Discussiones Mathematicae Graph Theory 32 (2012) 685–704 doi:10.7151/dmgt.1638

# MINIMAL TREES AND MONOPHONIC CONVEXITY

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#### Abstract

Let V be a finite set and  $\mathcal{M}$  a collection of subsets of V. Then  $\mathcal{M}$ is an alignment of V if and only if  $\mathcal{M}$  is closed under taking intersections and contains both V and the empty set. If  $\mathcal{M}$  is an alignment of V, then the elements of  $\mathcal{M}$  are called convex sets and the pair  $(V, \mathcal{M})$  is called an alignment or a convexity. If  $S \subseteq V$ , then the convex hull of S is the smallest convex set that contains S. Suppose  $X \in \mathcal{M}$ . Then  $x \in X$  is an extreme point for X if  $X \setminus \{x\} \in \mathcal{M}$ . A convex geometry on a finite set is an aligned space with the additional property that every convex set is the convex hull of its extreme points. Let G = (V, E) be a connected graph and U a set of vertices of G. A subgraph T of G containing U is a minimal U-tree if T is a tree and if every vertex of  $V(T) \setminus U$  is a cut-vertex of the subgraph induced by V(T). The monophonic interval of U is the collection of all vertices of G that belong to some minimal U-tree. Several graph convexities are defined using minimal U-trees and structural characterizations of graph classes for which the corresponding collection of convex sets is a convex geometry are characterized.

**Keywords:** minimal trees, monophonic intervals of sets, *k*-monophonic convexity, convex geometries.

2010 Mathematics Subject Classification: 05C75, 05C12, 05C17, 05C05.

<sup>&</sup>lt;sup>1</sup>Supported by an NSERC grant CANADA

### 1. INTRODUCTION

This paper is motivated by the results and ideas contained in [10, 11, 15]. We introduce new graph convexities and show how these give rise to structural characterizations of certain graph classes. For graph terminology we follow [4] and [9]. All graphs considered here are connected, finite, simple (i.e., without loops and multiple edges), unweighted and undirected. The structural characterizations of graphs that we describe are often given in terms of forbidden subgraphs. Let G and F be graphs. Then F is an *induced subgraph* of G if F is a subgraph of G and for every  $u, v \in V(F)$ ,  $uv \in E(F)$  if and only if  $uv \in E(G)$ . We say a graph G is F-free if it does not contain F as an induced subgraph. Suppose C is a collection of graphs. Then G is C-free if G is F-free for every  $F \in C$ . If F is a path or cycle that is a subgraph of G, then F has a *chord* if it is not an induced subgraph of G, i.e., F has two vertices that are adjacent in G but not in F. An induced cycle of length at least 5 is called a *hole*.

Let V be a finite set and  $\mathcal{M}$  a collection of subsets of V. Then  $\mathcal{M}$  is an alignment (or convexity) of V if and only if  $\mathcal{M}$  is closed under taking intersections and contains both V and the empty set. If  $\mathcal{M}$  is an alignment of V, then the elements of  $\mathcal{M}$  are called convex sets and the pair  $(V, \mathcal{M})$  is called an aligned space or a convexity. If  $S \subseteq V$ , then the convex hull of S is the smallest convex set that contains S. Suppose  $X \in \mathcal{M}$ . Then  $x \in X$  is an extreme point for X if  $X \setminus \{x\} \in \mathcal{M}$ . The collection of all extreme points of X is denoted by ex(X). A convex geometry on a finite set V is an aligned space  $(V, \mathcal{M})$  with the additional property that every convex set is the convex hull of its extreme points. This property is referred to as the Minkowski-Krein-Milman (MKM) property. For a more extensive overview of other abstract convex structures see [18]. Convexities associated with the vertex set of a graph are discussed for example in [4]. Their study is of interest in Computational Geometry and has applications in Game Theory [3].

Convexities on the vertex set of a graph are usually defined in terms of some type of 'intervals'. Suppose G is a connected graph and u, v two vertices of G. Then a u - v geodesic is a shortest u - v path in G. Such geodesics are necessarily induced paths. However, not all induced paths are geodesics. The *g*-interval (respectively, *m*-interval) between a pair u, v of vertices in a graph G is the collection of all vertices that lie on some u - v geodesic (respectively, induced u - v path) in G and is denoted by  $I_q[u, v]$  (respectively,  $I_m[u, v]$ ).

A subset S of vertices of a graph is said to be g-convex (m-convex) if it contains the g-interval (m-interval) between every pair of vertices in S. It is not difficult to see that the collection of all g-convex (m-convex) sets is an alignment of V. A vertex v is an extreme point for a g-convex (or m-convex) set S if and only if v is simplicial in the subgraph induced by S, i.e., every two neighbours of

v in S are adjacent. Of course the convex hull of the extreme points of a convex set S is contained in S, but equality holds only in special cases. In [11] those graphs for which the g-convex sets form a convex geometry are characterized as the chordal 3-fan-free graphs (see Figure 1). These are precisely the chordal, distance-hereditary graphs (see [2, 12]). In the same paper it is shown that the chordal graphs are precisely those graphs for which the m-convex sets form a convex geometry.

For what follows we use  $P_k$  to denote an induced path of order k. A vertex is simplicial in a set S of vertices if and only if it is not the centre vertex of an induced  $P_3$  in in the subgraph  $\langle S \rangle$  induced by S. Jamison and Olariu [13] relaxed this condition. They defined a vertex to be *semisimplicial* in S if and only if it is not a centre vertex of an induced  $P_4$  in  $\langle S \rangle$ .



Indicates a centre vertex

Figure 1. Special graphs.

Dragan, Nicolai and Brandstädt [10] introduced another convexity notion that relies on induced paths. The  $m^3$ -interval between a pair u, v of vertices in a graph G, denoted by  $I_{m^3}[u, v]$ , is the collection of all vertices of G that belong to an induced u - v path of length at least 3. Let G be a graph with vertex set V. A set  $S \subseteq V$  is  $m^3$ -convex if and only if for every pair u, v of vertices of S the vertices of the  $m^3$ -interval between u and v belong to S. As in the other cases the collection of all  $m^3$ -convex sets is an alignment. Note that an  $m^3$ -convex set is not necessarily connected. It is shown in [10] that the extreme points of an  $m^3$ -convex set are precisely the semisimplicial vertices of  $\langle S \rangle$ . Moreover, those graphs for which the  $m^3$ -convex sets form a convex geometry are characterized in [10] as the (house, hole, domino, A)-free (HHDA-free) graphs (see Figure 1). In the same paper several 'local' convexities related to the  $m^3$ -convexity were studied. For a set S of vertices in a graph G, N[S] is  $S \cup N(S)$  where N(S) is the collection of all vertices adjacent with some vertex of S. A set S of vertices in a graph is connected if  $\langle S \rangle$  is connected. The following result which we will use in this paper is established in [10].

**Theorem 1.** A graph G is (house, hole, domino)-free if and only if N[S] is  $m^3$ -convex for all connected sets S of vertices of G.

The (house, hole, domino)-free graphs also arise in the study of the induced path function (see for example, [7, 8]). We next look at more recently studied graph convexities that motivate the convexities studied in this paper. In [16] a graph convexity that generalizes q-convexity is introduced. Let S be a set of vertices in a graph G. A Steiner tree T for S is a connected subgraph of G that contains Sand has the smallest number of edges among all such subgraphs. The subgraph induced by the vertices of T may not be an induced subgraph; for example, if G is a net (i.e. the graph obtained by joining a new vertex to each of the three vertices in a  $K_3$ ) and S consists of the three leaves in G, then any spanning tree of G is a Steiner tree for S. The Steiner interval of a set S of vertices in a connected graph G, denoted by I(S), is the union of all vertices of G that lie on some Steiner tree for S. Steiner intervals have been studied, for example, in [14, 17]. A set S of vertices in a graph G is k-Steiner convex ( $g_k$ -convex) if the Steiner interval of every collection of k vertices of S is contained in S. Thus S is  $q_2$ -convex if and only if it is *q*-convex. The collection of  $q_k$ -convex sets forms an aligned space. We call an extreme point of a  $g_k$ -convex set a k-Steiner simplicial vertex, abbreviated kSS vertex.

The extreme points of  $g_3$ -convex set S, i.e., the 3SS vertices are characterized in [5] as those vertices that are **not** a centre vertex of an induced claw, paw or  $P_4$ , in  $\langle S \rangle$  see Figure 1. Thus a 3SS vertex is semisimplicial. Apart from the  $g_k$ convexity, for a fixed k, other graph convexities that (i) depend on more than one value of k and (ii) combine the  $g_3$  convexity and the geodesic counterpart of the  $m^3$ -convexity are introduced and studied in [15]. In particular characterizations of convex geometries for several of these graph convexities are given. We state here only those results that are used in this paper.

A graph G is a *replicated twin*  $C_4$  if it is isomorphic to any one of the four graphs shown in Figure 2(a), where any subset of the dashed edges may belong to G. The collection of the four replicated twin  $C_4$  graphs is denoted by  $\mathcal{R}_{C_4}$ .

**Theorem 2** [15]. Let G = (V, E) be a graph. Then the following are equivalent (1) G is  $(P_4, \mathcal{R}_{C_4})$ -free.

(2)  $(V, \mathcal{M}_{q_3}(G))$  is a convex geometry.

Convex geometries give rise to 'elimination orderings' of vertices in graphs and these are particularly useful for algorithmic purposes. Suppose  $\mathbf{P}$  is a property

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that a vertex in a graph G may possess. We say G has a **P**-elimination ordering if the vertices of G can be ordered as  $\{v_1, v_2, \ldots, v_n\}$  such that for every  $i, 1 \leq i \leq n$ , vertex  $v_i$  has property **P** in the subgraph induced by  $v_i, v_{i+1}, \ldots, v_n$ . If **P** is a property that characterizes the extreme vertices with respect to a graph convexity, then it is well known (see e.g. [11]) that if the corresponding convex sets form a convex geometry, then G has a **P**-elimination ordering. The extreme vertices for the convexities studied in this paper are also the 3SS vertices. Graphs for which every LexBFS ordering is a 3SS-elimination ordering are characterized in [5]. Interestingly this class of graphs is precisely the same as the class for which the monophonic intervals, satisfying a certain betweenness axiom, also satisfy the monotone property as shown in [8].

We now introduce the notion of a 'minimal U-tree' which extends both the definition of an induced path and that of a Steiner tree. This in turn gives rise to graph convexities that extend both the *m*-convexity and  $g_k$ -convexity. Let U be a set of at least two vertices in a connected graph G. A subgraph H containing Uis a minimal U-tree if H is a tree and if every vertex  $v \in V(H) \setminus U$  is a cut-vertex of  $\langle V(H) \rangle$ , i.e., H is a minimal U-tree if H connects U and if H is a minimal subgraph that connects U in the sense that for every  $v \in V(H) \setminus U$ , U is no longer connected in  $\langle V(H) \setminus \{v\}$ . Thus if  $U = \{u, v\}$ , then a minimal U-tree is just an induced u - v path. Moreover, every Steiner tree for a set U of vertices is a minimal U-tree. The collection of all vertices that belong to some minimal U-tree is called the monophonic interval of U and is denoted by  $I_m(U)$ . A set S of vertices is k-monophonic convex, abbreviated as  $m_k$ -convex, if it contains the monophonic interval of every subset U of k vertices of S. Thus a set of vertices in G is a monophonic convex set if and only if it is an  $m_2$ -convex set. For a set  $T = \{k_1, k_2, \dots, k_t\}$  of integers such that  $2 \leq k_1 < k_2 < \dots < k_t$  a set S of vertices of G is  $m_T$ -convex if G is  $m_{k_i}$ -convex for all  $1 \leq i \leq t$ . It is not difficult to see that the collection of  $m_T$ -convex sets is an alignment of V(G), called the  $m_T$ -convex alignment. We show that the class of graphs for which the  $m_T$ -convex alignment is a convex geometry is the same as the class of graphs for which the  $m_k$ -convex alignment is a convex geometry where k is the smallest value in T.

In this paper we give structural characterizations of those classes of graphs for which the  $m_3$ -convex alignment forms a convex geometry. It turns out that these graphs are precisely the same as the graphs for which the  $g_3$ -convex alignment forms a convex geometry and thus contains no induced  $P_4$ 's. However, by combining the  $m_3$ - convexity with the  $m^3$ -convexity introduced in [10], we obtain a graph convexity for which the convex geometries cover a larger more interesting class of graphs that has no restriction on the diameter. More specifically we define a set S of vertices in a connected graph to be  $m_3^3$ -convex if S is both  $m^3$ and  $m_3$ -convex. In this paper we give structural characterizations of those classes of graphs for which the  $m_3^3$ -convex alignment forms a convex geometry.

### 2. Convex Geometries

#### 2.1. $m_3$ -convex geometries

We begin by showing that the extreme vertices of 3-monophonic convex sets are precisely the 3SS vertices of the set. If v is an extreme vertex of a 3-monophonic set S, then v cannot be the centre of an induced claw, paw or  $P_4$  in  $\langle S \rangle$ ; otherwise, v is on a minimal tree for some set of three vertices in S. Hence v is a 3SS vertex. Suppose now that v is not an extreme vertex of a 3-monophonic set S. Then there is a set U of three vertices in  $S \setminus \{v\}$  such that v lies on a minimal U-tree H. So H - v is disconnected. If v has at least three neighbours in H - v, then v is the centre of a claw or a paw. (Let x, y, z be three neighbours of v from H - vsuch that x does not belong to the same component as y or z in H - v. Then  $\langle \{v, x, y, z\} \rangle$  is a claw or paw.) If v has two neighbours, say x, y, in H - v, then either the component of H - v containing x or the component of y in H - v. Then x is nonadjacent with both y and z. So v is not 3SS. Hence if v is a 3SSvertex of a 3-monophonic set S, then v is an extreme vertex of S.

Let G = (V, E) be a connected graph and  $\mathcal{M}_{m_3}(G)$  the collection of  $m_3$ convex sets and  $\mathcal{M}_{g_3}(G)$  the collection of  $g_3$ -convex sets. In this section we determine the class of connected graphs G for which  $(V, \mathcal{M}_{m_3}(G))$  is a convex geometry. Observe that every 2-element set of vertices in a graph is  $m_3$ -convex. Thus  $m_3$ -convex sets may induce disconnected graphs but only if they consist of two nonadjacent vertices.

We observe first that if  $(V, \mathcal{M}_{m_3}(G))$  is a convex geometry, then G has no induced path of length at least 3. To see this, suppose that G is a connected graph and suppose that  $P: (u = )v_0v_1 \dots v_d (= v)$  an induced u - v path of length at least 3. Let S be the  $m_3$ -convex hull of V(P). Since u and v are the only 3SS vertices of P, the 3SS vertices of S are a subset of  $\{u, v\}$ . But the  $m_3$ -convex hull of any subset R of  $\{u, v\}$  is just R and hence does not contain all the vertices of P and thus not all the vertices of S. Thus, if  $(V, \mathcal{M}_{m_3}(G))$  is a convex geometry, then G contains no induced path of length at least 3.

**Theorem 3.** Let G be a connected graph. Then  $(V, \mathcal{M}_{m_3}(G))$  is a convex geometry if and only if  $(V, \mathcal{M}_{g_3}(G))$  is a convex geometry.

**Proof.** Suppose that  $(V, \mathcal{M}_{g_3}(G))$  is a convex geometry. Let  $S \in \mathcal{M}_{m_3}(G)$ . Then  $S \in \mathcal{M}_{g_3}(G)$ , since S contains the Steiner interval of every set of three vertices of S. Since the 3SS vertices of a set are the extreme vertices of the set with respect to both the  $g_3$ - and  $m_3$ -convexity, S is the  $g_3$ -convex hull of the 3SS vertices and hence also the  $m_3$ -convex hull of these vertices. Thus  $(V, \mathcal{M}_{m_3}(G))$  is also a convex geometry. Conversely, suppose  $(V, \mathcal{M}_{m_3}(G))$  is a convex geometry. Then, by the above observation, G has no induced paths of length at least 3. Let S be a  $g_3$ -convex set of G and  $U = \{u, v, w\}$  any set of three vertices of S. Let H be any Utree. Then H is either a path or H is homeomorphic to  $K_{1,3}$  and its leaves are contained in U. Suppose first that H is a path, say a u - w path. Then the u - vsubpath and v - w subpath of H are necessarily induced paths of G. Since G has no induced paths of length at least 3, both these subpaths contain at most three vertices. If neither of the subpaths has an interior vertex, then H is a Steiner tree for  $\{u, v, w\}$ . If exactly one of the subpaths has no interior vertex, say  $uv \in E(G)$ , then H must be a Steiner tree for  $\{u, v, w\}$ ; otherwise, the vertex of H not in  $\{u, v, w\}$  is not a cut-vertex of  $\langle V(H) \rangle$ . If both have an interior vertex, then H is a path of length 4 and is thus not an induced subgraph of G. Let u' be the neighbour of u in H and w' the neighbour of w in H. Then, by the above observation, the only possible edges in  $\langle V(H) \rangle$  that do not belong to H are edges with one end in  $\{u, u'\}$  and the other end in  $\{w, w'\}$ . Since H is a minimal  $\{u, v, w\}$ -tree the only possible edge of  $\langle V(H) \rangle$  that does not belong to H is u'w'. However, in this case uu'w'w is an induced path of length 3 which is not possible. So this case cannot occur.

Suppose now that H is a homeomorphic with  $K_{1,3}$ . Let x be the vertex of degree 3 in H. The three paths from x to each of the leaves of H must necessarily be induced. We can argue as above that at most one of these three paths has length 2. Suppose that one of these paths, say the x - u path of H has length 2. Let u' be the neighbour of u on this path. If uw or uv is an edge of G, then u' is not a cut-vertex of H. So  $uw, uv \notin E(G)$ . Since G has no induced paths of length at least 3, u'v, u'w are necessarily edges of G. However, then x is not a cut-vertex of  $\langle V(H) \rangle$ . So this case cannot occur. Thus u, v, w are all adjacent with x. Now  $\langle \{u, v, w\} \rangle$  is not connected; otherwise, x is not a cut-vertex of H. Hence H is a Steiner tree for  $\{u, v, w\}$ . So S is  $g_3$ -convex. Thus the  $m_3$ -convex hull of the extreme points of S is the same as the  $g_3$ -convex hull of the extreme points of S. So  $(V, \mathcal{M}_{g_3}(G))$  is a convex geometry.

Recall that a graph G is a replicated twin  $C_4$  if it is isomorphic to any one of the four graphs shown in Figure 2(a), where any subset of the dotted edges may belong to G. The collection of the four replicated twin  $C_4$  graphs is denoted by  $\mathcal{R}_{C_4}$ .

From Theorem 2 we now obtain the following,

**Corollary 4.** Let G = (V, E) be a graph. Then the following are equivalent:

- (1) G is  $P_4$  and  $\mathcal{R}_{C_4}$ -free.
- (2)  $(V(G), \mathcal{M}_{m_3}(G))$  is a convex geometry.

### 2.2. $m_T$ -convex geometries

In this section we show that if  $T = \{k_1, k_2, \ldots, k_t\}$  is a collection of positive integers such that  $2 \leq k_1 < k_2 < \cdots < k_t$ , then the class of graphs for which the  $m_T$ -convex sets form a convex geometry is precisely the same as the class for which the  $m_{k_1}$ -convex sets form a convex geometry. For a connected graph G let  $\mathcal{M}_{m_T}(G)$  be the collection of all  $m_T$ -convex sets of G and for an integer  $k \geq 2$ , let  $\mathcal{M}_{m_k}(G)$  be the collection of all  $m_k$ -convex sets.

**Lemma 5.** Suppose that G is a graph and  $U = \{u_1, u_2, \ldots, u_l\}$  a set of  $l \ge 2$  vertices of G. Suppose that T is a minimal U-tree and k is an integer such that  $2 \le k \le l$ . If  $v \in V(T) \setminus U$ , then there exists a k element subset W of U and a minimal W-tree T' containing v.

**Proof.** If l = 2, the result follows immediately. Suppose thus that l > 2. For the remainder of the proof we proceed by induction on k. We prove a slightly stronger result than that stated in the lemma. We show that the W-tree T' can be chosen so that it has the property that it is a subtree of T and whenever F is a subtree of T that contains T', then every vertex of  $V(T') \setminus W$  is a cut-vertex of  $\langle V(F) \rangle$ . Since v is a cut-vertex of  $H = \langle V(T) \rangle$ , v is a cut-vertex of T. Let  $x_1$  and  $y_1$  be neighbours of v in T that belong to different components of H - v, say  $H_1$  and  $H_2$ , respectively. We first construct a subtree T' of T that contains v and exactly two vertices of U. This subtree will be an induced path of which one branch at v is contained in  $H_1$  and the other branch at v in  $H_2$ . Begin the construction of T' by starting with the vertices  $v, x_1, y_1$  and the edges  $vx_1, vy_1$ . If  $x_1 \in U$ , then the construction of the branch of T' contained in  $H_1$  is completed. If  $x_1 \notin U, x_1$ is a cut-vertex of H. Let  $x_2$  be a neighbour of  $x_1$  in T such that  $x_2$  belongs to a component of  $H - x_1$  that does not contain v. Add  $x_2$  and the edge  $x_1x_2$  to the tree that is being constructed. If  $x_2 \in U$ , the construction of the branch of T' at v contained in  $H_1$  is completed; otherwise, we let  $x_3 \neq x_1$  be a neighbour of  $x_2$  in T that belongs to a component of  $H - x_2$  that does not contain  $x_1$ . We continue in this manner constructing a sequence  $x_1, x_2, x_3 \dots$  of vertices in  $H_1$  such that for  $i \geq 2$ ,  $x_i$  is a neighbour of  $x_{i-1}$  in T and such that  $x_i$  belongs to a component of  $H - x_{i-1}$  that does not contain  $x_{i-2}$ . We stop with the smallest j such that  $x_i \in U$ . Such a j must exist since the path we are constructing in  $H_1$  is a path in T starting at v and ending necessarily at a vertex of U. (We may of course reach a vertex of U before we reach a leaf of T.) The branch of T' at v contained in  $H_1$  is the path  $vx_1x_2...x_j$ . We proceed in the same manner as for  $H_1$  and  $x_1$ when constructing the branch of T' contained in  $H_2$  that starts with  $vy_1$ . These two branches produce the tree T' which is necessarily a minimal W-tree for the set W consisting of the two leaves of T'. Note that T' is constructed in such a way that if F is a subtree of T that contains T' then every vertex of  $V(T') \setminus U$  is a cut-vertex of  $\langle V(F) \rangle$ .

Suppose now that  $2 < k \leq l$  and that we have constructed a subtree T'' of T containing v that is a minimal W'-tree for some k-1 element subset W' of U such that  $V(T'') \cap U = W'$ . Suppose also that if F is a subtree of T that contains T'', then every vertex of  $V(T'') \setminus U$  is a cut-vertex of  $\langle F \rangle$ . If some vertex v' of T'' has a neighbour z in T that belongs to U but not to T'', then we add z and the edge v'z to T'' to obtain a minimal  $W' \cup \{z\}$ -tree with the desired properties. Otherwise let  $z_1 \in N_T(V(T''))$  and let  $z_0$  be its neighbour in T that belongs to T". Add  $z_1$  and the edge  $z_0 z_1$  to the tree T". Now let  $z_2$  be a neighbour of  $z_1$  in T that does not belong to the component of  $H - z_1$  that contains  $z_0$  (and hence T"). Add  $z_2$  and the edge  $z_1 z_2$  to the current tree. If  $z_2 \in U$  we stop and let T' be the tree we constructed; otherwise, let  $z_3$  be a neighbour of  $z_2$  in T that does not belong to the component of  $H - z_2$  that contains  $z_1$ . Add  $z_3$  and the edge  $z_2 z_3$  to the current tree. We continue in this manner constructing a sequence  $z_0, z_1, z_2 \dots$  such that  $z_{i-1}z_i$  is an edge of T and of the tree we are constructing. We stop when we encounter for the first time a vertex that belongs to U. As was the case for k = 2 it is not difficult to see that this needs to happen since the sequence  $z_0 z_1 \dots$  we are constructing corresponds to a path in T which must eventually encounter a vertex of U. Let  $s \ge 2$  be the smallest integer such that  $z_s \in U$ . Let T' be the tree obtained from T'' by adding the path  $z_0 z_1 \dots z_s$  and let  $W = W' \cup \{z_s\}$ . Then T' is a W-tree with the desired properties.

**Theorem 6.** Let G = (V, E) be a connected graph and let  $T = \{k_1, k_2, \ldots, k_t\}$  be a collection of integers such that  $2 \leq k_1 < k_2 < \cdots < k_t$ . Then  $(V, \mathcal{M}_{m_T}(G))$  is a convex geometry if and only if  $(V, \mathcal{M}_{m_{k_1}}(G))$  is a convex geometry.

# Proof.

Claim 1.  $\mathcal{M}_{m_T}(G) = \mathcal{M}_{m_{k_1}}(G).$ 

**Proof.** Let S be a  $m_{k_1}$ -convex set. We show first that S is also an  $m_T$ -convex set. If this is not the case, then there is some i > 1 such that S is not  $m_{k_i}$ -convex. Thus there is a set U of  $k_i$  vertices in S such that the monophonic interval of U is not contained in S, i.e., for some vertex  $v \notin S$  there is a minimal U-tree containing v. By Lemma 5 there is a  $k_1$  element subset W of U and a minimal W-tree containing v. This is not possible since S is  $m_{k_1}$ -convex. So every  $m_{k_1}$  convex set is  $m_T$ -convex. By definition every  $m_T$ -convex set is  $m_{k_1}$ -convex. Hence  $\mathcal{M}_{m_T}(G) = \mathcal{M}_{m_{k_1}}(G)$ .

**Claim 2.** The extreme vertices of S with respect to the  $m_{k_1}$ -convexity are the same as the extreme vertices with respect to the  $m_T$ -convexity.

**Proof.** If x is an extreme vertex of S with respect to the  $m_T$ -convexity, then  $S \setminus \{x\}$  is an  $m_T$ -convex set and thus an  $m_{k_1}$ -convex set. So x is an extreme vertex of S with respect to the  $m_{k_1}$ -convexity. Suppose now that x is an extreme

vertex of S with respect to the  $m_{k_1}$ -convexity. If x is not an extreme vertex with respect to the  $m_T$ -convexity, then there is some i > 1 such that  $S \setminus \{x\}$  is not  $m_{k_i}$ -convex. Hence there is a  $k_i$  element subset U of  $S \setminus \{x\}$  such that x belongs to some minimal U-tree of U. By Lemma 5 there is a  $k_1$  element subset W of U and a minimal W-tree containing x. This contradicts the fact that  $S \setminus \{x\}$  is  $m_{k_1}$ -convex. So the extreme vertices of S with respect to the  $m_{k_1}$ -convexity are the same as the extreme vertices with respect to the  $m_T$ -convexity.

Suppose first that  $(V, \mathcal{M}_{m_T}(G))$  is a convex geometry. Let S be an  $m_{k_1}$ -convex set. By Claim 1, S is  $m_T$ -convex and by Claim 2 the extreme vertices of S with respect to  $m_{k_1}$  convexity are the same as the extreme vertices with respect to  $m_T$ convexity. The  $m_{k_1}$ -convex hull of the extreme vertices of S is a subset of S. If it is a proper subset of S, then this proper subset is, by Claim 1, also  $m_T$ -convex. Thus the  $m_T$ -convex hull of the extreme vertices of S is also a proper subset of S. This contradicts the hypothesis. Thus  $(V, \mathcal{M}_{m_{k_1}}(G))$  is a convex geometry.

For the converse suppose that  $(V, \mathcal{M}_{m_{k_1}}(G))$  is a convex geometry. Let S be an  $m_T$ -convex set. Then S is, by definition,  $m_{k_1}$ -convex. By Claim 2, the extreme vertices of S with respect to the  $m_{k_1}$  convexity are the same as the extreme vertices with respect to the  $m_T$  convexity. The convex hull with respect to the  $m_T$  convexity of S. If it is a proper subset of S, then this proper subset contains the convex hull of the extreme vertices with respect to  $m_{k_1}$  convexity. This contradicts the hypothesis. Thus  $(V, \mathcal{M}_{m_T}(G))$  is a convex geometry.

# 2.3. $m_3^3$ -convex geometries

Before characterizing the class of graphs for which the  $m_3^3$ -convex sets form a convex geometry, we introduce another useful result. Recall that the graphs for which the  $m^3$ -convex sets form a convex geometry are characterized in [10] as the (house, hole, domino, A)-free graphs. The proof of this characterization depends on the following result also proven in [10].

**Theorem 7.** If G is a (house, hole, domino, A)-free graph, then every vertex of G is either semisimplicial or lies on an induced path of length at least 3 between two semisimplicial vertices.

We now proceed to characterize those graphs for which the  $m_3^3$ -convex alignment forms a convex geometry. Let  $\mathcal{M}_{m_3^3}(G)$  be the  $m_3^3$ -convex alignment of a graph G. Recall that a graph F is a replicated twin  $C_4$  if it is isomorphic to one of the four graphs shown in Figure 2(a) where any subset of the dotted edges may be chosen to belong to F, and the collection of replicated twin  $C_4$ 's is denoted by  $\mathcal{R}_{C_4}$ . A graph F is a *tailed twin*  $C_4$  if it is isomorphic to one of the two graphs



Figure 2. Forbidden subgraphs for  $m_3^3$ -convex geometries.

shown in Figure 2(b) where again any subset of the dotted edges may be chosen to belong to F. We denote the collection of tailed twin  $C_4$ 's by  $\mathcal{T}_{C_4}$ .

**Theorem 8.** For a connected graph G = (V, E) the following are equivalent: (1) G is (house, hole, domino, A,  $\mathcal{R}_{C_4}$ ,  $\mathcal{T}_{C_4}$ )-free. (2)  $(V, \mathcal{M}_{m_3^3}(G))$  is a convex geometry.

**Proof.**  $(2) \rightarrow (1)$  Suppose F is a house, hole, domino, replicated twin  $C_4$  or a tailed twin  $C_4$ . Then F has at most one 3SS vertex. Suppose G is a graph that contains F as an induced subgraph. Then the set of extreme points of the convex hull of V(F) is contained in the collection of 3SS vertices of F. So the convex hull of the extreme points of the  $m_3^3$ -convex hull of V(F) is empty or consists of a single vertex and hence cannot contain all the vertices of F. So in this case the  $m_3^3$ -convex alignment of G does not form a convex geometry. It is shown in [6] that if the  $m_3^3$ -convex sets of a graph G form a convex geometry, then G does not contain an A.

 $(1) \rightarrow (2)$  It is not difficult to see that if G is a connected graph of order at most 4, then every  $m_3^3$ -convex set is the convex hull of its extreme points. Suppose now that there exists a connected (house, hole, domino, A,  $\mathcal{R}_{C_4}$ ,  $\mathcal{T}_{C_4}$ )-free graph G (abbreviated by  $HHDA\mathcal{R}_{C_4}\mathcal{T}_{C_4}$ -free graph G) for which  $(V, \mathcal{M}_{m_3^3}(G))$ is not a convex geometry. We may assume that G is such a graph of smallest possible order. Thus every proper connected induced subgraph of G has the property that its vertex set is the  $m_3^3$ -convex hull of its extreme points, i.e, the 3SS vertices. By assumption V is not the  $m_3^3$ -convex hull of its extreme points. Since V is  $m_3^3$ -convex it is  $m^3$ -convex; so, by Theorem 7, every vertex of G is either semisimplicial or lies on an induced path of length at least 3 between two semisimplicial vertices. Thus if every semisimplicial vertex is 3SS, then V is the  $m_3^3$ -convex hull of its extreme points, a contradiction. Let S be the  $m_3^3$ -convex hull of the 3SS vertices of G. By assumption  $V \setminus S \neq \emptyset$ .

Case 1.  $V \setminus S$  contains a vertex *a* that is not semisimplicial. Since *G* is *HHDA*-free and *V* is  $m^3$ -convex, Theorem 7 guarantees that *a* lies on an induced path of length at least 3 between two semisimplicial vertices w, w' of *G*.

Among all pairs  $\{w, w'\}$  of semisimplicial vertices such that  $a \in I_{m^3}[w, w']$  we will assume that  $\{v, v'\}$  is a pair that has a maximum number of 3SS vertices. At least one of v and v', say v, is not 3SS in G; otherwise, a lies on an induced path of length at least 3 between two extreme vertices of V. Since v is semisimplicial but not 3SS it must be the centre of an induced claw or paw in G. Let x, y, z be the peripheral vertices of a claw or paw containing v as centre where  $xz, yz \notin E$ .

Let  $I_{m^3}^{(a)}[v, v']$  be the collection of all vertices that lie in some induced v - v' path of length at least 3 that contains the vertex a.

Claim 1. None of x, y or z is in  $I_{m^3}^{(a)}[v, v']$ .

**Proof.** Assume, to the contrary, that P is an induced path of length at least 3 containing v, v', a and one of x, y, z. Suppose first that  $z \in I_{m^3}^{(a)}$ . Since P is an induced path  $x, y \notin V(P)$ . Let z' be the neighbour of z on P different from v. Then  $xz', yz' \in E(G)$ ; otherwise, xvzz' or yvzz' is an induced  $P_4$  which is not possible since v is semisimplicial. Let z'' be the neighbour of z' on P different from z. If  $z''x, z''y \notin E$ , then  $\langle \{v, x, y, z, z', z''\} \rangle$  is a tailed twin  $C_4$  which is forbidden. So we may assume  $xz'' \in E$ . Then  $\langle \{x, v, z, z', z''\} \rangle$  is a house which is forbidden. Similarly  $yz'' \notin E$ . Hence  $z \notin I_{m^3}^{(a)}[v, v']$ . Suppose now that x or y, say x, belongs to P. In that case we may assume  $xy \in E$ ; otherwise, we can argue as for z that G contains a forbidden subgraph as induced subgraph. Let x' be the neighbour of x on P different from v. Then  $zx' \in E$ ; otherwise, v is not semisimplicial. Also  $yx' \in E$ ; otherwise,  $\langle \{y, v, z, x, x'\} \rangle$  is a house which is forbidden. Let x''be the neighbour of x' on P different from x. If  $zx'' \in E$ , then  $\langle \{z, v, x, x', x''\} \rangle$ is a house which is forbidden. So  $zx'' \notin E$ . If  $yx'' \in E$ , then  $\langle \{y, v, z, x', x''\} \rangle$  is a house which is not possible. But then  $\langle \{y, v, z, x, x', x''\} \rangle$  induces a tailed twin  $C_4$  which is not possible. So we may assume that  $x, y, z \notin I_{m^3}^{(a)}$ . This completes the proof. 

**Claim 2.** If  $P : (v =)v_0v_1 \dots v_k (= v')$  is an induced v - v' path of length at least 3 containing a, then each of x, y, z is adjacent with  $v_1$  but with no  $v_i$  for  $2 \le i \le k$ .

**Proof.** If  $zv_i \in E$  for  $2 \leq i \leq k$ , then both  $xv_i, yv_i \in E$ ; otherwise,  $xvzv_i$  or  $yvzv_i$  is an induced  $P_4$  having v as centre which is not possible since v is semisimplicial. Similarly if  $xv_i \in E$  for some  $i, 2 \leq i \leq k$ , then  $zv_i \in E$  and thus  $yv_i \in E$ . Hence for every  $i, 2 \leq i \leq k$ , the vertices x, y, z are either all adjacent with  $v_i$  or all are non-adjacent with  $v_i$ . Also if there is an  $i, 2 \leq i < k$  such that x, y, z are all adjacent with  $v_i$ , then x, y, z are all adjacent with  $v_i$  there x, y, z are all adjacent with  $v_i$  for some  $2 \leq i < k$ , then  $\langle \{v, x, y, z, v_i, v_{i+1} \rangle$  is a tailed twin  $C_4$ . Thus, if x, y, z are all adjacent with  $v_i$  for some  $2 \leq i < k$ , then  $\langle \{v, x, y, z, v_i, v_{i+1} \rangle$  is a replicated twin  $C_4$ ; which is forbidden. We may thus assume x, y, z are all nonadjacent with  $v_i$  for  $2 \leq i < k$ . Since v is semisimplicial and  $zvv_1v_2$  is a  $P_4, zv_1 \in E$ . Similarly  $xv_1, yv_1 \in E$ . If

x, y, z are all adjacent with  $v_k$ , then  $\langle \{v, v_1, x, y, z, v_k\} \rangle$  is a replicated twin  $C_4$  which is forbidden. We have thus shown that x, y, z are all adjacent with  $v_1$  and that they are nonadjacent with  $v_i$  for  $2 \leq i \leq k$ . This completes the proof.

So  $zv_1v_2...v_k$  is an induced path of length at least 3 containing *a* as internal vertex. Now *z* is not a 3*SS* vertex; otherwise, we have a contradiction to our choice of the pair v, v'. So *z* is either not semisimplicial or the centre of an induced claw or paw.

### **Claim 3.** x, y, z can be chosen in such a way that z is semisimplicial.

**Proof.** Suppose that z is not semisimplicial. Then there exists an induced path wzrs having z as centre. If v is on this path, then v is w. Suppose w = v. Then  $\{x, y\} \cap \{r, s\} = \emptyset$  since v is adjacent with x and y but not r and s. Now xvzr (respectively, yvzr) is an induced  $P_4$  having v as centre unless xr (respectively, yr) is an edge of G. So  $xr, yr \in E$ . If  $xs \in E$ , then  $\langle \{x, v, z, r, s\} \rangle$  is a house which is forbidden. So  $xs \notin E$ . Similarly,  $ys \notin E$ . Since  $xs, ys \notin E$ ,  $\langle \{v, x, y, z, r, s\} \rangle$  is a tailed twin  $C_4$  which is forbidden. So  $w \neq v$ .

Since vzrs is a path of order 4 having z as centre and v as end-vertex, it follows from the above that it cannot be an induced path. So vr or vs is an edge of G. Suppose first that  $vr \notin E$ . Then  $vs \in E$ . Now wzvs is an induced  $P_4$ unless  $vw \in E$ . However, then  $\langle \{z, r, s, v, w, \} \rangle$  is a house which is forbidden. So  $vr \in E$ . If  $vs \notin E$ , then x and y are not on the path wzrs (i.e.,  $s \neq x, y$ ). Now xvrs is an induced  $P_4$ 's having v as centre unless xr or xs is an edge of G. If  $xr \notin E$ , then  $xs \in E$ . But then  $\langle \{x, v, r, s, z\} \rangle$  is a house. So  $xr \in E$ . Similarly  $yr \in E$ . If  $wv \in E$ , then wvrs is an induced  $P_4$  having v as centre. This is not possible. So  $vw \notin E$ . Now wzvx and wzvy are induced  $P_4$ 's unless  $wx, wy \in E$ . However, then  $\langle \{w, x, y, z, v, r\} \rangle$  is a replicated twin  $C_4$  which is forbidden. So  $vr, vs \in E$ . Now wzvs is an induced  $P_4$  having v as centre unless  $vw \in E$ . Note  $\langle \{v, w, r, s\} \rangle$  is a paw with v as centre. So as we argued for x, y, z, none of w, r, sis  $v_1$  and each of w, r, s is adjacent with  $v_1$  and to no  $v_i$  for  $2 \le i \le k$ .

We know, since G is HHDA-free, that z is an interior vertex of an induced path of length at least 3 between two semisimplicial vertices. Let  $Q: u_0u_1 \ldots u_m$ be such a path. Then  $z = u_i$  for some  $i \ (0 < i < m)$ . Thus  $u_{i-1}u_iu_{i+1}u_{i+2}$ or  $u_{i-1}u_{i-1}u_iu_{i+1}$  is an induced  $P_4$  having z as centre vertex, say the former. As we showed for the path wzrs,  $v \notin \{u_{i-1}, u_{i+1}, u_{i+2}\}$  and  $u_{i-1}, u_i, u_{i+1}$ , and  $u_{i+2}$  are each adjacent with both v and  $v_1$  but with no other vertex of P. If  $i - 1 \neq 0$ , we repeat the argument with  $u_{i-2}u_{i-1}u_iu_{i+1}$  and  $u_{i-1}$  instead of  $u_i$ since  $\langle \{v, u_{i-1}, u_{i+1}, u_{i+2}\} \rangle$  is a paw having v as centre. So  $u_{i-2}$  is adjacent with both v and  $v_1$  but with no  $v_j$  for  $2 \leq j \leq k$ . Continuing in this manner we see that for all  $j \ (0 \leq j \leq i+2)$ , vertex  $u_j$  is not on P and  $u_j$  is adjacent with both v and  $v_1$ . Similarly one can show if  $i+2 \neq m$ , then every vertex  $u_j$  for  $i+2 < j \leq m$  is not on P and  $u_j$  is adjacent with both v and  $v_1$  and with no  $v_l$  for  $3 \le l \le k$ . Hence v is the centre of the paw  $\langle \{v, u_0, u_2, u_3\} \rangle$  where  $u_0 u_2, u_0 u_3 \notin E$ . So we may assume  $z = u_0, x = u_2$  and  $y = u_3$ . This completes the proof.

The path  $zv_1v_2...v_k$  is an induced path of length at least 3 containing *a* as interior vertex and since  $z, v_k$  are both semisimplicial. Vertex *z* cannot be 3SS; otherwise, we have a contradiction to the choice of the pair v, v'. So *z* is the centre of a claw or paw whose peripheral vertices are, say r, s, t where  $tr, ts \notin E$ . By Claim 3 we may assume *t* is semisimplicial.

Claim 4.  $v \notin \{r, s, t\}$ .

**Proof.** Suppose first that v = t. Then rzvx and szvx are induced  $P_4$ 's having v as centre unless  $rx, sx \in E$ . Similarly  $ry, sy \in E$ . However, then  $\langle \{v, x, y, z, s, r\}$  is a replicated twin  $C_4$  which is forbidden. So  $v \neq t$ . Suppose now that v is r or s, say v = r. Thus we may assume  $rs \in E$ ; otherwise, we can repeat the argument we used for t. Then xvzt and yvzt are induced  $P_4$ 's unless  $xt, yt \in E$ . Now sztx and szty are induced  $P_4$ 's having z as centre unless  $sx, sy \in E$ . But then  $\langle \{t, x, y, z, v, s\} \rangle$  is a replicated twin  $C_4$  which is forbidden. Hence  $v \notin \{r, s, t\}$ . This completes the proof of Claim 4.

Claim 5. v is adjacent with each of r, s, t.

**Proof.** If v is nonadjacent with some  $b \in \{r, s, t\}$ , then bzvx and bzvy are induced  $P_4$ 's having v as centre vertex unless  $bx, by \in E$ . Thus if v is nonadjacent with two vertices in r, s, t, then these two vertices together with v and x, y, z induce a replicated twin  $C_4$  which is forbidden. So v is adjacent with at least two of the vertices r, s, t. Suppose v is nonadjacent with t. Then  $tx, ty, vs, vr \in E$  and txvr, txvs, tyvr and tyvs are induced  $P_4$ 's having v as centre vertex unless xr, xs, yr, ys, respectively are edges of G. However, then  $\langle \{t, x, y, z, r, s\} \rangle$  is a replicated twin  $C_4$ . If v is nonadjacent with r or s, say r, then  $vs, vt, rx, ry \in E$ . We may also assume  $rs \in E$ ; otherwise, we can argue as for t that G has a replicated twin  $C_4$ . Now tvxr and tvyr are induced  $P_4$ 's having v as centre vertex unless  $xt, yt \in E$ . However, then  $\langle \{r, x, y, z, v, t\} \rangle$  is a replicated twin  $C_4$  which is forbidden. This completes the proof of Claim 5.

Thus, by Claim 2, r, s, t are all adjacent with  $v_1$  and with none of the vertices  $v_j$  for  $2 \leq j \leq k$ . So  $tv_1v_2...v_k$  is an induced path of length at least three containing a as interior vertex. By our choice of the pair v, v' we know that t is not 3SS. So t is the centre of a claw or paw.

Since t is adjacent with z it is neither x nor y. By Claims 3, t is the centre of a claw or paw with peripheral vertices  $r_1, s_1, t_1$  such that  $t_1s_1, t_1r_1 \notin E$  and such that  $t_1$  is semisimplicial. Since both v and z are the centre of a

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claw or paw whose peripheral vertices are r, s, t, it follows from Claim 4 that  $v, z \notin \{r_1, s_1, t_1\}$  and by Claim 5, v and z are both adjacent with every vertex of  $\{r_1, s_1, t_1\}$ . Moreover,  $\{r_1, s_1, t_1\} \cap \{x, y, r, s\} = \emptyset$ . Now  $t_1$  is semisimplicial and  $r_1, s_1, t_1$  are adjacent with  $v_1$  but with no  $v_j$  for  $2 \leq j \leq k$ . Thus as for t we can argue that  $t_1$  is the centre of some induced claw or paw  $\langle \{t_1, r_2, s_2, t_2\} \rangle$  where we may assume  $t_2s_2, t_2r_2 \notin E(G)$ . Moreover, one can argue as before that  $v, z, x, y, r, s, t, r_1, s_1, t_1 \notin \{r_2, s_2, t_2\}$  and that v, z, t and  $t_1$  are all adjacent with  $r_2, s_2, t_2$ . Continuing in this manner we see that G has an infinite number of vertices which is not possible. So this case cannot occur.

Case 2. Every vertex of G that is not semisimplicial belongs to S.

Subcase 2.1. All vertices of G are semisimplicial. Then the extreme points of G are the vertices that are not the centre of a claw or paw in G and G has no induced path of length at least 3. So the  $m_3^3$ -convex sets are the  $m_3$ -convex sets and the  $m_3^3$ -convex hull of the extreme points is just the  $m_3$ -convex hull of the extreme points. Also since G has no induced path of length 3, G is 3-fan free. Since G is  $(P_4, \mathcal{R}_{C_4})$ -free it follows from Corollary 4 that  $(V, \mathcal{M}_{m_3^3}(G))$  is a convex geometry.

Subcase 2.2. There exist vertices that are not semisimplicial. From the case we are in, these vertices all belong to S. So S has vertices that are not 3SS. Thus  $\langle S \rangle$  has at least four vertices and is therefore connected.

We show first that G - S has exactly one component. Suppose G - S has at least two components. Let  $H_1, H_2, \ldots, H_l$  be the components of G - S. Then the 3SS vertices of G, which necessarily belong to S, are still 3SS vertices of  $G - V(H_1)$ . Moreover if  $G - V(H_1)$  contains any 3SS vertices that were not 3SS vertices of G these are also contained in S since such vertices are necessarily adjacent with vertices of  $H_1$ . But since all 3SS vertices of  $G - V(H_1)$  are contained in S, their  $m_3^3$ -convex hull is also contained in S since S is  $m_3^3$ -convex. However, by our choice of G, the  $m_3^3$ -convex hull of the 3SS vertices of  $G - V(H_1)$  is  $V(G - V(H_1)) \neq S$ . This contradiction shows that G - S has exactly one component, say H.

Since S contains vertices that are not 3SS, each such vertex v is either the interior vertex of an induced path of length at least 3 whose end vertices are in S or there exist three vertices x, y, z in S such that v is an interior vertex of a minimal  $\{x, y, z\}$ -tree. In the first case all the vertices on the induced path belong to S. In the second case  $\langle \{x, y, z\} \rangle$  is not connected. In either case S contains three vertices that induce a disconnected graph. Hence a vertex of G - S cannot be adjacent to all vertices of S; otherwise, it would belong to a minimal R-tree for some set R of three vertices of S. This is not possible since S is  $m_3^3$ -convex.

Observe that every vertex of G - S is adjacent with some vertex of S; otherwise, there is a vertex b distance 2 from S in G. Let bcS be a b - S path. Since c

is not adjacent to every vertex of S, there is some vertex  $d \in S$  and a neighbour d' of d in S such  $cd \in E$  and  $cd' \notin E$ . Thus bcdd' is an induced  $P_4$ . However, then c is not semisimplicial and thus by the case we are in  $c \in S$ . This contradiction shows that every vertex of G - S is adjacent with some vertex of S.

We now show that every vertex of S is adjacent with a vertex of H. Suppose some vertex v of S is not adjacent with any vertex of H. Suppose first that G-v is connected. By the minimality of G, V(G-v) is the  $m_3^3$ -convex hull of its extreme points, i.e., its 3SS vertices. Since the extreme points of G-vare contained in S and since S is  $m_3^3$ -convex, the  $m_3^3$ -convex hull of the extreme points of V(G-v) is contained in  $S \setminus \{v\}$ , a contradiction. Suppose next that G - v is disconnected. Then, by the minimality of G, the vertex set of each component is the  $m_3^3$ -convex hull of its extreme points. Since G - v has at least three vertices, there is a set R of three vertices of G - v such that v belongs to a minimal R-tree. (Pick the vertices of R in such a way that they belong to at least two distinct components of G - v.) However, since the extreme points of G - vare contained in S, the extreme points of each component of G - v are also in S. Thus the  $m_3^3$ -convex hull of the extreme points of each component is contained in S. Since v is also in S the  $m_3^3$ -convex hull of the union of the  $m_3^3$ -convex hulls of the components together with v is also contained in S, a contradiction. So each vertex of S is adjacent with some vertex of H.

Suppose first that  $\langle S \rangle$  contains an induced path of order 4, say wrst. From the above we know that w is adjacent with some vertex w' in H. Now w' cannot be adjacent with both s and t; otherwise, w would be on a minimal  $\{w, s, t\}$ -tree and thus in the  $m_3^3$ -convex hull of the extreme points of G. Moreover, w' cannot be adjacent with exactly one of s and t; otherwise, w' is not semisimplicial since either ww'st (if  $w's \in E$ ) or ww'ts (if  $w't \in E$ ) are induced  $P_4$ 's having w as centre vertex. Let t' be a neighbour of t in  $H_1$ . We have argued that  $t' \neq w'$ . Since H is connected there is an induced w' - t' path P' in H. Since all vertices of H are semisimplicial in G, P' has length 1 or 2. As we argued for w' we can show that t' is not adjacent with either w or r. If  $w't' \in E$ , then ww't't is an induced  $P_4$  containing w' and t' as centre vertices. This is not possible since all vertices of H are semisimplicial. Suppose thus that P' has length 2 and let w''be the common neighbour of w' and t' on P'. If w'' is adjacent with w, then it is nonadjacent with s and t (we argue as for w'). However, then ww''t't is an induced  $P_4$  containing w'' as centre vertex which is not possible. Similarly if w''is adjacent with t we can show that w'' is not semisimplicial. But now ww'w't'is an induced  $P_4$  containing w' as centre vertex. This is not possible.

Thus  $\langle S \rangle$  has no induced  $P_4$ . By the case we are considering, H contains no induced  $P_4$ 's. We know that G has an induced path of order 4 and that any such path necessarily contains vertices from H and S.

We show first that  $diam(G) \leq 2$ . Suppose  $diam(G) = d \geq 3$ . Let v, v' be

vertices such that d(v, v') = d. Let  $V_i$  be the vertices distance  $i, 1 \le i \le d$  from vin the *g*-interval between v and v'. Since H is connected and all vertices of H are semisimplicial, H contains no induced path of order at least 4. Moreover, since  $\langle S \rangle$  does not contain an induced path of order at least 4, one of v and v', say v, belongs to H and the other to S. In fact d = 3. Since the vertices of  $V_1$  and  $V_2$ are not semisimplicial, they belong to S. No neighbour x of v in H is adjacent with a vertex of  $V_i$  for i = 2 or 3; otherwise, x is either not semisimplicial or d(v, v') < d = 3; neither of these situations is possible. Moreover, such a neighbour is adjacent with every vertex of  $V_1$ ; otherwise, v is not semisimplicial.

Suppose now that v has a neighbour x in  $S \setminus V_1$ . Then x is not adjacent with a vertex of  $V_i$  for i = 2 or 3; otherwise, x is either in  $V_1$  or d(v, v') < d = 3, neither of which is possible. Moreover, such a vertex x is adjacent with every vertex of  $V_1$ ; otherwise, v is not semisimplicial. (Note that if  $y \in V_1$  is such that  $xy \notin E$  and that if  $s \in V_2$  is such that  $sy \in E$ , then xvys is an induced  $P_4$  having v as centre vertex.) But then  $\langle S \rangle$  contains an induced x - v' path of order 4 which is not possible in the case we are considering. So  $V_1$  is the collection of neighbours of v in S. Similarly every neighbour y of v in H is adjacent with precisely the vertices of  $V_1$  and with no other vertices of S. It is not difficult to see that d(y, v') = 3. So arguing as we did for v we see that every neighbour of y in H is adjacent to precisely the vertices of  $V_1$  and to no other vertices of S. Since H is connected and contains no induced  $P_4$ 's it follows that every vertex of H is adjacent with precisely the vertices of  $V_1$  and to no other vertices of S. But then not every vertex of S is adjacent with a vertex of H; a contradiction. So  $diam(G) \leq 2$ .

Let P : wrst be an induced path. Then P is not contained in  $\langle S \rangle$  and P is not contained in H. Since r and s are not semisimplicial they belong to S (from the case we are considering). Suppose first that w and t both belong to H. Since H is connected there is an induced w-t path in H having length at most 2. Let u be a common neighbour of w and t in H. Since G contains no house and hole, u is adjacent with r and s. Since r and s are not semisimplicial they are not 3SS vertices. But r and s belong to the  $m_3^3$ -convex hull of the extreme points of G. So they must be the centre of a claw or paw whose peripheral vertices belong to S. Let x, y, z be the peripheral vertices of such a claw or paw in  $\langle S \rangle$ having r as centre vertex. Vertex t cannot be adjacent to all three vertices x, y, z; otherwise, t belongs to the  $m_3^3$ -convex hull of the 3SS vertices of G, i.e.,  $t \in S$ , a contradiction. We may assume  $tx \notin E$ . Then xrut is an induced  $P_4$  having u as centre vertex unless  $ux \in E$ . Also u is not adjacent with each of the three vertices x, y, z; otherwise,  $u \in S$ . Suppose  $yu \notin E$ . Then yrut is an induced  $P_4$  having u as centre vertex unless  $yt \in E$ . If  $xy \notin E$ , then  $\langle \{y, r, u, t, x\} \rangle$  is a house which is forbidden. So  $xy \in E$ . Since x, y, z, are the peripheral vertices of a claw or paw it follows that  $zx, zy \notin E$ . Now zrut is an induced  $P_4$  having u as centre vertex unless one of zu or zt is and edge of G. If  $zt \notin E$ , then  $zu \in E$ and  $\langle \{t, u, r, y, z\} \rangle$  is a house which is forbidden. So  $zt \in E$ . But then ztyx is an induced  $P_4$  having t as centre vertex, which is not possible since  $t \in V(H)$ .

So we may assume that  $w \in V(H)$  and that  $t \in S$ . Of course  $r, s \in S$ . We show first that H contains a common neighbour of w and t. Suppose this is not the case. We know that t has a neighbour t' in H. Since  $diam(H) \leq 2$  there is a vertex u in H that is a common neighbour of w and t'. By assumption,  $ut \notin E$ . But then tt'uw is an induced  $P_4$  having u as centre vertex which is not possible. So there is a vertex u in H that is adjacent with w and t. Since G contains no house or hole,  $us, ur \in E$ . Since s is not semisimplicial but  $s \in S$  it must be the centre of an induced claw or paw in  $\langle S \rangle$ . Let x, y, z be the peripheral vertices of such an induced claw or paw having s as centre. Since  $w \in V(H)$ , w is not adjacent with all three of the vertices x, y, z. Suppose  $wx \notin E$ . Then xsuw is an induced  $P_4$  having u as centre vertex unless  $ux \in E$ . Similarly u is not adjacent with all three vertices x, y, z. We may assume  $uy \notin E$ . Then ysuw is an induced  $P_4$  unless  $wy \in E$ . If  $xy \notin E$ , then  $\langle \{w, u, s, y, x\} \rangle$  is a house which is forbidden. So  $xy \in E$ . Since x, y, z are the peripheral vertices of a claw or paw, we conclude that  $zx, zy \notin E$ . Now zsuw is an induced  $P_4$  having u as centre vertex unless one of zu, zw is in E. If  $zw \notin E$ , then  $\langle \{w, u, s, y, z\} \rangle$  is a house which is forbidden. So  $zw \in E$ . But then zwyx is an induced  $P_4$  having w as centre vertex. This completes the proof.

### 3. Concluding Remarks

In this paper we introduced the definition of a minimal U-tree where U is a set of vertices in a connected graph G and defined several graph convexities that use this concept. Of course every Steiner tree for U is a minimal U-tree but the converse does not hold. So the Steiner interval is contained in the monophonic interval for U. Two graph invariants have been studied that indicate the smallest number of vertices that "span" the vertex set of a graph using different interval notions. In particular, the geodetic number of a graph G, denoted by q(G), is the smallest number k of vertices in G for which there exists a set S of k vertices with the property that  $V(G) = \bigcup_{u,v \in S} I_a[u,v]$  and the Steiner geodetic number of G, denoted by sq(G), is the smallest number k for which there exists a set S of k vertices with V(G) = I(S). These invariants can be extended naturally if we replace geodetic (Steiner) intervals by monophonic intervals. Let m(G) be the smallest integer k for which there exists a set S of k vertices in G such that  $V(G) = \bigcup_{u,v \in S} I_m[u,v]$  and sm(G) the smallest integer k such that there exists a set S of k vertices in G with  $V(G) = I_m(S)$ . It was shown in [17] that in general there is no relationship between g(G) and sg(G) by showing that the

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ratio g(G)/sg(G) can be made arbitrarily large and arbitrarily small. However, such is not the case for the ratio m(G)/sm(G). It is not difficult to see that it can never exceed 1. To see this suppose that S is a set of vertices in G such that  $V(G) = I_m(S)$ . If  $w \in V(G) \setminus S$ , then w belongs to some minimal S-tree T. Thus w is a cut-vertex of  $H = \langle V(T) \rangle$ . Hence there exist two vertices u and v of S that belong to distinct components in H - w. Thus w lies on an induced u - v path. So S also has the property that  $V(G) = \bigcup_{u,v \in S} I_m[u,v]$ . But m(G)/sm(G) can be arbitrarily small. Take for example the complete bipartite graph  $K_{r,s}$  where  $2 \leq r \leq s$ . It is not difficult to see that  $m(K_{r,s}) = 4$  whereas sm(G) = r. Hence by choosing r sufficiently large the ratio m(G)/sm(G) can be made as small as we wish. The problem of finding g(G) is known to be NP-hard (see [1]). In view of the fact that the problem of finding Steiner trees for sets of vertices in a graph is NP-hard it is likely that the problem of finding sg(G) may also be NP-hard. However not much is known about the computability of m(G) and sm(G).

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Received 11 July 2011 Revised 20 December 2011 Accepted 21 December 2011