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# COLORING SOME FINITE SETS IN $\mathbb{R}^n$

József Balogh[1]

*Department of Mathematics, University of Illinois,*
*Urbana, IL 61801, USA*

**e-mail:** jobal@math.uiuc.edu

Alexandr Kostochka[2]

*Department of Mathematics, University of Illinois,*
*Urbana, IL, 61801, USA*
*Sobolev Institute of Mathematics,*
*Novosibirsk, Russia*

**e-mail:** kostochk@math.uiuc.edu

and

Andrei Raigorodskii[3]

*Department of Mechanics and Mathematics,*
*Moscow State University,*
*Leninskie gory, Moscow, 119991, Russia*
*Department of Discrete Mathematics,*
*Moscow Institute of Physics and Technology,*
*Dolgoprudny, Russia*

**e-mail:** mraigor@yandex.ru

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**Abstract**

This note relates to bounds on the chromatic number $\chi(\mathbb{R}^n)$ of the Euclidean space, which is the minimum number of colors needed to color all the points in $\mathbb{R}^n$ so that any two points at the distance 1 receive different colors. In [6] a sequence of graphs $G_n$ in $\mathbb{R}^n$ was introduced showing that $\chi(\mathbb{R}^n) \geq \chi(G_n) \geq (1+o(1))\frac{n^2}{6}$. For many years, this bound has been remaining the best known bound for the chromatic numbers of some low-dimensional spaces. Here we prove that $\chi(G_n) \sim \frac{n^2}{6}$ and find an exact formula for the chromatic number in the case of $n = 2^k$ and $n = 2^k - 1$.

**Keywords:** chromatic number, independence number, distance graph.

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## 1. Introduction

In this note, we study the classical chromatic number $\chi(\mathbb{R}^n)$ of the Euclidean space. The quantity $\chi(\mathbb{R}^n)$ is the minimum number of colors needed to color all the points in $\mathbb{R}^n$ so that any two points at a given distance $a$ receive different colors. By a well-known compactness result of Erdős and de Bruijn (see [1]), the value of $\chi(\mathbb{R}^n)$ is equal to the chromatic number of a *finite* distance graph $G = (V, E)$, where $V \subset \mathbb{R}^n$ and $E = \{\{\mathbf{x}, \mathbf{y}\} : |\mathbf{x} - \mathbf{y}| = a\}$.

Now we know that

$$(1.239\ldots + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n,$$

where the lower bound is due to the third author of this paper (see [8]) and the upper bound is due to Larman and Rogers (see [6]). Also, in [3] one can find an up-to-date table of estimates obtained for the dimensions $n \leq 12$.

It is worth noting that the linear bound $\chi(\mathbb{R}^n) \geq n + 2$ is quite simple, and the first superlinear bound was discovered by Larman, Rogers, Erdős, and Sós in [6]. They considered a family of graphs $G_n = (V_n, E_n)$ with

$$V_n = \{\mathbf{x} = (x_1, \ldots, x_n) : \ x_i \in \{0, 1\}, \ x_1 + \cdots + x_n = 3\},$$

$$E_n = \{\{\mathbf{x}, \mathbf{y}\} : \ |\mathbf{x} - \mathbf{y}| = 2\}.$$

In other words, the vertices of $G_n$ are all the 3-subsets of the set $[n] = \{1, \ldots, n\}$ and two vertices $A, B$ are connected with an edge iff $|A \cap B| = 1$. Larman and Rogers [6] used an earlier result by Zs. Nagy who proved the following theorem.

**Theorem 1** [6]**.** *Let $s$ and $t \leq 3$ be nonnegative integers and let $n = 4s + t$. Then*

$$\alpha(G_n) = \begin{cases} n, & \text{if } t = 0, \\ n - 1, & \text{if } t = 1, \\ n - 2, & \text{if } t = 2 \text{ or } t = 3. \end{cases}$$

The standard inequality $\chi(G_n) \geq \frac{|V_n|}{\alpha(G_n)}$ combined with the above theorem gives an obvious corollary.

**Corollary 2** [6]**.** *Let $s$ and $t \leq 3$ be nonnegative integers and let $n = 4s + t$. Then*

$$\chi(G_n) \geq \begin{cases} \frac{(n-1)(n-2)}{6}, & \text{if } t = 0, \\ \frac{n(n-2)}{6}, & \text{if } t = 1, \\ \frac{n(n-1)}{6}, & \text{if } t = 2 \text{ or } t = 3. \end{cases}$$

The bounds from the corollary are applied to estimate from below the chromatic numbers $\chi(\mathbb{R}^{n-1})$, since the vertices of $G_n$ lie in the hyperplane $x_1 + \cdots + x_n = 3$. Now all these bounds are surpassed due to the consideration of some other distance graphs (see [3]). However, it could happen that actually $\chi(G_n)$ is much bigger than the ratio $\frac{|V_n|}{\alpha(G_n)}$. It turns out that this is not the case, and the main result of this note is as follows.

**Theorem 3.** *If $n = 2^k$ for some integer $k \geq 2$, then*

$$\chi(G_n) = \frac{(n-1)(n-2)}{6}.$$

*Additionally, if $n = 2^k - 1$ for some integer $k \geq 2$, then*

$$\chi(G_n) = \frac{n(n-1)}{6}.$$

*Finally, there is a constant $c$ such that for every $n$,*

$$\chi(G_n) \leq \frac{(n-1)(n-2)}{6} + cn.$$

Our proof yields that $c \leq 5.5$. With some more work we could prove that $c \leq 4.5$. On the other hand, since $n(n-1)/6 - (n-1)(n-2)/6 = (n-1)/3$, we have $c \geq 1/3$.

In the next section, we prove Theorem 3.

## 2. PROOF OF THEOREM 3

Easily, $\chi(G_3) = 1, \quad \chi(G_4) = 1, \quad \chi(G_5) = 3$.

Let $f(n) := \frac{(n-1)(n-2)}{6}$. We show by induction on $k$ that $\chi(G_{2^k}) = f(2^k)$. For $k = 2$ it is trivial. Assume that for some $k$ we established the induction hypothesis. Partition the set $[n] = \left[2^{k+1}\right]$ into the equal parts $A_1 = \left[\frac{n}{2}\right]$, $A_2 = [n] \setminus \left[\frac{n}{2}\right]$ of size $2^k$. Denote by $U_1$ and $U_2$ the sets of vertices of $G = G_{2^{k+1}}$ lying in the sets

$A_1$ and $A_2$ respectively. By the induction assumption, each of the induced subgraphs $G[U_1]$ and $G[U_2]$ can be properly colored with at most $f(2^k)$ colors. Cover all pairs of the $2k$ elements of $A_1$ with disjoint perfect matchings $N_1, \dots, N_{2^k-1}$ and all pairs of the $2^k$ elements of $A_2$ with matchings $M_1, \dots, M_{2^k-1}$. We form a color class $C(i,j)$ for $1 \le i \le 2^k - 1, 1 \le j \le 2^{k-1}$ as follows. Consider the matchings $N_i, M_i$ and assume that the edges are $\{u_1, u_2\}, \{u_3, u_4\}, \dots$ in $N_i$ and $\{v_1, v_2\}, \{v_3, v_4\}, \dots$ in $M_i$. For $j = 1, \dots, 2^{k-1}$ let $D(i,j)$ denote the following set of 4-tuples (indices are considered modulo $2^k$):

$$\{u_1, u_2, v_{2j-1}, v_{2j}\}, \{u_3, u_4, v_{2j+1}, v_{2j+2}\}, \dots, \{u_{2^k-1}, u_{2^k}, v_{2j-3}, v_{2j-2}\}.$$

For $i = 1, \dots, 2^k - 1$ and $j = 1, \dots, 2^{k-1}$, the color class $C(i,j)$ is formed by the collection of triples contained in the members of $D(i,j)$. The intersection sizes are all 0 or 2, so the triples in $C(i,j)$ form an independent set in $G$. Moreover, each triple is contained in a member of some $D(i,j)$. The total number of used colors is

$$2^{k-1}(2^k - 1) + f(2^k) = 2^{2k-1} - 2^{k-1} + \tfrac{(2^k-1)(2^k-2)}{6} = f(2^{k+1}).$$

This proves the first statement of the theorem. Since $\chi(G_n) \le \chi(G_{n+1})$, this together with Corollary 2 also implies the statement of the theorem for $n = 2^k - 1$.

It remains to show that there exists a constant $c$ such that $\chi(G_n) \le \frac{n^2}{6} + cn$ for every $n$. Consider our coloring in steps.

*Step* 1: Let $n = 4s_1 + t_1$ where $t_1 \le 3$. First, color all triples containing the elements $4s_1 + 1, \dots, 4s_1 + t_1$ with at most $t_1(n-1) < 3n$ colors. Now consider the set $[4s_1]$ and all the triples in this set. Partition $[4s_1]$ into $A_1 = [2s_1]$ and $A_2 = [4s_1] - [2s_1]$ and color the triples intersecting both $A_1$ and $A_2$ with $s_1(2s_1 - 1) < \frac{n}{4}\left(\frac{n}{2} - 1\right)$ colors as above.

*Step* 2: Since the triples contained in $A_1$ are disjoint from the triples contained in $A_2$, we will use for coloring the triples contained in $A_2$ the same colors and the same procedure as for the triples contained in $A_1$. Consider $A_1$. Let $n_1 = |A_1| = 2s_1 = 4s_2 + t_2$ where $t_2 \le 3$. Since $2s_1$ is even, $t_2 \le 2$. By construction, $n_1 \le \frac{n}{2}$. Similarly to Step 1, color all triples containing the elements $4s_2 + 1, \dots, 4s_2 + t_2$ with at most $t_2(n_1 - 1) < 2n_1$ colors. Partition $[4s_2]$ into $A_{1,1} = [2s_2]$ and $A_{1,2} = [4s_2] - [2s_2]$ and color the triples intersecting both $A_{1,1}$ and $A_{1,2}$ with at most $\frac{n}{8}\left(\frac{n}{4} - 1\right)$ new colors.

*Step* i (for $i \ge 3$): If $2s_{i-1} \le 2$, then Stop. Otherwise, repeat Step 2 with $[2s_{i-1}]$ in place of $[2s_1]$.

Altogether, we use at most $(3n + \frac{n}{4}(\frac{n}{2} - 1)) + (\frac{2n}{2} + \frac{n}{8}(\frac{n}{4} - 1)) + (\frac{2n}{4} + \frac{n}{16}(\frac{n}{8} - 1)) + \cdots < 5n + \frac{n^2}{8} \cdot \frac{4}{3} = \frac{n^2}{6} + 5n = \frac{(n-1)(n-2)}{6} + 5.5n - 1/3$ colors. The theorem is proved. ■

## 3. DISCUSSION

As we have already said, the constant 5 in the bound $\chi(G_n) \le \frac{n^2}{6} + 5n$ is not the best possible and can be improved. However, to find the exact value of the chromatic number is still interesting. For example, we know that $\chi(\mathbb{R}^{12}) \ge 27$ (see [3]). At the same time, $\chi(G_{13}) \ge \left\lceil \frac{\binom{13}{3}}{12} \right\rceil = 24$ (due to Corollary 2), and the proof of Theorem 3 applied for $n = 13$ yields a bound $\chi(G_{13}) \le 31$.

It would be quite interesting to study more general graphs. Let $G(n, r, s)$ be the graph whose set of vertices consists of all the $r$-subsets of the set $[n]$ and whose set of edges is formed by all possible pairs of vertices $A, B$ with $|A \cap B| = s$. Larman proved in [5] that

$$\chi(\mathbb{R}^n) \ge \chi(G(n,5,2)) \ge \frac{\binom{n}{5}}{\alpha(G(n,5,2))} \ge (1+o(1))\frac{\binom{n}{5}}{1485n^2} \sim \frac{n^3}{178200}.$$

Thus, the main result of Larman was in finding the bound $\alpha(G(n,5,2)) \le (1 + o(1))1485n^2$. However, the so-called linear algebra method ([2], see also [8]) can be directly applied here to show that $\alpha(G(n,5,2)) \le (1+o(1))\binom{n}{2} \sim \frac{n^2}{2}$. This substantially improves Larman's estimate and gives $\chi(G(n,5,2)) \ge (1 + o(1))\frac{n^3}{60}$. We do not know any further improvements on this result. On the other hand, observe that for any 3-set $A$, the collection of 5-sets containing $A$ forms an independent set in $G(n,5,2)$, yielding $\chi(G(n,5,2)) \le \binom{n}{3} \sim \frac{n^3}{6}$. It is plausible that $\chi(G(n,5,2)) \sim cn^3$ with a constant $c \in [1/60, 1/6]$, but this constant is not yet found and even no better bounds for $c$ have been published.

Furthermore, the graphs $G(n,5,3)$ have been studied, since the best known lower bound $\chi(\mathbb{R}^9) \ge 21$ is due to the fact that $\chi(G(10,5,3)) = 21$ (see [4]). No related results concerning the case of $n \to \infty$ have apparently been published.

Although for combinatorial geometry small values of $n$ are of greater interest, we see that the consideration of graphs $G(n,r,s)$ with small $r, s$ and *growing* $n$ is of its intrinsic interest, too. So assume that $r, s$ are fixed and $n \to \infty$. We have

$$\chi(G(n,r,s)) \le \min\{O(n^{r-s}), O(n^{s+1})\}.$$

The first bound follows from Brooks' theorem, since the maximum degree of $G(n,r,s)$ is

$$\binom{r}{s}\binom{n-r}{r-s} = (1+o(1))\frac{r!}{s!(r-s)!(r-s)!}n^{r-s}.$$

The second bound is a simple generalization of the above-mentioned bound $\chi(G(n,5,2)) \le (1+o(1))n^3/6$.

Note that the second bound can be somewhat improved. Assume $s < r/2$, so $q := \lceil (r-1)/s \rceil$ is at least 2. Assuming that $q$ divides $n$, partition $[n]$ into $q$

equal classes, $A_1, \dots, A_q$. Let $\mathcal{C}$ be the family of $(s+1)$-sets that are subsets of some $A_i$. For each $B \in \mathcal{C}$, the $r$-sets containing $B$ form an independent set in $G(n,r,s)$, and by the pigeonhole principle every $r$-set contains such $B$, hence

$$\chi(G(n,r,s)) \le |\mathcal{C}| = q\binom{n/q}{s+1} = (1+o(1))\frac{n^{s+1}}{q^s(s+1)!}.$$

In particular, $\chi(G(n,5,2)) \le (1+o(1))\frac{n^3}{24}$, which improves the previous bound $\frac{n^3}{6}$.

It is worthwhile to look at the construction in Section 2 from a different point of view. For $n = 2^k$ we constructed a 4-uniform hypergraph $\mathcal{H}$ with the property that every 3-subset of vertices is covered exactly once. Note that $e(\mathcal{H}) = \binom{n}{3}/4$. Then we decomposed $E(\mathcal{H})$ into $\binom{n}{3}$ perfect matchings. Each matching gives a color class of our coloring. Note that instead of providing the explicit decomposition, we could have used a classical theorem of Pippenger and Spencer [7], which claims the existence of $(1+o(1))\binom{n}{3}$ covering matchings.

This motivates the following possible approach to the case $r = 2s+1$. The discussion here is not a proof, it is just a sketch of a possible way to generalize our argument. Assume that we managed to construct an $(r+s)$-uniform hypergraph $\mathcal{H}$ that covers every $r$-set exactly once. Then $e(\mathcal{H}) = \binom{n}{r}/\binom{r+s}{s}$. Assume that $\mathcal{H}$ can be decomposed into $t$ hypergraphs, $\mathcal{N}_1, \dots, \mathcal{N}_t$, such that for every $i$ and every $A, B \in \mathcal{N}_i$ we have $|A \cap B| \le s-1$. Then the $r$-sets covered by sets in $\mathcal{N}_i$ form an independent set, yielding $\chi(G(n,r,s)) \le t$. Probably a generalization of the theorem of Pippenger and Spencer [7] would give $t \le (1+o(1))\binom{n}{r}/\binom{n}{s} = (1+o(1))(s!/r!)n^{r-s}$. This bound, if true, would be asymptotically best possible, since the already mentioned linear algebra method (see [2, 8]) ensures that $\alpha(G(n,2s+1,s)) \le (1+o(1))\binom{n}{s}$ and so $\chi(G(n,2s+1,s)) \ge (1+o(1))\binom{n}{r}/\binom{n}{s}$, provided $s+1$ is a prime power. In particular, we would get $\chi(G(n,5,2)) \sim \frac{n^2}{60}$.

The case of simultaneously growing $n, r, s$ has also been studied. Namely, $r \sim r'n$ and $s \sim s'n$ with any $r' \in (0,1)$ and $s' \in (0,r')$ have been considered. This is due to the fact that the first exponential estimate to the quantity $\chi(\mathbb{R}^n)$, $\chi(\mathbb{R}^n) \ge (1.207\cdots + o(1))^n$, was obtained by Frankl and Wilson in [2] with the help of some graphs $G(n,r,s)$ having $r \sim r'n$ and $s \sim \frac{r'}{2}n$. Lower bounds are usually based on the linear algebra (see [8]) and upper bounds can be found in [9].

## References

[1] N.G. de Bruijn and P. Erdős, *A colour problem for infinite graphs and a problem in the theory of relations*, Proc. Koninkl. Nederl. Acad. Wet. (A) **54** (1951) 371–373.

[2] P. Frankl and R. Wilson, *Intersection theorems with geometric consequences*, Combinatorica **1** (1981) 357–368.
doi:10.1007/BF02579457

[3] A.B. Kupavskiy, *On coloring spheres embedded into* $\mathbb{R}^n$, Sb. Math. **202(6)** (2011) 83–110.

[4] A.B. Kupavskiy and A.M. Raigorodskii, *On the chromatic number of* $\mathbb{R}^9$, J. Math. Sci. **163(6)** (2009) 720–731.
doi:10.1007/s10958-009-9708-4

[5] D.G. Larman, *A note on the realization of distances within sets in Euclidean space*, Comment. Math. Helv. **53** (1978) 529–535.
doi:10.1007/BF02566096

[6] D.G. Larman and C.A. Rogers, *The realization of distances within sets in Euclidean space*, Mathematika **19** (1972) 1–24.
doi:10.1112/S0025579300004903

[7] N. Pippenger and J. Spencer, *Asymptotic behavior of the chromatic index for hypergraphs*, J. Combin. Theory (A) **51** (1989) 24–42.
doi:10.1016/0097-3165(89)90074-5

[8] A.M. Raigorodskii, *On the chromatic number of a space*, Russian Math. Surveys **55** (2000) N2, 351–352.
doi:10.1070/RM2000v055n02ABEH000281

[9] A.M. Raigorodskii, *The problems of Borsuk and Grünbaum on lattice polytopes*, Izv. Math. **69(3)** (2005) 81–108. English transl. Izv. Math. **69(3)** (2005) 513–537.
doi:10.1070/IM2005v069n03ABEH000537

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