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MINIMAL TREES AND MONOPHONIC CONVEXITY

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Abstract

Let V be a finite set and $\mathcal{M}$ a collection of subsets of V. Then $\mathcal{M}$ is an alignment of V if and only if $\mathcal{M}$ is closed under taking intersections and contains both V and the empty set. If $\mathcal{M}$ is an alignment of V, then the elements of $\mathcal{M}$ are called convex sets and the pair $(V, \mathcal{M})$ is called an alignment or a convexity. If $S \subseteq V$, then the convex hull of S is the smallest convex set that contains S. Suppose $X \in \mathcal{M}$. Then $x \in X$ is an extreme point for X if $X \setminus \{x\} \in \mathcal{M}$. A convex geometry on a finite set is an aligned space with the additional property that every convex set is the convex hull of its extreme points. Let $G = (V, E)$ be a connected graph and U a set of vertices of G. A subgraph T of G containing U is a minimal U-tree if T is a tree and if every vertex of $V(T) \setminus U$ is a cut-vertex of the subgraph induced by $V(T)$. The monophonic interval of U is the collection of all vertices of G that belong to some minimal U-tree. Several graph convexities are defined using minimal U-trees and structural characterizations of graph classes for which the corresponding collection of convex sets is a convex geometry are characterized.

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1. Introduction

This paper is motivated by the results and ideas contained in [10, 11, 15]. We introduce new graph convexities and show how these give rise to structural characterizations of certain graph classes. For graph terminology we follow [4] and [9]. All graphs considered here are connected, finite, simple (i.e., without loops and multiple edges), unweighted and undirected. The structural characterizations of graphs that we describe are often given in terms of forbidden subgraphs. Let G and F be graphs. Then F is an *induced subgraph* of G if F is a subgraph of G and for every $u, v \in V(F)$, $uv \in E(F)$ if and only if $uv \in E(G)$. We say a graph G is *F-free* if it does not contain F as an induced subgraph. Suppose $\mathcal{C}$ is a collection of graphs. Then G is $\mathcal{C}$-free if G is F-free for every $F \in \mathcal{C}$. If F is a path or cycle that is a subgraph of G, then F has a *chord* if it is not an induced subgraph of G, i.e., F has two vertices that are adjacent in G but not in F. An induced cycle of length at least 5 is called a *hole*.

Let V be a finite set and $\mathcal{M}$ a collection of subsets of V. Then $\mathcal{M}$ is an *alignment* (or *convexity*) of V if and only if $\mathcal{M}$ is closed under taking intersections and contains both V and the empty set. If $\mathcal{M}$ is an alignment of V, then the elements of $\mathcal{M}$ are called *convex sets* and the pair $(V, \mathcal{M})$ is called an *aligned space* or a *convexity.* If $S \subseteq V$, then the *convex hull* of S is the smallest convex set that contains S. Suppose $X \in \mathcal{M}$. Then $x \in X$ is an *extreme point* for X if $X \setminus \{x\} \in \mathcal{M}$. The collection of all extreme points of X is denoted by $ex(X)$. A *convex geometry* on a finite set V is an aligned space $(V, \mathcal{M})$ with the additional property that every convex set is the convex hull of its extreme points. This property is referred to as the *Minkowski-Krein-Milman* (MKM) property. For a more extensive overview of other abstract convex structures see [18]. Convexities associated with the vertex set of a graph are discussed for example in [4]. Their study is of interest in Computational Geometry and has applications in Game Theory [3].

Convexities on the vertex set of a graph are usually defined in terms of some type of 'intervals'. Suppose G is a connected graph and u, v two vertices of G. Then a $u - v$ *geodesic* is a shortest $u - v$ path in G. Such geodesics are necessarily induced paths. However, not all induced paths are geodesics. The *g-interval* (respectively, *m-interval*) between a pair u, v of vertices in a graph G is the collection of all vertices that lie on some $u - v$ geodesic (respectively, induced $u - v$ path) in G and is denoted by $I_g[u, v]$ (respectively, $I_m[u, v]$).

A subset S of vertices of a graph is said to be *g-convex* (*m-convex*) if it contains the g-interval (m-interval) between every pair of vertices in S. It is not difficult to see that the collection of all g-convex (m-convex) sets is an alignment of V. A vertex v is an extreme point for a g-convex (or m-convex) set S if and only if v is simplicial in the subgraph induced by S, i.e., every two neighbours of

v in S are adjacent. Of course the convex hull of the extreme points of a convex set S is contained in S, but equality holds only in special cases. In [11] those graphs for which the g-convex sets form a convex geometry are characterized as the chordal 3-fan-free graphs (see Figure 1). These are precisely the chordal, distance-hereditary graphs (see [2, 12]). In the same paper it is shown that the chordal graphs are precisely those graphs for which the m-convex sets form a convex geometry.

For what follows we use P_k to denote an induced path of order k. A vertex is simplicial in a set S of vertices if and only if it is not the centre vertex of an induced P_3 in in the subgraph $\langle S\rangle$ induced by S. Jamison and Olariu [13] relaxed this condition. They defined a vertex to be *semisimplicial* in S if and only if it is not a centre vertex of an induced P_4 in $\langle S\rangle$.

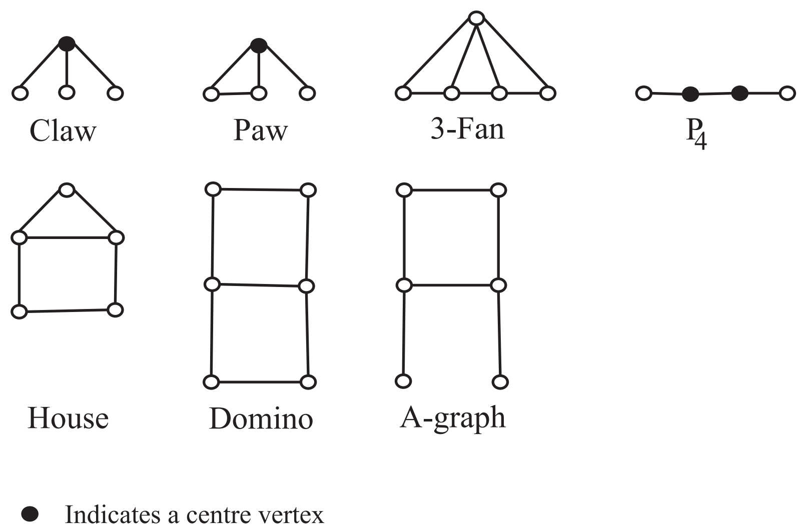

Figure 1. Special graphs.

Dragan, Nicolai and Brandstädt [10] introduced another convexity notion that relies on induced paths. The m^3-*interval* between a pair u, v of vertices in a graph G, denoted by $I_{m^3}[u, v]$, is the collection of all vertices of G that belong to an induced $u - v$ path of length at least 3. Let G be a graph with vertex set V. A set $S \subseteq V$ is m^3-*convex* if and only if for every pair u, v of vertices of S the vertices of the m^3-interval between u and v belong to S. As in the other cases the collection of all m^3-convex sets is an alignment. Note that an m^3-convex set is not necessarily connected. It is shown in [10] that the extreme points of an m^3-convex set are precisely the semisimplicial vertices of $\langle S\rangle$. Moreover, those graphs for which the m^3-convex sets form a convex geometry are characterized in [10] as the (house, hole, domino, A)-free ($HHDA$-free) graphs (see Figure 1).

In the same paper several 'local' convexities related to the m^3-convexity were studied. For a set S of vertices in a graph G, $N[S]$ is $S \cup N(S)$ where $N(S)$ is the collection of all vertices adjacent with some vertex of S. A set S of vertices in a graph is connected if $\langle S \rangle$ is connected. The following result which we will use in this paper is established in [10].

Theorem 1. *A graph G is (house, hole, domino)-free if and only if $N[S]$ is m^3-convex for all connected sets S of vertices of G.*

The (house, hole, domino)-free graphs also arise in the study of the induced path function (see for example, [7, 8]). We next look at more recently studied graph convexities that motivate the convexities studied in this paper. In [16] a graph convexity that generalizes g-convexity is introduced. Let S be a set of vertices in a graph G. A *Steiner tree* T for S is a connected subgraph of G that contains S and has the smallest number of edges among all such subgraphs. The subgraph induced by the vertices of T may not be an induced subgraph; for example, if G is a net (i.e. the graph obtained by joining a new vertex to each of the three vertices in a K_3) and S consists of the three leaves in G, then any spanning tree of G is a Steiner tree for S. The *Steiner interval* of a set S of vertices in a connected graph G, denoted by $I(S)$, is the union of all vertices of G that lie on some *Steiner tree* for S. Steiner intervals have been studied, for example, in [14, 17]. A set S of vertices in a graph G is *k-Steiner convex* (g_k-convex) if the Steiner interval of every collection of k vertices of S is contained in S. Thus S is g_2-convex if and only if it is g-convex. The collection of g_k-convex sets forms an aligned space. We call an extreme point of a g_k-convex set a *k-Steiner simplicial* vertex, abbreviated kSS vertex.

The extreme points of g_3-convex set S, i.e., the $3SS$ vertices are characterized in [5] as those vertices that are **not** a centre vertex of an induced claw, paw or P_4, in $\langle S \rangle$ see Figure 1. Thus a $3SS$ vertex is semisimplicial. Apart from the g_k-convexity, for a fixed k, other graph convexities that (i) depend on more than one value of k and (ii) combine the g_3 convexity and the geodesic counterpart of the m^3-convexity are introduced and studied in [15]. In particular characterizations of convex geometries for several of these graph convexities are given. We state here only those results that are used in this paper.

A graph G is a *replicated twin* C_4 if it is isomorphic to any one of the four graphs shown in Figure 2(a), where any subset of the dashed edges may belong to G. The collection of the four replicated twin C_4 graphs is denoted by $\mathcal{R}_{C_4}$.

Theorem 2 [15]**.** *Let $G = (V, E)$ be a graph. Then the following are equivalent*

(1) *G is (P_4, $\mathcal{R}_{C_4}$)-free.*

(2) *(V, $\mathcal{M}_{g_3}(G)$) is a convex geometry.*

Convex geometries give rise to 'elimination orderings' of vertices in graphs and these are particularly useful for algorithmic purposes. Suppose **P** is a property that a vertex in a graph G may possess. We say G has a **P**-*elimination ordering* if the vertices of G can be ordered as $\{v_1, v_2, \ldots, v_n\}$ such that for every i, $1 \leq i \leq n$, vertex v_i has property **P** in the subgraph induced by $v_i, v_{i+1}, \ldots, v_n$. If **P** is a property that characterizes the extreme vertices with respect to a graph convexity, then it is well known (see e.g. [11]) that if the corresponding convex sets form a convex geometry, then G has a **P**-elimination ordering. The extreme vertices for the convexities studied in this paper are also the $3SS$ vertices. Graphs for which every LexBFS ordering is a $3SS$-elimination ordering are characterized in [5]. Interestingly this class of graphs is precisely the same as the class for which the monophonic intervals, satisfying a certain betweenness axiom, also satisfy the monotone property as shown in [8].

We now introduce the notion of a 'minimal U-tree' which extends both the definition of an induced path and that of a Steiner tree. This in turn gives rise to graph convexities that extend both the m-convexity and g_k-convexity. Let U be a set of at least two vertices in a connected graph G. A subgraph H containing U is a *minimal* U*-tree* if H is a tree and if every vertex $v \in V(H) \setminus U$ is a cut-vertex of $\langle V(H) \rangle$, i.e., H is a minimal U-tree if H connects U and if H is a minimal subgraph that connects U in the sense that for every $v \in V(H) \setminus U$, U is no longer connected in $\langle V(H) \setminus \{v\} \rangle$. Thus if $U = \{u, v\}$, then a minimal U-tree is just an induced $u - v$ path. Moreover, every Steiner tree for a set U of vertices is a minimal U-tree. The collection of all vertices that belong to some minimal U-tree is called the *monophonic interval of* U and is denoted by $I_m(U)$. A set S of vertices is k*-monophonic convex*, abbreviated as m_k-convex, if it contains the monophonic interval of every subset U of k vertices of S. Thus a set of vertices in G is a monophonic convex set if and only if it is an m_2-convex set. For a set $T = \{k_1, k_2, \ldots, k_t\}$ of integers such that $2 \leq k_1 < k_2 < \cdots < k_t$ a set S of vertices of G is m_T-convex if G is m_{k_i}-convex for all $1 \leq i \leq t$. It is not difficult to see that the collection of m_T-convex sets is an alignment of $V(G)$, called the m_T-convex alignment. We show that the class of graphs for which the m_T-convex alignment is a convex geometry is the same as the class of graphs for which the m_k-convex alignment is a convex geometry where k is the smallest value in T.

In this paper we give structural characterizations of those classes of graphs for which the m_3-convex alignment forms a convex geometry. It turns out that these graphs are precisely the same as the graphs for which the g_3-convex alignment forms a convex geometry and thus contains no induced P_4's. However, by combining the m_3- convexity with the m^3-convexity introduced in [10], we obtain a graph convexity for which the convex geometries cover a larger more interesting class of graphs that has no restriction on the diameter. More specifically we define a set S of vertices in a connected graph to be m^3_3*-convex* if S is both m^3- and m_3-convex. In this paper we give structural characterizations of those classes of graphs for which the m^3_3-convex alignment forms a convex geometry.

2. Convex Geometries

2.1. m_3-convex geometries

We begin by showing that the extreme vertices of 3-monophonic convex sets are precisely the $3SS$ vertices of the set. If v is an extreme vertex of a 3-monophonic set S, then v cannot be the centre of an induced claw, paw or P_4 in $\langle S\rangle$; otherwise, v is on a minimal tree for some set of three vertices in S. Hence v is a $3SS$ vertex. Suppose now that v is not an extreme vertex of a 3-monophonic set S. Then there is a set U of three vertices in $S \setminus \{v\}$ such that v lies on a minimal U-tree H. So $H - v$ is disconnected. If v has at least three neighbours in $H - v$, then v is the centre of a claw or a paw. (Let x, y, z be three neighbours of v from $H - v$ such that x does not belong to the same component as y or z in $H - v$. Then $\langle\{v, x, y, z\}\rangle$ is a claw or paw.) If v has two neighbours, say x, y, in $H - v$, then either the component of $H - v$ containing x or the component of $H - v$ containing y has at least two vertices, say the latter. Let z be a neighbour of y in $H - v$. Then x is nonadjacent with both y and z. So v is not $3SS$. Hence if v is a $3SS$ vertex of a 3-monophonic set S, then v is an extreme vertex of S.

Let $G = (V, E)$ be a connected graph and $\mathcal{M}_{m_3}(G)$ the collection of m_3-convex sets and $\mathcal{M}_{g_3}(G)$ the collection of g_3-convex sets. In this section we determine the class of connected graphs G for which $(V,\ \mathcal{M}_{m_3}(G))$ is a convex geometry. Observe that every 2-element set of vertices in a graph is m_3-convex. Thus m_3-convex sets may induce disconnected graphs but only if they consist of two nonadjacent vertices.

We observe first that if $(V,\ \mathcal{M}_{m_3}(G))$ is a convex geometry, then G has no induced path of length at least 3. To see this, suppose that G is a connected graph and suppose that $P : (u = \ )v_0 v_1 \ldots v_d(\ = v)$ an induced $u - v$ path of length at least 3. Let S be the m_3-convex hull of $V(P)$. Since u and v are the only $3SS$ vertices of P, the $3SS$ vertices of S are a subset of $\{u, v\}$. But the m_3-convex hull of any subset R of $\{u, v\}$ is just R and hence does not contain all the vertices of P and thus not all the vertices of S. Thus, if $(V,\ \mathcal{M}_{m_3}(G))$ is a convex geometry, then G contains no induced path of length at least 3.

Theorem 3. *Let G be a connected graph. Then $(V,\ \mathcal{M}_{m_3}(G))$ is a convex geometry if and only if $(V,\ \mathcal{M}_{g_3}(G))$ is a convex geometry.*

Proof. Suppose that $(V,\ \mathcal{M}_{g_3}(G))$ is a convex geometry. Let $S \in \mathcal{M}_{m_3}(G)$. Then $S \in \mathcal{M}_{g_3}(G)$, since S contains the Steiner interval of every set of three vertices of S. Since the $3SS$ vertices of a set are the extreme vertices of the set with respect to both the g_3- and m_3-convexity, S is the g_3-convex hull of the $3SS$ vertices and hence also the m_3-convex hull of these vertices. Thus $(V,\ \mathcal{M}_{m_3}(G))$ is also a convex geometry.

Conversely, suppose $(V,\ \mathcal{M}_{m_3}(G))$ is a convex geometry. Then, by the above observation, G has no induced paths of length at least 3. Let S be a g_3-convex set of G and $U = \{u, v, w\}$ any set of three vertices of S. Let H be any U-tree. Then H is either a path or H is homeomorphic to $K_{1,3}$ and its leaves are contained in U. Suppose first that H is a path, say a $u-w$ path. Then the $u-v$ subpath and $v-w$ subpath of H are necessarily induced paths of G. Since G has no induced paths of length at least 3, both these subpaths contain at most three vertices. If neither of the subpaths has an interior vertex, then H is a Steiner tree for $\{u, v, w\}$. If exactly one of the subpaths has no interior vertex, say $uv \in E(G)$, then H must be a Steiner tree for $\{u, v, w\}$; otherwise, the vertex of H not in $\{u, v, w\}$ is not a cut-vertex of $\langle V(H)\rangle$. If both have an interior vertex, then H is a path of length 4 and is thus not an induced subgraph of G. Let u' be the neighbour of u in H and w' the neighbour of w in H. Then, by the above observation, the only possible edges in $\langle V(H)\rangle$ that do not belong to H are edges with one end in $\{u, u'\}$ and the other end in $\{w, w'\}$. Since H is a minimal $\{u, v, w\}$-tree the only possible edge of $\langle V(H)\rangle$ that does not belong to H is $u'w'$. However, in this case $uu'w'w$ is an induced path of length 3 which is not possible. So this case cannot occur.

Suppose now that H is a homeomorphic with $K_{1,3}$. Let x be the vertex of degree 3 in H. The three paths from x to each of the leaves of H must necessarily be induced. We can argue as above that at most one of these three paths has length 2. Suppose that one of these paths, say the $x-u$ path of H has length 2. Let u' be the neighbour of u on this path. If uw or uv is an edge of G, then u' is not a cut-vertex of H. So $uw, uv \notin E(G)$. Since G has no induced paths of length at least 3, $u'v, u'w$ are necessarily edges of G. However, then x is not a cut-vertex of $\langle V(H)\rangle$. So this case cannot occur. Thus u, v, w are all adjacent with x. Now $\langle\{u, v, w\}\rangle$ is not connected; otherwise, x is not a cut-vertex of H. Hence H is a Steiner tree for $\{u, v, w\}$. So S is g_3-convex. Thus the m_3-convex hull of the extreme points of S is the same as the g_3-convex hull of the extreme points of S. So $(V, \mathcal{M}_{g_3}(G))$ is a convex geometry. ■

Recall that a graph G is a replicated twin C_4 if it is isomorphic to any one of the four graphs shown in Figure 2(a), where any subset of the dotted edges may belong to G. The collection of the four replicated twin C_4 graphs is denoted by $\mathcal{R}_{C_4}$.

From Theorem 2 we now obtain the following,

Corollary 4. *Let $G = (V, E)$ be a graph. Then the following are equivalent:*

(1) *G is P_4- and $\mathcal{R}_{C_4}$-free.*

(2) *$(V(G),\ \mathcal{M}_{m_3}(G))$ is a convex geometry.*

2.2. m_T-convex geometries

In this section we show that if $T = \{k_1, k_2, \ldots, k_t\}$ is a collection of positive integers such that $2 \le k_1 < k_2 < \cdots < k_t$, then the class of graphs for which the m_T-convex sets form a convex geometry is precisely the same as the class for which the m_{k_1}-convex sets form a convex geometry. For a connected graph G let $\mathcal{M}_{m_T}(G)$ be the collection of all m_T-convex sets of G and for an integer $k \ge 2$, let $\mathcal{M}_{m_k}(G)$ be the collection of all m_k-convex sets.

Lemma 5. *Suppose that G is a graph and $U = \{u_1, u_2, \ldots, u_l\}$ a set of $l \ge 2$ vertices of G. Suppose that T is a minimal U-tree and k is an integer such that $2 \le k \le l$. If $v \in V(T) \setminus U$, then there exists a k element subset W of U and a minimal W-tree T' containing v.*

Proof. If $l = 2$, the result follows immediately. Suppose thus that $l > 2$. For the remainder of the proof we proceed by induction on k. We prove a slightly stronger result than that stated in the lemma. We show that the W-tree T' can be chosen so that it has the property that it is a subtree of T and whenever F is a subtree of T that contains T', then every vertex of $V(T') \setminus W$ is a cut-vertex of $\langle V(F)\rangle$. Since v is a cut-vertex of $H = \langle V(T)\rangle$, v is a cut-vertex of T. Let x_1 and y_1 be neighbours of v in T that belong to different components of $H - v$, say H_1 and H_2, respectively. We first construct a subtree T' of T that contains v and exactly two vertices of U. This subtree will be an induced path of which one branch at v is contained in H_1 and the other branch at v in H_2. Begin the construction of T' by starting with the vertices v, x_1, y_1 and the edges vx_1, vy_1. If $x_1 \in U$, then the construction of the branch of T' contained in H_1 is completed. If $x_1 \notin U$, x_1 is a cut-vertex of H. Let x_2 be a neighbour of x_1 in T such that x_2 belongs to a component of $H - x_1$ that does not contain v. Add x_2 and the edge x_1x_2 to the tree that is being constructed. If $x_2 \in U$, the construction of the branch of T' at v contained in H_1 is completed; otherwise, we let $x_3 \neq x_1$ be a neighbour of x_2 in T that belongs to a component of $H - x_2$ that does not contain x_1. We continue in this manner constructing a sequence $x_1, x_2, x_3 \ldots$ of vertices in H_1 such that for $i \ge 2$, x_i is a neighbour of x_{i-1} in T and such that x_i belongs to a component of $H - x_{i-1}$ that does not contain x_{i-2}. We stop with the smallest j such that $x_j \in U$. Such a j must exist since the path we are constructing in H_1 is a path in T starting at v and ending necessarily at a vertex of U. (We may of course reach a vertex of U before we reach a leaf of T.) The branch of T' at v contained in H_1 is the path $vx_1x_2 \ldots x_j$. We proceed in the same manner as for H_1 and x_1 when constructing the branch of T' contained in H_2 that starts with vy_1. These two branches produce the tree T' which is necessarily a minimal W-tree for the set W consisting of the two leaves of T'. Note that T' is constructed in such a way that if F is a subtree of T that contains T' then every vertex of $V(T') \setminus U$ is a cut-vertex of $\langle V(F)\rangle$.

Suppose now that $2 < k \leq l$ and that we have constructed a subtree T'' of T containing v that is a minimal W'-tree for some $k-1$ element subset W' of U such that $V(T'') \cap U = W'$. Suppose also that if F is a subtree of T that contains T'', then every vertex of $V(T'') \setminus U$ is a cut-vertex of $\langle F \rangle$. If some vertex v' of T'' has a neighbour z in T that belongs to U but not to T'', then we add z and the edge $v'z$ to T'' to obtain a minimal $W' \cup \{z\}$-tree with the desired properties. Otherwise let $z_1 \in N_T(V(T''))$ and let z_0 be its neighbour in T that belongs to T''. Add z_1 and the edge z_0z_1 to the tree T''. Now let z_2 be a neighbour of z_1 in T that does not belong to the component of $H - z_1$ that contains z_0 (and hence T''). Add z_2 and the edge z_1z_2 to the current tree. If $z_2 \in U$ we stop and let T' be the tree we constructed; otherwise, let z_3 be a neighbour of z_2 in T that does not belong to the component of $H - z_2$ that contains z_1. Add z_3 and the edge z_2z_3 to the current tree. We continue in this manner constructing a sequence $z_0, z_1, z_2 \ldots$ such that $z_{i-1}z_i$ is an edge of T and of the tree we are constructing. We stop when we encounter for the first time a vertex that belongs to U. As was the case for $k = 2$ it is not difficult to see that this needs to happen since the sequence $z_0z_1 \ldots$ we are constructing corresponds to a path in T which must eventually encounter a vertex of U. Let $s \geq 2$ be the smallest integer such that $z_s \in U$. Let T' be the tree obtained from T'' by adding the path $z_0z_1 \ldots z_s$ and let $W = W' \cup \{z_s\}$. Then T' is a W-tree with the desired properties. ■

Theorem 6. *Let $G = (V, E)$ be a connected graph and let $T = \{k_1, k_2, \ldots k_t\}$ be a collection of integers such that $2 \leq k_1 < k_2 < \cdots < k_t$. Then $(V, \mathcal{M}_{m_T}(G))$ is a convex geometry if and only if $(V, \mathcal{M}_{m_{k_1}}(G))$ is a convex geometry.*

Proof.

Claim 1. $\mathcal{M}_{m_T}(G) = \mathcal{M}_{m_{k_1}}(G)$.

Proof. Let S be a m_{k_1}-convex set. We show first that S is also an m_T-convex set. If this is not the case, then there is some $i > 1$ such that S is not m_{k_i}-convex. Thus there is a set U of k_i vertices in S such that the monophonic interval of U is not contained in S, i.e., for some vertex $v \notin S$ there is a minimal U-tree containing v. By Lemma 5 there is a k_1 element subset W of U and a minimal W-tree containing v. This is not possible since S is m_{k_1}-convex. So every m_{k_1} convex set is m_T-convex. By definition every m_T-convex set is m_{k_1}-convex. Hence $\mathcal{M}_{m_T}(G) = \mathcal{M}_{m_{k_1}}(G)$. □

Claim 2. *The extreme vertices of S with respect to the m_{k_1}-convexity are the same as the extreme vertices with respect to the m_T-convexity.*

Proof. If x is an extreme vertex of S with respect to the m_T-convexity, then $S \setminus \{x\}$ is an m_T-convex set and thus an m_{k_1}-convex set. So x is an extreme vertex of S with respect to the m_{k_1}-convexity. Suppose now that x is an extreme vertex of S with respect to the m_{k_1}-convexity. If x is not an extreme vertex with respect to the m_T-convexity, then there is some $i > 1$ such that $S \setminus \{x\}$ is not m_{k_i}-convex. Hence there is a k_i element subset U of $S \setminus \{x\}$ such that x belongs to some minimal U-tree of U. By Lemma 5 there is a k_1 element subset W of U and a minimal W-tree containing x. This contradicts the fact that $S \setminus \{x\}$ is m_{k_1}-convex. So the extreme vertices of S with respect to the m_{k_1}-convexity are the same as the extreme vertices with respect to the m_T-convexity. □

Suppose first that $(V, \mathcal{M}_{m_T}(G))$ is a convex geometry. Let S be an m_{k_1}-convex set. By Claim 1, S is m_T-convex and by Claim 2 the extreme vertices of S with respect to m_{k_1} convexity are the same as the extreme vertices with respect to m_T convexity. The m_{k_1}-convex hull of the extreme vertices of S is a subset of S. If it is a proper subset of S, then this proper subset is, by Claim 1, also m_T-convex. Thus the m_T-convex hull of the extreme vertices of S is also a proper subset of S. This contradicts the hypothesis. Thus $(V, \mathcal{M}_{m_{k_1}}(G))$ is a convex geometry.

For the converse suppose that $(V, \mathcal{M}_{m_{k_1}}(G))$ is a convex geometry. Let S be an m_T-convex set. Then S is, by definition, m_{k_1}-convex. By Claim 2, the extreme vertices of S with respect to the m_{k_1} convexity are the same as the extreme vertices with respect to the m_T convexity. The convex hull with respect to the m_T convexity of the extreme vertices of S is a subset of S. If it is a proper subset of S, then this proper subset contains the convex hull of the extreme vertices with respect to m_{k_1} convexity. This contradicts the hypothesis. Thus $(V, \mathcal{M}_{m_T}(G))$ is a convex geometry. ■

2.3. m_3^3-convex geometries

Before characterizing the class of graphs for which the m_3^3-convex sets form a convex geometry, we introduce another useful result. Recall that the graphs for which the m^3-convex sets form a convex geometry are characterized in [10] as the (house, hole, domino, A)-free graphs. The proof of this characterization depends on the following result also proven in [10].

Theorem 7. *If G is a (house, hole, domino, A)-free graph, then every vertex of G is either semisimplicial or lies on an induced path of length at least 3 between two semisimplicial vertices.*

We now proceed to characterize those graphs for which the m_3^3-convex alignment forms a convex geometry. Let $\mathcal{M}_{m_3^3}(G)$ be the m_3^3-convex alignment of a graph G. Recall that a graph F is a replicated twin C_4 if it is isomorphic to one of the four graphs shown in Figure 2(a) where any subset of the dotted edges may be chosen to belong to F, and the collection of replicated twin C_4's is denoted by $\mathcal{R}_{C_4}$. A graph F is a *tailed twin* C_4 if it is isomorphic to one of the two graphs

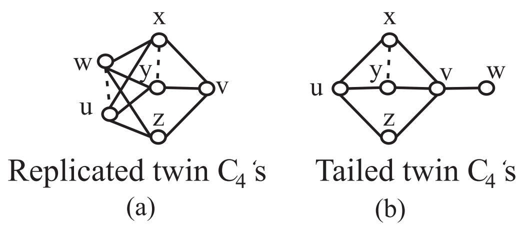

Figure 2. Forbidden subgraphs for m_3^3-convex geometries.

shown in Figure 2(b) where again any subset of the dotted edges may be chosen to belong to F. We denote the collection of tailed twin C_4's by $\mathcal{T}_{C_4}$.

Theorem 8. *For a connected graph $G = (V, E)$ the following are equivalent:*
(1) *G is (house, hole, domino, A, $\mathcal{R}_{C_4}$, $\mathcal{T}_{C_4}$)-free.*
(2) *$(V, \mathcal{M}_{m_3^3}(G))$ is a convex geometry.*

Proof. (2) $\to$ (1) Suppose F is a house, hole, domino, replicated twin C_4 or a tailed twin C_4. Then F has at most one $3SS$ vertex. Suppose G is a graph that contains F as an induced subgraph. Then the set of extreme points of the convex hull of $V(F)$ is contained in the collection of $3SS$ vertices of F. So the convex hull of the extreme points of the m_3^3-convex hull of $V(F)$ is empty or consists of a single vertex and hence cannot contain all the vertices of F. So in this case the m_3^3-convex alignment of G does not form a convex geometry. It is shown in [6] that if the m_3^3-convex sets of a graph G form a convex geometry, then G does not contain an A.

(1) $\to$ (2) It is not difficult to see that if G is a connected graph of order at most 4, then every m_3^3-convex set is the convex hull of its extreme points. Suppose now that there exists a connected (house, hole, domino, A, $\mathcal{R}_{C_4}$, $\mathcal{T}_{C_4}$)-free graph G (abbreviated by $HHDA\mathcal{R}_{C_4}\mathcal{T}_{C_4}$-free graph G) for which $(V, \mathcal{M}_{m_3^3}(G))$ is not a convex geometry. We may assume that G is such a graph of smallest possible order. Thus every proper connected induced subgraph of G has the property that its vertex set is the m_3^3-convex hull of its extreme points, i.e, the $3SS$ vertices. By assumption V is not the m_3^3-convex hull of its extreme points. Since V is m_3^3-convex it is m^3-convex; so, by Theorem 7, every vertex of G is either semisimplicial or lies on an induced path of length at least 3 between two semisimplicial vertices. Thus if every semisimplicial vertex is $3SS$, then V is the m_3^3-convex hull of its extreme points, a contradiction. Let S be the m_3^3-convex hull of the $3SS$ vertices of G. By assumption $V \setminus S \neq \emptyset$.

Case 1. $V \setminus S$ contains a vertex a that is not semisimplicial. Since G is $HHDA$-free and V is m^3-convex, Theorem 7 guarantees that a lies on an induced path of length at least 3 between two semisimplicial vertices w, w' of G.

Among all pairs $\{w, w'\}$ of semisimplicial vertices such that $a \in I_{m^3}[w, w']$ we will assume that $\{v, v'\}$ is a pair that has a maximum number of $3SS$ vertices. At least one of v and v', say v, is not $3SS$ in G; otherwise, a lies on an induced path of length at least 3 between two extreme vertices of V. Since v is semisimplicial but not $3SS$ it must be the centre of an induced claw or paw in G. Let x, y, z be the peripheral vertices of a claw or paw containing v as centre where $xz, yz \notin E$.

Let $I^{(a)}_{m^3}[v, v']$ be the collection of all vertices that lie in some induced $v - v'$ path of length at least 3 that contains the vertex a.

Claim 1. *None of x, y or z is in $I^{(a)}_{m^3}[v, v']$.*

Proof. Assume, to the contrary, that P is an induced path of length at least 3 containing v, v', a and one of x, y, z. Suppose first that $z \in I^{(a)}_{m^3}$. Since P is an induced path $x, y \notin V(P)$. Let z' be the neighbour of z on P different from v. Then $xz', yz' \in E(G)$; otherwise, $xvzz'$ or $yvzz'$ is an induced P_4 which is not possible since v is semisimplicial. Let z'' be the neighbour of z' on P different from z. If $z''x, z''y \notin E$, then $\langle\{v, x, y, z, z', z''\}\rangle$ is a tailed twin C_4 which is forbidden. So we may assume $xz'' \in E$. Then $\langle\{x, v, z, z', z''\}\rangle$ is a house which is forbidden. Similarly $yz'' \notin E$. Hence $z \notin I^{(a)}_{m^3}[v, v']$. Suppose now that x or y, say x, belongs to P. In that case we may assume $xy \in E$; otherwise, we can argue as for z that G contains a forbidden subgraph as induced subgraph. Let x' be the neighbour of x on P different from v. Then $zx' \in E$; otherwise, v is not semisimplicial. Also $yx' \in E$; otherwise, $\langle\{y, v, z, x, x'\}\rangle$ is a house which is forbidden. Let x'' be the neighbour of x' on P different from x. If $zx'' \in E$, then $\langle\{z, v, x, x', x''\}\rangle$ is a house which is forbidden. So $zx'' \notin E$. If $yx'' \in E$, then $\langle\{y, v, z, x', x''\}\rangle$ is a house which is not possible. But then $\langle\{y, v, z, x, x', x''\}\rangle$ induces a tailed twin C_4 which is not possible. So we may assume that $x, y, z \notin I^{(a)}_{m^3}$. This completes the proof. □

Claim 2. *If $P : (v =)v_0v_1 \ldots v_k(= v')$ is an induced $v - v'$path of length at least 3 containing a, then each of x, y, z is adjacent with v_1 but with no v_i for $2 \le i \le k$.*

Proof. If $zv_i \in E$ for $2 \le i \le k$, then both $xv_i, yv_i \in E$; otherwise, $xvzv_i$ or $yvzv_i$ is an induced P_4 having v as centre which is not possible since v is semisimplicial. Similarly if $xv_i \in E$ for some i, $2 \le i \le k$, then $zv_i \in E$ and thus $yv_i \in E$. Hence for every i, $2 \le i \le k$, the vertices x, y, z are either all adjacent with v_i or all are non-adjacent with v_i. Also if there is an i, $2 \le i < k$ such that x, y, z are all adjacent with v_i, then x, y, z are all adjacent with v_{i+1}; otherwise, $\langle\{v, x, y, z, v_i, v_{i+1}\rangle$ is a tailed twin C_4. Thus, if x, y, z are all adjacent with v_i for some $2 \le i < k$, then $\langle\{v, x, y, z, v_i, v_{i+1}\}\rangle$ is a replicated twin C_4; which is forbidden. We may thus assume x, y, z are all nonadjacent with v_i for $2 \le i < k$. Since v is semisimplicial and zvv_1v_2 is a P_4, $zv_1 \in E$. Similarly $xv_1, yv_1 \in E$. If

x, y, z are all adjacent with v_k, then $\langle\{v, v_1, x, y, z, v_k\}\rangle$ is a replicated twin C_4 which is forbidden. We have thus shown that x, y, z are all adjacent with v_1 and that they are nonadjacent with v_i for $2 \leq i \leq k$. This completes the proof. □

So $zv_1v_2 \ldots v_k$ is an induced path of length at least 3 containing a as internal vertex. Now z is not a $3SS$ vertex; otherwise, we have a contradiction to our choice of the pair v, v'. So z is either not semisimplicial or the centre of an induced claw or paw.

Claim 3. *x, y, z can be chosen in such a way that z is semisimplicial.*

Proof. Suppose that z is not semisimplicial. Then there exists an induced path $wzrs$ having z as centre. If v is on this path, then v is w. Suppose $w = v$. Then $\{x, y\} \cap \{r, s\} = \emptyset$ since v is adjacent with x and y but not r and s. Now $xvzr$ (respectively, $yvzr$) is an induced P_4 having v as centre unless xr (respectively, yr) is an edge of G. So $xr, yr \in E$. If $xs \in E$, then $\langle\{x, v, z, r, s\}\rangle$ is a house which is forbidden. So $xs \notin E$. Similarly, $ys \notin E$. Since $xs, ys \notin E$, $\langle\{v, x, y, z, r, s\}\rangle$ is a tailed twin C_4 which is forbidden. So $w \neq v$.

Since $vzrs$ is a path of order 4 having z as centre and v as end-vertex, it follows from the above that it cannot be an induced path. So vr or vs is an edge of G. Suppose first that $vr \notin E$. Then $vs \in E$. Now $wzvs$ is an induced P_4 unless $vw \in E$. However, then $\langle\{z, r, s, v, , w, \}\rangle$ is a house which is forbidden. So $vr \in E$. If $vs \notin E$, then x and y are not on the path $wzrs$ (i.e., $s \neq x, y$). Now $xvrs$ is an induced P_4's having v as centre unless xr or xs is an edge of G. If $xr \notin E$, then $xs \in E$. But then $\langle\{x, v, r, s, z\}\rangle$ is a house. So $xr \in E$. Similarly $yr \in E$. If $wv \in E$, then $wvrs$ is an induced P_4 having v as centre. This is not possible. So $vw \notin E$. Now $wzvx$ and $wzvy$ are induced P_4's unless $wx, wy \in E$. However, then $\langle\{w, x, y, z, v, r\}\rangle$ is a replicated twin C_4 which is forbidden. So $vr, vs \in E$. Now $wzvs$ is an induced P_4 having v as centre unless $vw \in E$. Note $\langle\{v, w, r, s\}\rangle$ is a paw with v as centre. So as we argued for x, y, z, none of w, r, s is v_1 and each of w, r, s is adjacent with v_1 and to no v_i for $2 \leq i \leq k$.

We know, since G is $HHDA$-free, that z is an interior vertex of an induced path of length at least 3 between two semisimplicial vertices. Let $Q : u_0u_1 \ldots u_m$ be such a path. Then $z = u_i$ for some i $(0 < i < m)$. Thus $u_{i-1}u_iu_{i+1}u_{i+2}$ or $u_{i-1}u_{i-1}u_iu_{i+1}$ is an induced P_4 having z as centre vertex, say the former. As we showed for the path $wzrs$, $v \notin \{u_{i-1}, u_{i+1}, u_{i+2}\}$ and u_{i-1}, u_i, u_{i+1}, and u_{i+2} are each adjacent with both v and v_1 but with no other vertex of P. If $i - 1 \neq 0$, we repeat the argument with $u_{i-2}u_{i-1}u_iu_{i+1}$ and u_{i-1} instead of u_i since $\langle\{v, u_{i-1}, u_{i+1}, u_{i+2}\}\rangle$ is a paw having v as centre. So u_{i-2} is adjacent with both v and v_1 but with no v_j for $2 \leq j \leq k$. Continuing in this manner we see that for all j $(0 \leq j \leq i+2)$, vertex u_j is not on P and u_j is adjacent with both v and v_1. Similarly one can show if $i+2 \neq m$, then every vertex u_j for $i+2 < j \leq m$

is not on P and u_j is adjacent with both v and v_1 and with no v_l for $3 \leq l \leq k$. Hence v is the centre of the paw $\langle\{v, u_0, u_2, u_3\}\rangle$ where $u_0u_2, u_0u_3 \notin E$. So we may assume $z = u_0$, $x = u_2$ and $y = u_3$. This completes the proof. □

The path $zv_1v_2 \ldots v_k$ is an induced path of length at least 3 containing a as interior vertex and since z, v_k are both semisimplicial. Vertex z cannot be $3SS$; otherwise, we have a contradiction to the choice of the pair v, v'. So z is the centre of a claw or paw whose peripheral vertices are, say r, s, t where $tr, ts \notin E$. By Claim 3 we may assume t is semisimplicial.

Claim 4. *$v \notin \{r, s, t\}$.*

Proof. Suppose first that $v = t$. Then $rzvx$ and $szvx$ are induced P_4's having v as centre unless $rx, sx \in E$. Similarly $ry, sy \in E$. However, then $\langle\{v, x, y, z, s, r\}$ is a replicated twin C_4 which is forbidden. So $v \neq t$. Suppose now that v is r or s, say $v = r$. Thus we may assume $rs \in E$; otherwise, we can repeat the argument we used for t. Then $xvzt$ and $yvzt$ are induced P_4's unless $xt, yt \in E$. Now $sztx$ and $szty$ are induced P_4's having z as centre unless $sx, sy \in E$. But then $\langle\{t, x, y, z, v, s\}\rangle$ is a replicated twin C_4 which is forbidden. Hence $v \notin \{r, s, t\}$. This completes the proof of Claim 4. □

Claim 5. *v is adjacent with each of r, s, t.*

Proof. If v is nonadjacent with some $b \in \{r, s, t\}$, then $bzvx$ and $bzvy$ are induced P_4's having v as centre vertex unless $bx, by \in E$. Thus if v is nonadjacent with two vertices in r, s, t, then these two vertices together with v and x, y, z induce a replicated twin C_4 which is forbidden. So v is adjacent with at least two of the vertices r, s, t. Suppose v is nonadjacent with t. Then $tx, ty, vs, vr \in E$ and $txvr$, $txvs$, $tyvr$ and $tyvs$ are induced P_4's having v as centre vertex unless xr, xs, yr, ys, respectively are edges of G. However, then $\langle\{t, x, y, z, r, s\}\rangle$ is a replicated twin C_4. If v is nonadjacent with r or s, say r, then $vs, vt, rx, ry \in E$. We may also assume $rs \in E$; otherwise, we can argue as for t that G has a replicated twin C_4. Now $tvxr$ and $tvyr$ are induced P_4's having v as centre vertex unless $xt, yt \in E$. However, then $\langle\{r, x, y, z, v, t\}\rangle$ is a replicated twin C_4 which is forbidden. This completes the proof of Claim 5. □

Thus, by Claim 2, r, s, t are all adjacent with v_1 and with none of the vertices v_j for $2 \leq j \leq k$. So $tv_1v_2 \ldots v_k$ is an induced path of length at least three containing a as interior vertex. By our choice of the pair v, v' we know that t is not $3SS$. So t is the centre of a claw or paw.

Since t is adjacent with z it is neither x nor y. By Claims 3, t is the centre of a claw or paw with peripheral vertices r_1, s_1, t_1 such that $t_1s_1, t_1r_1 \notin E$ and such that t_1 is semisimplicial. Since both v and z are the centre of a claw or paw whose peripheral vertices are r, s, t, it follows from Claim 4 that $v, z \notin \{r_1, s_1, t_1\}$ and by Claim 5, v and z are both adjacent with every vertex of $\{r_1, s_1, t_1\}$. Moreover, $\{r_1, s_1, t_1\} \cap \{x, y, r, s\} = \emptyset$. Now t_1 is semisimplicial and r_1, s_1, t_1 are adjacent with v_1 but with no v_j for $2 \leq j \leq k$. Thus as for t we can argue that t_1 is the centre of some induced claw or paw $\langle\{t_1, r_2, s_2, t_2\}\rangle$ where we may assume $t_2s_2, t_2r_2 \notin E(G)$. Moreover, one can argue as before that $v, z, x, y, r, s, t, r_1, s_1, t_1 \notin \{r_2, s_2, t_2\}$ and that v, z, t and t_1 are all adjacent with r_2, s_2, t_2. Continuing in this manner we see that G has an infinite number of vertices which is not possible. So this case cannot occur.

Case 2. Every vertex of G that is not semisimplicial belongs to S.

Subcase 2.1. All vertices of G are semisimplicial. Then the extreme points of G are the vertices that are not the centre of a claw or paw in G and G has no induced path of length at least 3. So the m_3^3-convex sets are the m_3-convex sets and the m_3^3-convex hull of the extreme points is just the m_3-convex hull of the extreme points. Also since G has no induced path of length 3, G is 3-fan free. Since G is (P_4, $\mathcal{R}_{C_4}$)-free it follows from Corollary 4 that $(V, \mathcal{M}_{m_3^3}(G))$ is a convex geometry.

Subcase 2.2. There exist vertices that are not semisimplicial. From the case we are in, these vertices all belong to S. So S has vertices that are not $3SS$. Thus $\langle S\rangle$ has at least four vertices and is therefore connected.

We show first that $G - S$ has exactly one component. Suppose $G - S$ has at least two components. Let $H_1, H_2, \ldots, H_l$ be the components of $G - S$. Then the $3SS$ vertices of G, which necessarily belong to S, are still $3SS$ vertices of $G - V(H_1)$. Moreover if $G - V(H_1)$ contains any $3SS$ vertices that were not $3SS$ vertices of G these are also contained in S since such vertices are necessarily adjacent with vertices of H_1. But since all $3SS$ vertices of $G - V(H_1)$ are contained in S, their m_3^3-convex hull is also contained in S since S is m_3^3-convex. However, by our choice of G, the m_3^3-convex hull of the $3SS$ vertices of $G - V(H_1)$ is $V(G - V(H_1)) \neq S$. This contradiction shows that $G - S$ has exactly one component, say H.

Since S contains vertices that are not $3SS$, each such vertex v is either the interior vertex of an induced path of length at least 3 whose end vertices are in S or there exist three vertices x, y, z in S such that v is an interior vertex of a minimal $\{x, y, z\}$-tree. In the first case all the vertices on the induced path belong to S. In the second case $\langle\{x, y, z\}\rangle$ is not connected. In either case S contains three vertices that induce a disconnected graph. Hence a vertex of $G - S$ cannot be adjacent to all vertices of S; otherwise, it would belong to a minimal R-tree for some set R of three vertices of S. This is not possible since S is m_3^3-convex.

Observe that every vertex of $G - S$ is adjacent with some vertex of S; otherwise, there is a vertex b distance 2 from S in G. Let bcS be a $b - S$ path. Since c

is not adjacent to every vertex of S, there is some vertex $d \in S$ and a neighbour d' of d in S such $cd \in E$ and $cd' \notin E$. Thus $bcdd'$ is an induced P_4. However, then c is not semisimplicial and thus by the case we are in $c \in S$. This contradiction shows that every vertex of $G - S$ is adjacent with some vertex of S.

We now show that every vertex of S is adjacent with a vertex of H. Suppose some vertex v of S is not adjacent with any vertex of H. Suppose first that $G - v$ is connected. By the minimality of G, $V(G - v)$ is the m_3^3-convex hull of its extreme points, i.e., its $3SS$ vertices. Since the extreme points of $G - v$ are contained in S and since S is m_3^3-convex, the m_3^3-convex hull of the extreme points of $V(G - v)$ is contained in $S \setminus \{v\}$, a contradiction. Suppose next that $G - v$ is disconnected. Then, by the minimality of G, the vertex set of each component is the m_3^3-convex hull of its extreme points. Since $G - v$ has at least three vertices, there is a set R of three vertices of $G - v$ such that v belongs to a minimal R-tree. (Pick the vertices of R in such a way that they belong to at least two distinct components of $G - v$.) However, since the extreme points of $G - v$ are contained in S, the extreme points of each component of $G - v$ are also in S. Thus the m_3^3-convex hull of the extreme points of each component is contained in S. Since v is also in S the m_3^3-convex hull of the union of the m_3^3-convex hulls of the components together with v is also contained in S, a contradiction. So each vertex of S is adjacent with some vertex of H.

Suppose first that $\langle S \rangle$ contains an induced path of order 4, say $wrst$. From the above we know that w is adjacent with some vertex w' in H. Now w' cannot be adjacent with both s and t; otherwise, w would be on a minimal $\{w, s, t\}$-tree and thus in the m_3^3-convex hull of the extreme points of G. Moreover, w' cannot be adjacent with exactly one of s and t; otherwise, w' is not semisimplicial since either $ww'st$ (if $w's \in E$) or $ww'ts$ (if $w't \in E$) are induced P_4's having w as centre vertex. Let t' be a neighbour of t in H_1. We have argued that $t' \neq w'$. Since H is connected there is an induced $w' - t'$ path P' in H. Since all vertices of H are semisimplicial in G, P' has length 1 or 2. As we argued for w' we can show that t' is not adjacent with either w or r. If $w't' \in E$, then $ww't't$ is an induced P_4 containing w' and t' as centre vertices. This is not possible since all vertices of H are semisimplicial. Suppose thus that P' has length 2 and let w'' be the common neighbour of w' and t' on P'. If w'' is adjacent with w, then it is nonadjacent with s and t (we argue as for w'). However, then $ww''t't$ is an induced P_4 containing w'' as centre vertex which is not possible. Similarly if w'' is adjacent with t we can show that w'' is not semisimplicial. But now $ww'w''t'$ is an induced P_4 containing w' as centre vertex. This is not possible.

Thus $\langle S \rangle$ has no induced P_4. By the case we are considering, H contains no induced P_4's. We know that G has an induced path of order 4 and that any such path necessarily contains vertices from H and S.

We show first that $diam(G) \leq 2$. Suppose $diam(G) = d \geq 3$. Let v, v' be vertices such that $d(v,v') = d$. Let V_i be the vertices distance i, $1 \le i \le d$ from v in the g-interval between v and v'. Since H is connected and all vertices of H are semisimplicial, H contains no induced path of order at least 4. Moreover, since $\langle S\rangle$ does not contain an induced path of order at least 4, one of v and v', say v, belongs to H and the other to S. In fact $d = 3$. Since the vertices of V_1 and V_2 are not semisimplicial, they belong to S. No neighbour x of v in H is adjacent with a vertex of V_i for $i = 2$ or 3; otherwise, x is either not semisimplicial or $d(v,v') < d = 3$; neither of these situations is possible. Moreover, such a neighbour is adjacent with every vertex of V_1; otherwise, v is not semisimplicial.

Suppose now that v has a neighbour x in $S \setminus V_1$. Then x is not adjacent with a vertex of V_i for $i = 2$ or 3; otherwise, x is either in V_1 or $d(v,v') < d = 3$, neither of which is possible. Moreover, such a vertex x is adjacent with every vertex of V_1; otherwise, v is not semisimplicial. (Note that if $y \in V_1$ is such that $xy \notin E$ and that if $s \in V_2$ is such that $sy \in E$, then $xvys$ is an induced P_4 having v as centre vertex.) But then $\langle S\rangle$ contains an induced $x - v'$ path of order 4 which is not possible in the case we are considering. So V_1 is the collection of neighbours of v in S. Similarly every neighbour y of v in H is adjacent with precisely the vertices of V_1 and with no other vertices of S. It is not difficult to see that $d(y,v') = 3$. So arguing as we did for v we see that every neighbour of y in H is adjacent to precisely the vertices of V_1 and to no other vertices of S. Since H is connected and contains no induced P_4's it follows that every vertex of H is adjacent with precisely the vertices of V_1 and to no other vertices of S. But then not every vertex of S is adjacent with a vertex of H; a contradiction. So $diam(G) \le 2$.

Let $P : wrst$ be an induced path. Then P is not contained in $\langle S\rangle$ and P is not contained in H. Since r and s are not semisimplicial they belong to S (from the case we are considering). Suppose first that w and t both belong to H. Since H is connected there is an induced $w - t$ path in H having length at most 2. Let u be a common neighbour of w and t in H. Since G contains no house and hole, u is adjacent with r and s. Since r and s are not semisimplicial they are not $3SS$ vertices. But r and s belong to the m_3^3-convex hull of the extreme points of G. So they must be the centre of a claw or paw whose peripheral vertices belong to S. Let x, y, z be the peripheral vertices of such a claw or paw in $\langle S\rangle$ having r as centre vertex. Vertex t cannot be adjacent to all three vertices x, y, z; otherwise, t belongs to the m_3^3-convex hull of the $3SS$ vertices of G, i.e., $t \in S$, a contradiction. We may assume $tx \notin E$. Then $xrut$ is an induced P_4 having u as centre vertex unless $ux \in E$. Also u is not adjacent with each of the three vertices x, y, z; otherwise, $u \in S$. Suppose $yu \notin E$. Then $yrut$ is an induced P_4 having u as centre vertex unless $yt \in E$. If $xy \notin E$, then $\langle\{y, r, u, t, x\}\rangle$ is a house which is forbidden. So $xy \in E$. Since x, y, z, are the peripheral vertices of a claw or paw it follows that $zx, zy \notin E$. Now $zrut$ is an induced P_4 having u

as centre vertex unless one of zu or zt is and edge of G. If $zt \notin E$, then $zu \in E$ and $\langle\{t, u, r, y, z\}\rangle$ is a house which is forbidden. So $zt \in E$. But then $ztyx$ is an induced P_4 having t as centre vertex, which is not possible since $t \in V(H)$.

So we may assume that $w \in V(H)$ and that $t \in S$. Of course $r, s \in S$. We show first that H contains a common neighbour of w and t. Suppose this is not the case. We know that t has a neighbour t' in H. Since $diam(H) \leq 2$ there is a vertex u in H that is a common neighbour of w and t'. By assumption, $ut \notin E$. But then $tt'uw$ is an induced P_4 having u as centre vertex which is not possible. So there is a vertex u in H that is adjacent with w and t. Since G contains no house or hole, $us, ur \in E$. Since s is not semisimplicial but $s \in S$ it must be the centre of an induced claw or paw in $\langle S\rangle$. Let x, y, z be the peripheral vertices of such an induced claw or paw having s as centre. Since $w \in V(H)$, w is not adjacent with all three of the vertices x, y, z. Suppose $wx \notin E$. Then $xsuw$ is an induced P_4 having u as centre vertex unless $ux \in E$. Similarly u is not adjacent with all three vertixes x, y, z. We may assume $uy \notin E$. Then $ysuw$ is an induced P_4 unless $wy \in E$. If $xy \notin E$, then $\langle\{w, u, s, y, x\}\rangle$ is a house which is forbidden. So $xy \in E$. Since x, y, z are the peripheral vertices of a claw or paw, we conclude that $zx, zy \notin E$. Now $zsuw$ is an induced P_4 having u as centre vertex unless one of zu, zw is in E. If $zw \notin E$, then $\langle\{w, u, s, y, z\}\rangle$ is a house which is forbidden. So $zw \in E$. But then $zwyx$ is an induced P_4 having w as centre vertex. This completes the proof. ■

3. Concluding Remarks

In this paper we introduced the definition of a minimal U-tree where U is a set of vertices in a connected graph G and defined several graph convexities that use this concept. Of course every Steiner tree for U is a minimal U-tree but the converse does not hold. So the Steiner interval is contained in the monophonic interval for U. Two graph invariants have been studied that indicate the smallest number of vertices that "span" the vertex set of a graph using different interval notions. In particular, the geodetic number of a graph G, denoted by $g(G)$, is the smallest number k of vertices in G for which there exists a set S of k vertices with the property that $V(G) = \cup_{u,v \in S} I_g[u, v]$ and the Steiner geodetic number of G, denoted by $sg(G)$, is the smallest number k for which there exists a set S of k vertices with $V(G) = I(S)$. These invariants can be extended naturally if we replace geodetic (Steiner) intervals by monophonic intervals. Let $m(G)$ be the smallest integer k for which there exists a set S of k vertices in G such that $V(G) = \cup_{u,v \in S} I_m[u, v]$ and $sm(G)$ the smallest integer k such that there exists a set S of k vertices in G with $V(G) = I_m(S)$. It was shown in [17] that in general there is no relationship between $g(G)$ and $sg(G)$ by showing that the ratio $g(G)/sg(G)$ can be made arbitrarily large and arbitrarily small. However, such is not the case for the ratio $m(G)/sm(G)$. It is not difficult to see that it can never exceed 1. To see this suppose that S is a set of vertices in G such that $V(G) = I_m(S)$. If $w \in V(G) \setminus S$, then w belongs to some minimal S-tree T. Thus w is a cut-vertex of $H = \langle V(T) \rangle$. Hence there exist two vertices u and v of S that belong to distinct components in $H - w$. Thus w lies on an induced $u - v$ path. So S also has the property that $V(G) = \cup_{u,v \in S} I_m[u, v]$. But $m(G)/sm(G)$ can be arbitrarily small. Take for example the complete bipartite graph $K_{r,s}$ where $2 \leq r \leq s$. It is not difficult to see that $m(K_{r,s}) = 4$ whereas $sm(G) = r$. Hence by choosing r sufficiently large the ratio $m(G)/sm(G)$ can be made as small as we wish. The problem of finding $g(G)$ is known to be NP-hard (see [1]). In view of the fact that the problem of finding Steiner trees for sets of vertices in a graph is NP-hard it is likely that the problem of finding $sg(G)$ may also be NP-hard. However not much is known about the computability of $m(G)$ and $sm(G)$.

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