having $r \sim r^{\prime} n$ and $s \sim \frac{r^{\prime}}{2} n$. Lower bounds are usually based on the linear algebra (see [8]) and upper bounds can be found in [9].

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