Discussiones Mathematicae
Graph Theory 33 (2013) 181–192
doi:10.7151/dmgt.1640

Dedicated to Mietek Borowiecki on the occasion of his seventieth birthday

RAINBOW CONNECTION IN SPARSE GRAPHS

Arnfried Kemnitz

Computational Mathematics, Technische Universität Braunschweig
38 023 Braunschweig, Germany

e-mail: a.kemnitz@tu-bs.de

Jakub Przybyło[1]

AGH University of Science and Technology
al. A. Mickiewicza 30, 30-059 Krakow, Poland

e-mail: przybylo@wms.mat.agh.edu.pl

Ingo Schiermeyer[2]

Institut für Diskrete Mathematik und Algebra
Technische Universität Bergakademie Freiberg
09 596 Freiberg, Germany

e-mail: Ingo.Schiermeyer@tu-freiberg.de

and

Mariusz Woźniak[1]

AGH University of Science and Technology
al. A. Mickiewicza 30, 30-059 Krakow, Poland

e-mail: e-mail: mwozniak@agh.edu.pl

Abstract

An edge-coloured connected graph $G = (V, E)$ is called *rainbow-connected* if each pair of distinct vertices of G is connected by a path whose edges have distinct colours. The *rainbow connection number* of G, denoted by $\mathrm{rc}(G)$, is the minimum number of colours such that G is rainbow-connected. In this

[1]Research was partly supported by the Polish Ministry of Science and Higher Education.
[2]Part of this work was performed while the author was staying at AGH University of Science and Technology in Krakow as a visiting professor.

paper we prove that $\mathrm{rc}(G) \leq k$ if $|V(G)| = n$ and $|E(G)| \geq \binom{n-k+1}{2} + k - 1$ for all integers n and k with $n - 6 \leq k \leq n - 3$. We also show that this bound is tight.

Keywords: rainbow-connected graph, rainbow colouring, rainbow connection number.

2010 Mathematics Subject Classification: 05C15.

1. Introduction

We use [1] for terminology and notation not defined here and consider finite and simple graphs only.

An edge-coloured connected graph G is called *rainbow-connected* if each pair of distinct vertices of G is connected by a rainbow path, that is, by a path whose edges have pairwise distinct colours. Note that the edge colouring need not to be proper. The *rainbow connection number* of G, denoted by $\mathrm{rc}(G)$, is the minimum number of colours such that G is rainbow-connected.

This concept of rainbow connection in graphs was introduced by Chartrand et al. in [5]. An easy observation is that if G has n vertices then $\mathrm{rc}(G) \leq n - 1$, since one may colour the edges of a given spanning tree of G with different colours and colour the remaining edges with one of the already used colours. Chartrand et al. determined the precise rainbow connection number of several graph classes including complete multipartite graphs [5]. The rainbow connection number has been studied for further graph classes in [2] and for graphs with fixed minimum degree in [2, 10, 15].

There are different applications for such edge colourings of graphs. One interesting example is the secure transfer of classified information between agencies (see, e.g.[6]).

The computational complexity of rainbow connectivity has been studied in [3, 11]. It is proved that the computation of $\mathrm{rc}(G)$ is NP-hard ([3, 11]). In fact, it is already NP-complete to decide whether $\mathrm{rc}(G) = 2$. It is also NP-complete to decide whether a given edge-coloured graph (with an unbounded number of colours) is rainbow-connected [3]. More generally, it has been shown in [11] that for any fixed $k \geq 2$ it is NP-complete to decide whether $\mathrm{rc}(G) = k$.

For the rainbow connection numbers of graphs the following results are known (and obvious).

Proposition 1. *Let G be a connected graph of order n. Then*

(1) $1 \leq \mathrm{rc}(G) \leq n - 1$,

(2) $\mathrm{rc}(G) \geq \mathrm{diam}(G)$,

(3) $\mathrm{rc}(G) = 1$ *if and only if G is complete,*

(4) $\mathrm{rc}(G) = n - 1$ *if and only if G is a tree.*

2. Rainbow Connection and Size of Graphs

In [9] the following problem was introduced.

Problem 2. *For all integers n and k with $1 \le k \le n-1$ compute and minimize the function $f(n,k)$ with the following property: If $|V(G)| = n$ and $|E(G)| \ge f(n,k)$, then $\mathrm{rc}(G) \le k$.*

The following lower bound for $f(n,k)$ has been shown.

Proposition 3 [9]**.** *For n and k with $1 \le k \le n-1$ it holds that $f(n,k) \ge \binom{n-k+1}{2} + k - 1$.*

This lower bound is tight what can be seen by the construction of a graph G_k as follows: Take a $K_{n-k+1} - e$ and denote the two vertices of degree $n-k-1$ with u_1 and u_2. Now take a path P_k by vertices labeled $w_1, w_2, \ldots, w_k$ and identify the vertices u_2 and w_1. The resulting graph G_k has order n and size $|E(G_k)| = \binom{n-k+1}{2} + k - 2$. For its diameter we obtain $d(u_1, w_k) = \mathrm{diam}(G_k) = k+1 = \mathrm{rc}(G_k)$.

Problem 4. *Determine all values of n and k such that*

$$f(n,k) = \binom{n-k+1}{2} + k - 1. \tag{1}$$

It has been shown in [9] that $f(n,k) = \binom{n-k+1}{2} + k - 1$ for $k = 1, 2, n-2$, and $n-1$ and in [12] for $k = 3$ and 4. This is summarized in the following theorem.

Theorem 5. *For all integers n and k with $k = 1, 2, 3, 4, n-2, n-1$ it holds that $f(n,k) = \binom{n-k+1}{2} + k - 1$.*

The main result of this paper is the solution of Problem 4 for all graphs of order n for k satisfying $n-6 \le k \le n-3$.

The proof of this result consists of several parts. First, we prove for 2-connected graphs G of order n and size at least $\binom{n-k+1}{2} + k - 1$ that $\mathrm{rc}(G) \le k$ if $n-5 \le k \le n-3$. In the second step we prove this for $k = n-6$ and 2-connected graphs G where the case $n = 11$ and $k = 6$ covers most effort. Finally, we prove in the third step the statement for $n-6 \le k \le n-3$ and connected graphs that are not 2-connected.

Recently an improved upper bound for the rainbow connection number of 2-connected graphs has been shown.

Lemma 6 [7]**.** *Let G be a 2-connected graph with n vertices. Then $\mathrm{rc}(G) \le \lceil \frac{n}{2} \rceil$.*

Theorem 7. *If G is a 2-connected graph with $|V(G)| = n$ and $|E(G)| \ge \binom{n-k+1}{2} + k - 1$, then $\mathrm{rc}(G) \le k$ if $n-5 \le k \le n-3$ or if $k = n-6$ and $n \ge 12$.*

Proof. We may assume that $k \geq 5$ since (1) holds for $1 \leq k \leq 4$. This implies
$n \geq 8$ if $k = n-3$ and thus $\mathrm{rc}(G) \leq \lceil \frac{n}{2} \rceil \leq n-4 < n-3$,
$n \geq 9$ if $k = n-4$ and thus $\mathrm{rc}(G) \leq \lceil \frac{n}{2} \rceil \leq n-4$,
$n \geq 10$ if $k = n-5$ and thus $\mathrm{rc}(G) \leq \lceil \frac{n}{2} \rceil \leq n-5$,
$n \geq 11$ if $k = n-6$ and thus $\mathrm{rc}(G) \leq \lceil \frac{n}{2} \rceil \leq n-6$ for all $n \geq 12$. ■

3. $f(11,5) = 25$

In the proof of the next theorem we will use the following result.

Lemma 8 [4]. *Let G be a connected graph with $\delta(G) \geq 2$ and D be a dominating set of G such that $G[D]$ is connected. Then* $\mathrm{rc}(G) \leq \mathrm{rc}(G[D]) + 3$.

Let an *rc colouring* of a graph G be an edge colouring such that G is rainbow-connected.

Lemma 9. *Let G be a connected graph with a partition of its vertex set $V(G)$ into two subsets V_1, V_2 such that $G[V_1]$ and $G[V_2]$ are both connected and V_1 is a dominating set in G. If* $\mathrm{rc}(G[V_1]) \leq k$ *and* $\mathrm{rc}(G[V_2]) \leq l$ *for integers $k, l \geq 1$ then* $\mathrm{rc}(G) \leq \max\{k, l\} + 1$.

Proof. Take an rc colouring of $G[V_1]$ with k colours and an rc colouring of $G[V_2]$ with l colours and colour all edges between V_1 and V_2 with an additional colour. Then G is rainbow-connected and therefore $\mathrm{rc}(G) \leq \max\{k, l\} + 1$. ■

Theorem 10. *Let $G = (V, E)$ be a 2-connected graph of order $|V| = 11$, size $|E| = 25$ and maximum degree $\Delta(G) \geq 7$. Then* $\mathrm{rc}(G) \leq 5$.

Proof. If $\Delta(G) = 10$ then G contains a dominating vertex and hence $\mathrm{rc}(G) \leq 3$ by Lemma 8. If $\Delta(G) = 9$ then G contains a dominating K_2 and therefore $\mathrm{rc}(G) \leq 1 + 3 = 4$. If $\Delta(G) = 8$ then G always contains a dominating P_3 and hence $\mathrm{rc}(G) \leq 2 + 3 = 5$.

Suppose now that $\Delta(G) = 7$. Let w be a vertex with $d(w) = 7$ and let $F = G[N[w]]$ and $H = G[V \setminus N[w]]$ which implies $|V(H)| = 3$.

First assume that $H \cong 3P_1$ or $H \cong P_2 \cup P_1$. If there is a vertex $w_1 \in N(w)$ such that $N_H(w_1) = V(H)$ then the vertices w and w_1 induce a dominating P_2 implying that $\mathrm{rc}(G) \leq 1+3 = 4 < 5$. If there are two vertices $w_1, w_2 \in N(w)$ such that $N_H(w_1) \cup N_H(w_2) = V(H)$ then (w_1, w, w_2) is a dominating P_3 implying that $\mathrm{rc}(G) \leq 2 + 3 = 5$. Otherwise, $|E(F, H)| \leq 7$. Then $|E(F)| \geq 25 - (7 + 1) = 17$ and so $\mathrm{rc}(F) \leq 3$ by Theorem 5. Take an rc colouring of F with colours $1, 2, 3$. If $H \cong 3P_1$ then colour all edges incident with a vertex of H with colours 4 and 5 such that each colour occurs at least once at every vertex of H. If $H \cong P_2 \cup P_1$, say $V(P_2) = \{u_1, u_2\}$, $V(P_1) = \{u_3\}$, then colour the edge u_1u_2 with colour 1, all

other edges incident to u_1 or u_2 with colour 4, and all edges incident to u_3 with colour 5 to obtain a rainbow colouring of G with 5 colours.

Next assume that $H \cong K_3$. Then there is a dominating P_3 implying $\mathrm{rc}(G) \leq 2+3=5$ by Lemma 8.

Finally, assume that $H \cong P_3$ with $P_3 = (u_1, u_2, u_3)$. If $w_1u_2 \in E(G)$ for a vertex $w_1 \in N(w)$ then $\{w, w_1, u_2\}$ is a dominating set implying that $\mathrm{rc}(G) \leq 2+3=5$. Hence we may assume that $N(u_2) \cap N(w) = \emptyset$. Then $N(u_i) \cap N(w) \neq \emptyset$ for $i=1$ and $i=3$ since G is 2-connected. If $w_1u_1, w_1u_3 \in E(G)$ for a vertex $w_1 \in N(W)$ then $\{w, w_1, u_1\}$ is a dominating set implying that $\mathrm{rc}(G) \leq 2+3=5$.

Hence we may assume that $N(u_1) \cap N(u_3) \cap N(w) = \emptyset$. Then $|E(F)| \geq 25-(7+2)=16$. Let $d_F(u_i) = d_i$ for $i=1,3$ and $1 \leq d_1 \leq d_3$. Then $d_1 \leq 3$. Let $X = \{u_1, u_2\}$ and $Y = V(G) \setminus X$. Then $G[X]$ and $G[Y]$ are connected, Y dominates X, and $|E(G[Y])| \geq 25-5=20$. We then obtain $\mathrm{rc}(G[Y]) \leq 4$ by Theorem 5, and thus Lemma 9 implies that $\mathrm{rc}(G) \leq 5$. ■

We will use the following lemma, which is just a special case of a very strong theorem characterizing k-connected graphs proven independently by Győri [8] and Lovász [14].

Lemma 11. *If $G=(V,E)$ is a 2-connected graph of order n, then for every pair of vertices v_1, v_2 and for every pair of positive integers n_1, n_2 with $n_1+n_2=n$ there exists a partition of V into two subsets V_1, V_2 such that $v_1 \in V_1, v_2 \in V_2$, $|V_1|=n_1$, $|V_2|=n_2$ and the induced graphs $G[V_1]$, $G[V_2]$ are connected.*

Theorem 12. *Let $G=(V,E)$ be a 2-connected graph of order $|V|=11$, size $|E|=25$, maximum degree $\Delta(G) \leq 6$, and minimum degree $\delta(G)=2$. Then $\mathrm{rc}(G) \leq 5$.*

Proof. Let $w \in V(G)$ be a vertex with $d(w)=2$ and w_1, w_2 be its two neighbours. Then $d(w_i) \leq 6$ for $i=1,2$ since $\Delta(G) \leq 6$. By Lemma 11 there is a partition of V into two subsets V_1, V_2 such that $w \in V_1$, $|V_1|=2$, $|V_2|=9$ and the induced graphs $G[V_1]$, $G[V_2]$ are connected. We conclude that $|E(G[V_2])| \geq 25-7=18$. Hence $\mathrm{rc}(G[V_2]) \leq 4$ by Theorem 5. Clearly, $\mathrm{rc}(G[V_1]) = 1 < 4$ and V_2 is a dominating set in G which induces a connected subgraph. Therefore, $\mathrm{rc}(G) \leq 5$ by Lemma 9. ■

To organize the proof of the next theorem we provide a couple of lemmas. The following facts are just special cases of Theorem 5.

Proposition 13. *Let G_1 be a connected graph of order 5 and G_2 be a connected graph of order 6. Then*

(1) $\mathrm{rc}(G_1) \leq 3$ *if* $|E(G_1)| \geq 5$,

(2) $\mathrm{rc}(G_1) \leq 2$ *if* $|E(G_1)| \geq 7$,

(3) $\mathrm{rc}(G_2) \leq 3$ *if* $|E(G_2)| \geq 8$,

(4) $\mathrm{rc}(G_2) \leq 2$ *if* $|E(G_2)| \geq 11$.

Lemma 14. *Let $G = (V, E)$ be a 2-connected graph of order 11, size 25 and $\delta(G) \geq 3$. Then there exists a partition of the vertex set V into two sets, X and Y, such that $|X| = 5$ and $|Y| = 6$, the graphs $G_1 = G[X]$ and $G_2 = G[Y]$ are connected, G_2 is of size at least 6.*

Proof. By Lemma 11 there exists a partition of V into sets X and Y such that $|X| = 5$, $|Y| = 6$ and $G_1 = G[X]$, $G_2 = G[Y]$ are connected, and hence $|E(G_1)| \geq 4$, $|E(G_2)| \geq 5$. If $|E(G_2)| \geq 6$, then we are done, so assume that $|E(G_2)| = 5$, hence G_2 is a tree. Let y be a leaf of G_2. Since $\delta(G) \geq 3$, y must have at least two neighbours in X, and thus $Y \setminus \{y\}$ and $X \cup \{y\}$ form a desired partition of V. ■

Let $G = (V, E)$ be a 2-connected graph of order 11, size 25 and $\delta(G) \geq 3$. Our aim is to give a rainbow colouring of the edges of G with five colours. Let X, Y be a partition as in Lemma 14. Then by Theorem 5 four colours, say $1, 2, 3, 4$, suffice for an rc colouring of G_1 and G_2, respectively. The fifth colour, say 5, will be used for the edges of the set $E(X, Y)$, the set of edges between vertices of X and vertices of Y. If X dominates Y or Y dominates X then we are done by Lemma 9. If this is not the case then denote by X' the set of those vertices of X that are not connected with Y, and, similarly, denote by Y' the set of those vertices of Y that are not connected with X. Since G is 2-connected, we have $1 \leq |X'| \leq 3$ and $1 \leq |Y'| \leq 4$. Moreover, since $\delta(G) \geq 3$, the vertices in X' and Y' are of degree at least three in $G_1 = G[X]$ and $G_2 = G[Y]$, respectively.

Let f be an rc colouring of the graph G_1 with $X' \neq \emptyset$. A colour of an edge of a rainbow path connecting a vertex of X' with any vertex of $X \setminus X'$ is called *transit*. We are interested in the minimum number of transit colours sufficient to go from every vertex of X' to any (i.e., at least one) vertex of $X \setminus X'$ by a rainbow path using these colours. The minimum is taken over all possible rc colourings of G_1 with (at most) **four** colours. This number will be denoted by $t(X')$. We define analogously $t(Y')$, where the minimum is taken over all possible rc colourings of G_2 with **four** colours. Evidently, if $|X'| = 1$, then $t(X') = 1$.

Lemma 15 (Transit Lemma)**.** *Let G be a 2-connected graph of order 11, size 25 and $\delta(G) \geq 3$, and let X, Y be a partition of V as in Lemma 14 with $G_1 = G[X]$ and $G_2 = G[Y]$. Then there is an rc colouring of G with five colours if*

(1) $t(X') + \mathrm{rc}(G_2) \leq 4$ *or*

(2) $\mathrm{rc}(G_1) + t(Y') \leq 4$.

Proof. We shall prove only (1). The proof of (2) is analogous. Let f_1 be an rc colouring of G_1 that minimizes the parameter $t(X')$ and let f_2 be an rc colouring

of G_2 that minimizes the parameter $\mathrm{rc}(G_2)$. Without loss of generality, we may assume that the colours used as transit colours in G_1 and the colours used by f_2 form disjoint sets. Let now x be a vertex of X'. By definition of $t = t(X')$ we are able to reach the set $X'' = X \setminus X'$ (from $x \in X'$) by a rainbow path of length at most t, i.e., using at most t colours. Next, we go to a vertex $y \in Y$ by an edge belonging to $E(X, Y)$, i.e., coloured by the fifth colour. From y we are able to reach all remaining vertices of Y using at most $\mathrm{rc}(G_2)$ colours. If $x \in X \setminus X'$ then we go directly to Y. ■

In order to apply the transit lemma above, we shall first estimate the parameter t in distinct cases.

Lemma 16. *Let G_1 be a connected graph of order 5 with vertex set X and let X' be a nonempty subset of X. Suppose that all vertices of X' are of degree at least three in G_1.*

(1) *If $|X'| = 2$, then $t(X') = 1$ and $|E(G_1)| \geq 5$.*

(2) *If $|X'| = 3$, then $t(X') \leq 2$ and $|E(G_1)| \geq 6$.*

Proof. The claimed size of $|E(G_1)|$ is simply a consequence of the fact that every vertex in X' is of degree at least 3, while every vertex in $X'' = X \setminus X'$ must have degree at least 1 by connectedness of G_1.

Assume first that $|X'| = 2$ and $X' = \{x_1, x_2\}$, hence $|E(G_1)| \geq 5$. Then, since $d(x_1), d(x_2) \geq 3$, x_1 and x_2 must have a common neighbour, say x, such that x_1, x, x_2 lie on a common cycle, i.e., a triangle if x_1 and x_2 are adjacent, or a square otherwise. Thus we may temporarily remove the edge xx_2 and find an rc colouring of the remaining graph with four colours. Then it is sufficient to use the same colour for xx_2 as is used for xx_1 to obtain an rc colouring of G_1 with $t(X') = 1$.

Suppose then that $|X'| = 3$ and $X' = \{x_1, x_2, x_3\}$, hence $|E(G_1)| \geq 6$. Then, since $d(x_1)$, $d(x_2)$, $d(x_3) \geq 3$, it is easy to see that there must exist $x \in X''$ forming a triangle with two vertices of X'. Indeed, this is obvious if X' induces a complete graph in G_1. Otherwise, if, e.g.(without loss of generality) $x_1x_2 \in E(G_1)$ and $x_2x_3 \notin E(G_1)$, then x_1 must have a neighbour $x \in X''$, while x_2 is adjacent with both vertices from X'' (in particular with x). Then, analogously as above, we first fix an rc colouring of G_1 with the edge xx_2 removed, and then colour xx_2 with the colour of xx_1. Since x_3 must have at least one neighbour in X'', the proof is completed. ■

Lemma 17. *Let G_2 be a connected graph of order 6 and size ≥ 6 with vertex set Y and let Y' be a nonempty subset of Y. Suppose that all vertices of Y' are of degree at least three in G_2.*

(1) *If $|Y'| = 2$, then $t(Y') = 1$.*

(2) *If* $|Y'| = 3$, *then* $t(Y') \leq 2$.

(3) *If* $|Y'| = 4$, *then* $t(Y') \leq 3$.

Proof. First, note that if G_2 contains a triangle, we may construct an rc colouring of the edges of G_2 with four colours as follows. We contract the edges of this triangle (ignoring multiple edges and loops), then we choose an rc colouring of the resulting connected graph of order 4 with colours $2, 3, 4$, subsequently we reverse the contraction process returning to the original graph G_2 (where the meanwhile ignored multiple edges copy the colour of their retained corespondents), and finally we colour the edges of the triangle with colour 1. If, on the other hand, G_2 contains a square, we analogously construct an rc colouring by contracting the set of edges of this square, then using colours $3, 4$ for the resulting graph, and finally colouring the edges of the square alternately $1, 2, 1, 2$ (and putting arbitrary of the colours on the possibly remaining edges).

Suppose $|Y'| = 2$, $Y' = \{y_1, y_2\}$. Since $d(y_1), d(y_2) \geq 3$ then, if $y_1y_2 \notin E(G_2)$, G_2 contains a square with (opposite) vertices y_1, y_2. Then the assertion follows by the contraction construction above. If, on the other hand, $y_1y_2 \in E(G_2)$ then, if G_2 contains a triangle or a square including y_1 and y_2, we are again done by the construction above. Otherwise, $N(y_1)$ and $N(y_2)$ are disjoint (hence $|N(y_1) \setminus \{y_2\}| = 2$ and $|N(y_2) \setminus \{y_1\}| = 2$) and there are no edges between $N(y_1) \setminus \{y_2\}$ and $N(y_2) \setminus \{y_1\}$. Since $|E(G_2)| \geq 6$, the vertices from $N(y_1) \setminus \{y_2\}$ or (symmetrically) $N(y_2) \setminus \{y_1\}$ must form an edge in G_2. Then colour the edges incident to y_1 but not to y_2 with colours 1 and 2, also the edges incident to y_2 but not to y_1 with colours 1 and 2, the edge connecting the two vertices of $N(y_1) \setminus \{y_2\}$ (or of $N(y_2) \setminus \{y_1\}$) with colour 3, and the edge y_1y_2 with colour 4.

Suppose now that $|Y'| = 3$ and $Y' = \{y_1, y_2, y_3\}$. If Y' induces a triangle in G_2, then we apply the contraction construction above using this triangle. Then from every y_i we can reach $Y'' = Y \setminus Y'$ by (optionally) first going through an edge coloured with 1 and then through any (previously fixed) edge joining Y' and Y''. Otherwise, since $d(y_1)$, $d(y_2)$, $d(y_3) \geq 3$, it is very easy to verify that G_2 must contain a triangle with two vertices from Y' or a square with two elements from Y' as opposite vertices, hence we are done by the contraction construction above.

Let finally $|Y'| = 4$, $Y' = \{y_1, y_2, y_3, y_4\}$ and $Y'' = Y \setminus Y' = \{y_5, y_6\}$. Without loss of generality, assume that $d(y_6) \leq d(y_5)$. Then it is easy to verify that the graph $G_2 - y_6$ is connected and has size at least 5 (since $d(y_1)$, $d(y_2)$, $d(y_3)$, $d(y_4) \geq 3$), hence, by Proposition 13, we may choose an rc colouring using colours $1, 2, 3$ which implies $t(Y') \leq 3$. Then we are done by using colour 4 on the edges incident to y_6. ■

Theorem 18. *Let* $G = (V, E)$ *be a* 2*-connected graph of order* $|V| = 11$, *size* $|E| = 25$, *and minimum degree* $\delta(G) \geq 3$. *Then* $\mathrm{rc}(G) \leq 5$.

Proof. Let X, Y be a partition as in Lemma 14 with X', Y' being the subsets of those vertices from X, Y which have no neighbours in Y, X, respectively. Set $X'' = X \setminus X'$, $Y'' = Y \setminus Y'$. If $X' = \emptyset$ or $Y' = \emptyset$, then the result is obvious. Assume then that $|X'| \geq 1$, $|Y'| \geq 1$ ($|X'| \leq 3$, $|Y'| \leq 4$ since G is 2-connected). We will consider four cases with respect to the size of $|Y'|$.

Case 1. $|Y'| = 4$. Consider first the case where $|X'| = 1$ or $|X'| = 2$. Then $t(X') = 1$ by Lemma 16. If $\mathrm{rc}(G_2) \leq 3$, we are done by Lemma 15. So, we may assume that either $\mathrm{rc}(G_2) \leq 3$ and $|X'| \geq 3$ or $\mathrm{rc}(G_2) = 4$.

If $\mathrm{rc}(G_2) \leq 3$ and $|X'| \geq 3$, then $|E(X, Y)| \leq 4$ (since $|Y'| = 4$) and, in consequence, $|E(G_2)| \geq 11$ (since $|E(G_1)| \leq 10$). Then $\mathrm{rc}(G_2) \leq 2$ by Proposition 13. So, we can apply Lemma 15(1) since $t(X') \leq 2$.

If, on the other hand, $\mathrm{rc}(G_2) = 4$, then $|E(G_2)| \leq 7$ by Proposition 13. Since $|E(X, Y)| \leq 8$ (because of $|X'| \geq 1$) we have $|E(G_1)| = 10$, i.e., $G_1 \cong K_5$. Then $\mathrm{rc}(G_1) = 1$. Moreover, by Lemma 17, $t(Y') \leq 3$, hence we are done by Lemma 15(1).

Case 2. $|Y'| = 3$. Then $t(Y') \leq 2$, and, if $\mathrm{rc}(G_1) \leq 2$, then we are done by Lemma 15(2). So, assume that $\mathrm{rc}(G_1) \geq 3$. Then, in particular, $|E(G_1)| \leq 6$ by Proposition 13.

Consider first the case $|X'| = 3$. Then $|E(G_1)| + |E(X, Y)| \leq 12$, which implies that $|E(G_2)| \geq 13$. But then $\mathrm{rc}(G_2) \leq 2$ by Proposition 13 and, since $t(X') \leq 2$, we can again apply Lemma 15(1).

Now suppose that $|X'| = 2$. Then $t(X') = 1$ by Lemma 17 and we are done if $\mathrm{rc}(G_2) \leq 3$. If not, then $\mathrm{rc}(G_2) = 4$ and $|E(G_2)| \leq 7$ by Proposition 13. But then $|E(G_1)| \geq 9$, and hence $\mathrm{rc}(G_1) \leq 2$ again by Proposition 13. Thus we are done by Lemma 15(2).

Finally, consider the case $|X'| = 1$ (which implies $t(X') = 1$). Let $X' = \{a\}$ and denote by x one of its neighbours, $x \in X''$. As above, we are done if $\mathrm{rc}(G_2) \leq 3$. So, suppose that $\mathrm{rc}(G_2) = 4$. By Proposition 13 we have $|E(G_2)| \leq 7$. Thus $|E(G_1)| \geq 6$. If $|E(G_1)| \geq 7$, then $\mathrm{rc}(G_1) \leq 2$ and, since $t(Y') \leq 2$, we are done. So, $|E(G_1)| = 6$, and thus the set $E(X'', Y'')$ has all possible edges. In consequence, x is connected by an edge to all vertices in Y'', and the result follows.

Case 3. $|Y'| = 2$. Then $t(Y') = 1$ by Lemma 17 and we are done if $\mathrm{rc}(G_1) \leq 3$. So, assume that $\mathrm{rc}(G_1) = 4$. Thus G_1 is a tree, $|E(G_1)| = 4$. But then, by Lemma 16, $|X'| = 1$. Let $X' = \{a\}$ and $Y' = \{b, c\}$. By Lemma 17 we may choose an rc colouring of G_2 with four colours such that we have two edges by_1, cy_2 with $y_1, y_2 \in Y''$ coloured the same (in particular we may have $y_1 = y_2$). As above, we are done if $\mathrm{rc}(G_2) \leq 3$. So, suppose that G_2 has less than 8 edges. Then there are at least 14 edges between X and Y. So, we can have at most two edges 'missing' in $E(X'', Y'')$, hence at least one of (at least) three neighbours of

a, say $x \in X''$, is joined to both y_1 and y_2, and the result follows.

Case 4. $|Y'| = 1$. Let $Y' = \{b\}$ and denote by y any neighbour of b, $y \in Y''$. As above, we are done if $\mathrm{rc}(G_1) \leq 3$. So, assume that $\mathrm{rc}(G_1) = 4$. Thus G_1 is a tree, and hence $|X'| = 1$ by Lemma 16. Say $X' = \{a\}$. Since $d(a) \geq 3$, for each vertex x of X'' there is a path of length at most 2 (i.e., with at most 2 colours in any rainbow colouring of the tree G_1) from this vertex x to a. On the other hand, by the definition of Y', at least one vertex from X'' is joined to y, and again the result follows. ■

4. G Is Not 2-connected

Lemma 19 (Bridge reduction)**.** *Let G be a connected graph, $e = uw$ be a bridge of G, and let G' be the graph obtained from G by contracting the edge e. If $\mathrm{rc}(G') = p$, then $\mathrm{rc}(G) \leq p + 1$.*

Proof. Choose an rc colouring of G' with colours $1, 2, \ldots, p$. Now colour the edge e with colour $p + 1$. Then G is rainbow-connected and $\mathrm{rc}(G) \leq p + 1$. ■

Theorem 20. *If G is a connected graph which is not 2-connected with $|V(G)| = n$ and $|E(G)| \geq \binom{n-k+1}{2} + k - 1$, then $\mathrm{rc}(G) \leq k$ if $n - 6 \leq k \leq n - 3$.*

Proof. Setting $g(n, k) = \binom{n-k+1}{2} + k - 1$ and $k = n - t$ we obtain $g(n, n - t) = n + \binom{t}{2} - 1$.

The proof will be by induction on the number n of vertices. For $n = t + 1$ we obtain $g(t + 1, 1) = \binom{t+1}{2} = t + 1 + \binom{t}{2} - 1$.

Consider for the induction step from n to $n+1$ a graph G of order $n+1$ with $n + 1 + \binom{t}{2} - 1$ edges.
If G contains a bridge, say e, then we apply the bridge reduction of Lemma 19 and obtain $|E(G')| = (n + 1 + \binom{t}{2} - 1) - 1 = g(n, n - t)$. Hence $\mathrm{rc}(G') \leq n - t$ by induction hypothesis and therefore $\mathrm{rc}(G) \leq n + 1 - t$ by Lemma 19.

If G contains no bridge but a cut vertex, say w, then G can be decomposed into two subgraphs G_1, G_2 such that $V(G_1) \cup V(G_2) = V(G)$ and $V(G_1) \cap V(G_2) = \{w\}$. Let $n_i = |V(G_i)|$ and $m_i = |E(G_i)| = n_i + t_i$ for $i = 1, 2$. Since G has no bridges we conclude that $n_i \geq 3$ and $t_i \geq 0$ for $i = 1, 2$.
For each t_i there exists an integer $s_i \geq 2$ such that $t_i + 1 \geq \binom{s_i}{2}$. Choose s_i maximal with this property. Then $t_i + 1 \leq \binom{s_i+1}{2} - 1$.

Assume that $s_1 + s_2 \leq t$. With $|E(G)| = n + 1 + \binom{t}{2} - 1 = (n_1 + t_1) + (n_2 + t_2)$ we obtain

$$\begin{aligned} t_1 + t_2 + 1 &= \binom{t}{2} = \sum_{j=1}^{t-1} j \geq \sum_{j=1}^{s_1+s_2-1} j = \sum_{j=1}^{s_1} j + \sum_{j=s_1+1}^{s_1+s_2-1} j \\ &> \sum_{j=1}^{s_1-1} j + s_1 + \sum_{j=s_1+1}^{s_1+s_2-1} (j - (s_1 - 1)) \\ &= \sum_{j=1}^{s_1-1} j + (s_1 - 1) + \sum_{j=1}^{s_2} j \\ &> \sum_{j=1}^{s_1-1} j + (s_1 - 1) + \sum_{j=1}^{s_2-1} j + (s_2 - 1) \\ &= \binom{s_1+1}{2} - 1 + \binom{s_2+1}{2} - 1 \geq (t_1 + 1) + (t_2 + 1), \end{aligned}$$

a contradiction. Hence we have $s_1 + s_2 \geq t + 1$.

Now we apply the induction hypothesis. We have $|E(G_i)| = n_i + t_i = n_i + (t_i+1)-1 \geq n_i + \binom{s_i}{2} - 1$. Hence $\mathrm{rc}(G_i) \leq n_i - s_i$ for $i = 1, 2$. Choose for $i = 1, 2$ an rc colouring of G_i with $n_i - s_i$ distinct colours where the colour sets for G_1 and G_2 are disjoint. Note that if $s_i = 2$ this is possible by Theorem 5 and if $3 \leq s_i \leq t$ this is possible by induction if G_i is not 2-connected and by Theorems 7, 10, 12, 18 if G_i is 2-connected. This colouring is an rc colouring of G. Moreover, $\mathrm{rc}(G) \leq (n_1 - s_1) + (n_2 - s_2) = (n + 1) - (s_1 + s_2) \leq (n + 1) - (t + 1) = n - t$ which concludes the proof. ■

Summarizing the results of Theorems 7, 10, 12, 18, 20 we obtain together with the remark after Proposition 3 our main theorem.

Theorem 21. *For all integers n and k with $n - 6 \leq k \leq n - 3$ it holds that $f(n, k) = \binom{n-k+1}{2} + k - 1$.*

It would be an interesting task to determine additional values of n and k (beside those of Theorems 5 and 21) such that $f(n, k) = \binom{n-k+1}{2} + k - 1$. Of course, partial results can be obtained by applying Lemma 6.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory (Springer, 2008). doi:10.1007/978-1-84628-970-5

[2] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, and R. Yuster, *On rainbow connection*, Electron. J. Combin. **15** (2008) #57.

[3] S. Chakraborty, E. Fischer, A. Matsliah, and R. Yuster, *Hardness and algorithms for rainbow connectivity*, J. Comb. Optim. **21** (2011) 330–347. doi:10.1007/s10878-009-9250-9

[4] L.S. Chandran, A. Das, D. Rajendraprasad, and N.M. Varma, *Rainbow connection number and connected dominating sets*, J. Graph Theory **71** (2012) 206–218. doi:10.1002/jgt.20643

[5] G. Chartrand, G.L. Johns, K.A. McKeon, and P. Zhang, *Rainbow connection in graphs*, Math. Bohemica **133** (2008) 85–98.

[6] A.B. Ericksen, A matter of security, Graduating Engineer & Computer Careers (2007) 24–28.

[7] J. Ekstein, P. Holub, T. Kaiser, M. Koch, S. Matos Camacho, Z. Ryjáček and I. Schiermeyer, *The rainbow connection number in* 2*-connected graphs*, Discrete Math. doi:10.1016/j.disc.2012.04.022

[8] E. Győri, *On division of graphs to connected subgraphs*, Combinatorics, Vol. I, pp. 485–494, Colloq. Math. Soc. János Bolyai, 18, North-Holland, Amsterdam, 1978.

[9] A. Kemnitz and I. Schiermeyer, *Graphs with rainbow connection number two*, Discuss. Math. Graph Theory **31** (2011) 313–320. doi:10.7151/dmgt.1547

[10] M. Krivelevich and R. Yuster, *The rainbow connection of a graph is* (*at most*) *reciprocal to its minimum degree*, J. Graph Theory **63** (2010) 185–191.

[11] V.B. Le and Zs. Tuza, *Finding optimal rainbow connection is hard*, Preprint 2009.

[12] X. Li, M. Liu, and I. Schiermeyer, *Rainbow connection number of dense graphs*, to appear in Discuss. Math. Graph Theory. arXiv: 1110.5772v1 [math.CO] 2011

[13] X. Li and Y. Sun, *Rainbow Connections of Graphs*, Springer Briefs in Math., Springer, New York, 2012.

[14] L. Lovász, *A homology theory for spanning trees of a graph*, Acta Math. Hungar. **30** (1977) 241–251. doi:10.1007/BF01896190

[15] I. Schiermeyer, *Rainbow connection in graphs with minimum degree three*, Lecture Notes Computer Science **5874** (2009) 432–437. doi:10.1007/978-3-642-10217-2_42

Received 11 April 2012
Revised 11 July 2012
Accepted 11 July 2012